# Holographic SK contours and EFTs for Fluids

Julio Virrueta

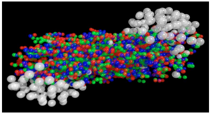
FSU Jena

Based on: T. He, R. Loganayagam, M. Rangamani, JV, S. Zhou [2108.03244], [2205.03415], [2211.07683], [2306.01055].

# An Old, Beautiful, Story: Fluid dynamics

Fluid dynamics, as an effective theory, has a very wide range of applicability.





- In a fluid system, one assumes *local* equilibrium: $T(x)$, $\mu(x)$, ....
- The equations of motion are given by conservation: $\nabla_\mu T^{\mu\nu} = F^{\nu\rho} J_\rho$, $\nabla_\mu J^\mu = 0$, ...
- The basic input of fluid dynamics are the constitutive relations: $T_{\mu\nu} = \epsilon u_\mu u_\nu + p P_{\mu\nu} + \Pi_{\mu\nu}$,...

# Some immediate questions:

- Relation between microscopic observables and fluid data.
- This has been long addressed by the Kubo formula:

$$G_{\mathsf{R}}(w,k) \sim \frac{F(w,k)}{-iw + \mathcal{D}k^2}$$

- The role of stochastic (and quantum) fluctuations.
- This can be addressed with a Lagrangian description [Crossley, Glorioso, and Liu; Loganayagam, Rangamani]:

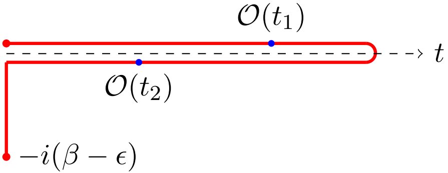
$$\int \mathcal{D}\chi \mathcal{D}\phi e^{iS_{\mathsf{micro}}[\chi,\phi]} \sim \int \mathcal{D}\chi e^{iS_{\mathsf{hydro}}[\chi]}$$

# A matter of order: the SK contour

- A key observable are the real-time thermal correlators.
- The order of operators is important (causality).
- A familiar choice is to compute time-ordered correlators: $\langle \mathcal{T}\mathcal{O}(t_1)\mathcal{O}(t_2)...\mathcal{O}(t_n)\rangle$.
- Other orderings are useful, for instance, retarded correlators: $\theta(t-t')\,\langle[\mathcal{O}(t),\mathcal{O}(t')]\rangle$.

- All the orderings can be taken into account by considering time-fold contours.

$$\langle \mathcal{O}(t_2)\mathcal{O}(t_1)\rangle_\beta =$$




- Up to 3-point (thermal) correlators, one only needs a two-fold contour.

# Many birds with one shutgun

The in-in formalism addresses many problems at once:

- Causal ordering.
- The lack of adiabaticity in strongly coupled, densed, quantum field theory.
- Correlation functions with respect of mixed states or in open systems.

# The open system paradigm for fluid actions

The dissipative nature of fluid dynamics emerges from unitarity theories by considering open quantum system[Caldeira-Leggett]:

$$\mathcal{L}_{\text{total}}[J, \mathcal{O}] = \mathcal{L}_{\text{sys.}}[J] + \mathcal{L}_{\text{env.}}[\mathcal{O}] + \mathcal{L}_{\text{int.}}[J, \mathcal{O}]$$

After integrating out the environment (fluid), one can obtain an effective description for the system of interest (thermometer, barometer, etc.).

- Using the in-in formalism:

$$\mathcal{Z} = \int DJ_{L/R} D\mathcal{O}_{L/R} \exp \Big( iS_{\text{sys.}}[J_R] - iS_{\text{sys.}}[J_L] + iS_{\text{env.}}[\mathcal{O}_R] - iS_{\text{env.}}[\mathcal{O}_L] + i \int \left( J_R \mathcal{O}_R - J_L \mathcal{O}_L \right) \Big)$$

$$= \int DJ_{L/R} \exp \left( iS_{\text{sys.}}[J_R] - iS_{\text{sys.}}[J_L] + iS_{\text{EFT}}[J_R, J_L] \right)$$

- Notice that, in general, $S_{\text{EFT}}[J_R, J_L]$ is not $R/L$ diagonal.

- In practice, we don't need to keep close watch of the system's internal dynamics:

$$\mathcal{Z}[J_R, J_L] = \langle e^{i\int (J_R \mathcal{O}_R - J_L \mathcal{O}_L)} \rangle_\beta = e^{iW[J_R, J_L]}$$

- However, it is not a good idea to fully integrate out the environmental degrees of freedom.
- Instead, one should compute the *Wilsonian influence functional*:

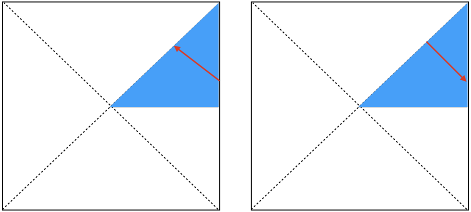
$$e^{iW[J_R, J_L]} = \int D\varphi_{R/L}\, e^{i\mathcal{S}_{\mathsf{WIF}}[J_{R/L}, \varphi_{R/L}]}$$

# Holographic real-time correlators

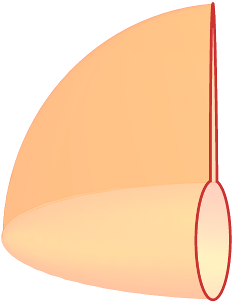
- The evaluation of $\mathcal{S}_{\text{WIF}}$ is a challenging endeavor, specially for strongly coupled theories.
- It is possible to learn many lessons by considering a holographic environment (for instance $\mathcal{N} = 4$ SYM).
- Long history of real-time holography [Herzog, Son, Starinets, Skenderis, van Rees...].

# Gravitational SK geometry

- An unified prescription for the holographic dual of the SK Keldysh contour has been proposed [Glorioso, Crossley, Liu]

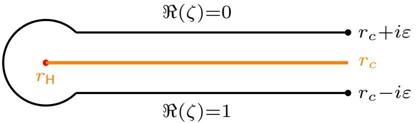


- Equivalently, one may consider a complex, two-sheeted, solution of Einstein equations[Jana, Loganayagam, Rangamani].



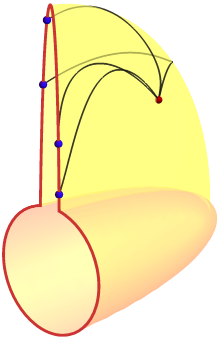
- Concretely:

$$ds^2 = -r^2 f dv^2 + i\beta r^2 f dv d\zeta + r^2 d\vec{x}^2, \qquad \frac{d\zeta}{dr} = \frac{2}{i\beta}\frac{1}{r^2 f}$$




- The non-trivial monodromy of $\zeta(r)$ precisely captures the Euclidean evolution around the thermal circle.

- Correlation functions are then computed as Witten diagrams in this geometry.



# Gravitational perturbations in SK contours

- Consider Einstein-Maxwell gravity:

$$S_{\text{Bulk}} = \int d^{d+1}x\sqrt{-g}\left(R + d(d-1) - \frac{1}{2}F_{AB}F^{AB}\right)$$

- We want to consider perturbations around equilibrium:

$$ds^2 = ds^2_{(0)} + ds^2_{(1)} \qquad A_B dx^B = -a(r)dv + \mathcal{A}_B dx^B \,.$$

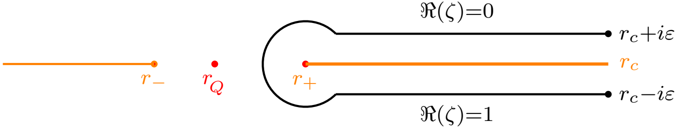
- The equilibrium configuration is taken to be the RN black hole:

$$
\begin{aligned}
ds^2_{(0)} &= -r^2 f(r) dv^2 + 2 dv dr + r^2 d\vec{x}^2 \,, \\
f(r) &= 1 - (1+Q^2)\left(\frac{r_+}{r}\right)^d + Q^2 \left(\frac{r_+}{r}\right)^{2(d-1)} \,, \\
a(r) &= \sqrt{\frac{d-1}{d-2}} Q \frac{r_+^{d-1}}{r^{d-2}}
\end{aligned}
$$

- With the usual relations between the black hole parameter and the thermodynamic variables:

$$
T = \frac{d-(d-2)Q^2}{4\pi} r_+ \,, \quad \mu = \sqrt{\frac{d-1}{d-2}} Q r_+
$$

- Due to the inner horizon, the SK geometry is a bit different:




- The parameter $r_Q^{d-2} = \frac{d-1}{d}\frac{2Q^2}{1+Q^2}r_+^{d-2}$ is related to the DC conductivity as

$$\sigma_{\mathrm{DC}} = r_+^{d-3}h(r_+)^2\,, \quad h(r) = 1 - \frac{r_Q^{d-2}}{r^{d-2}}\,.$$

# Repackaging gravity: designer scalars

- Classify the perturbations in terms of $SO(d-2)$ representations [Kodama, Ishibashi] [Gubser, Pufu]:

$$ds^2_{(1)} = ds^2_{\text{Tensor}} + ds^2_{\text{Vector}} + ds^2_{\text{Scalar}}$$

$$\mathcal{A}_B dx^B = \mathcal{A}_B^{\text{Vector}} dx^B + \mathcal{A}_B^{\text{Scalar}} dx^B$$

- Expand the perturbations in terms of harmonic functions of $SO(d-2)$.

# The executive summary

- Metric perturbations:
  - $ds^2_{\text{Tensor}} = r^2 \sum_{\sigma=1}^{N_T} \int_k \Phi_\sigma(r, v|\vec{k}) \mathbb{T}^\sigma_{ij}(\vec{x}|\vec{k}) dx^i dx^j$.
  - $ds^2_{\text{Vector}} = 2r^2 \sum_{\alpha=1}^{N_V} \int_k \left(\Psi^\alpha_r dr + \Psi^\alpha_v dv\right) \mathbb{V}^\alpha_i dx^i$.
  - $ds^2_{\text{Scalar}} = \frac{\Phi_{\mathsf{E}} - rf\Phi_{\mathsf{W}}}{r^{d-3}} dv^2 + \frac{2}{r^{d-1}f} \left(\Phi_{\mathsf{O}} - \Phi_{\mathsf{E}} + rf\Phi_{\mathsf{W}}\right) dv dr + \frac{\Phi_{\mathsf{W}}}{r^{d-4}} d\vec{x}^2 - \frac{2(\Phi_{\mathsf{O}} - \Phi_{\mathsf{E}}) + (d-1)rf\Phi_{\mathsf{W}}}{r^{d+1}f^2} dr^2$.
- Gauge field perturbations
  - $\mathcal{A}^{\text{Vector}}_B dx^B = \sum_{\alpha=1}^{N_V} \int_k \Xi_\alpha(v, r) \mathbb{V}^\alpha_i dx^i$.
  - $\mathcal{A}^{\text{Scalar}}_B dx^B = \frac{1}{r^{d-3}} \left(dv \mathbb{D}_+ - dr \frac{d}{dr}\right) \mathcal{V}, \qquad \mathbb{D}_+ = r^2 f \partial_r + \partial_v$.

# The physical summary

- Not all of the fields above are independent, due to gauge symmetry and the associated constraint equations.
- We can solve these constraints in terms of a few master fields:
  - Tensor sector: $\Phi_\sigma$.
  - Vector sector: $X_\alpha, Y_\alpha$.
  - Scalar sector: $\mathcal{Z}, \mathcal{V}$.
- Finally, we can introduce a diagonal basis $(\mathsf{X}_\alpha, \mathsf{Y}_\alpha)$ and $(\mathsf{Z}, \mathsf{V})$, where all the interactions decouple.

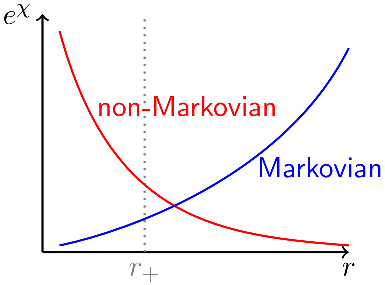
- Linear response in gravity is then all packaged in a few scalars:

$$S_{\mathcal{M}} = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}e^{\chi(r)}\left[\nabla_A\Phi_{\mathcal{M}}\nabla^A\Phi_{\mathcal{M}} + V(\Phi_{\mathcal{M}})\right] + S_{\text{bdy.}}$$

- Tensors: $e^{\chi(r)} = 1$.
- Vectors: $e^{\chi(r)} = \frac{1}{r^{2(d-1)}}$ , $e^{\chi(r)} = \frac{h^2}{r^2}$.
- Scalars: $e^{\chi(r)} = \frac{h^2}{r^{2(d-2)}\Lambda_k^2}$ , $e^{\chi(r)} = \frac{1}{h^2 r^{2(d-2)}}$ ,

  $\Lambda_k = k^2 + \frac{1}{2}(d-1)r^3 f'$.

- The dilaton makes all the difference:




- The asymptotic behavior determines whether a perturbation is long or short-lived.

# Dynamic of designer scalars in SK

- From our previous discussion, let us study the system:

$$S_{\text{designer}} = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\, r^{\mathcal{M}-d+1}\nabla_A\Phi_{\mathcal{M}}\nabla^A\Phi_{\mathcal{M}} + S_{\text{bdy.}}$$

- In Fourier domain, the equation of motion is

$$r^{-\mathcal{M}}\mathbb{D}_+\left(r^{\mathcal{M}}\mathbb{D}_+\Phi_{\mathcal{M}}\right) + (\omega^2 - k^2 f)\Phi_{\mathcal{M}} \equiv \mathcal{D}_{\mathcal{M}}\Phi_{\mathcal{M}} = 0\,.$$

- For $d > 2$, this equation can be solved in a gradient expansion:

$$\mathfrak{w}, \mathfrak{q} = \frac{\omega}{r_+}, \frac{k}{r_+} \ll 1$$

- At zeroth order, the solution behaves as:

$$\Phi_{\mathcal{M}} \sim c_1 + \frac{c_2}{r^{\mathcal{M}+1}}$$

- For $\mathcal{M} > -1$, we impose Dirichlet conditions ($S_{\text{bdy.}} = 0$).
- For $\mathcal{M} < -1$, compute the conjugate momentum:

$$\pi_{\mathcal{M}} = -r^{\mathcal{M}}\mathbb{D}_+\Phi_{\mathcal{M}} \sim \tilde{c}_1 + \tilde{c}_2 r^{\mathcal{M}}$$

- Quantization is done using Neumann conditions ($S_{\text{bdy.}} = \int d^d x \Phi_{\mathcal{M}}\pi_{\mathcal{M}}$).

# Solutions and actions: $\mathcal{M} > -1$

- The ingoing bulk-to-boundary propagator:

$$\mathcal{D}_{\mathcal{M}} G^{\text{in}}_{\mathcal{M}}(r, \omega, k) = 0\,, \quad G^{\text{in}}_{\mathcal{M}}(r \to \infty, \omega, k) = 1\,, \quad G^{\text{in}}_{\mathcal{M}}(r_+, \omega, k) = \text{reg.}$$

- The outgoing propagator is obtained by time-reversal symmetry: $G^{\text{out}}_{\mathcal{M}}(r, \omega, k) = e^{\beta\omega\zeta} G^{\text{in}}_{\mathcal{M}}(r_+, -\omega, k)$
- The most general solution (in SK geometry):

$$\Phi_{\mathcal{M}}(\zeta, \omega, k) = c_1 G^{\text{in}}_{\mathcal{M}}(r, \omega, k) + c_2 G^{\text{out}}_{\mathcal{M}}(r, \omega, k)$$

- Imposing Dirichlet bdy. conditions:

$$\lim_{\zeta\to 0}\Phi_{\mathcal{M}} = J_R\,, \quad \lim_{\zeta\to 1}\Phi_{\mathcal{M}} = J_L$$

- The solution in grSK geometry is

$$\Phi_{\mathcal{M}}(\zeta, w, \vec{k}) = J_a G^{\mathsf{in}}_{\mathcal{M}} + \left[\left(n_\beta + \frac{1}{2}\right) G^{\mathsf{in}}_{\mathcal{M}} - n_\beta e^{\beta(\zeta-1)} G^{\mathsf{rev}}_{\mathcal{M}}\right] J_d\,,$$

with $J_d = J_R - J_L$ and $J_a = \frac{1}{2}\left(J_R + J_L\right)$.

- Finally, we can evaluate the on-shell action:

$$W[J_a, J_d] = S[\Phi_{\mathcal{M}}]_{\text{on-shell}}$$
$$= -\int_k J_d^\dagger \mathcal{K}^{\text{in}}_{\mathcal{M}} \left[ J_a + \left( n_\beta + \frac{1}{2} \right) J_d \right]$$

$$\mathcal{K}^{\text{in}}_{\mathcal{M}} = -iw + \frac{k^2}{1-\mathcal{M}} + \Delta^{2,0}_{\mathcal{M}}(r_+)w^2 + ...$$

where $\mathcal{K}^{\text{in}}_{\mathcal{M}}$ is obtained as a boundary limit of the conjugate momentum.

# Solutions and actions: $\mathcal{M} < -1$

- Same story could be repeated with Neumann conditions in order to compute $W[J_a, J_d]$.
- But let us be smarter: for long-lived modes we should compute $\mathcal{S}_{\text{WIF}}[\check{\Phi}_a, \check{\Phi}_d]$.
- The relation between these two objectes is a Legendre transform: change boundary conditions from Neumann to Dirichlet!

- Then:

$$\Phi_{\mathcal{M}}(\zeta, w, \vec{k}) = \check{\Phi}_a G^{\text{in}}_{\mathcal{M}} + \left[\left(n_\beta + \frac{1}{2}\right) G^{\text{in}}_{\mathcal{M}} - n_\beta e^{\beta(\zeta-1)} G^{\text{rev}}_{\mathcal{M}}\right] \check{\Phi}_d$$

$$\begin{aligned}\mathcal{S}_{\text{WIF}}[\check{\Phi}_a, \check{\Phi}_d] &= S[\Phi_{\mathcal{M}}]_{\text{on-shell}} \\ &= -\int_k \check{\Phi}^\dagger_d \mathcal{K}^{\text{in}}_{-\mathcal{M}} \left[\check{\Phi}_a + \left(n_\beta + \frac{1}{2}\right) \check{\Phi}_d\right]\end{aligned}$$

$$\mathcal{K}_{-\mathcal{M}} = -iw + \frac{k^2}{1+\mathcal{M}} - \Delta^{2,0}_{\mathcal{M}}(r_+) w^2 + \ldots$$

- In order to compute correlators, one needs to go back to the generating functional, then $\langle \mathcal{O}\mathcal{O} \rangle \sim \frac{1}{\mathcal{K}_{-\mathcal{M}}}$.

# Back to EFTs for fluids

- We can run this machinery for every mode in the gravitational perturbation.
- $\Phi_\sigma$ and $Y_\alpha$ are completely markovian, hence boring.
- $X_\alpha$, Z, and V are non-markovian, they correspond to the physical degrees of freedom of the fluid.

- In total:

$$
\begin{aligned}
\mathcal{S}_{\mathsf{WIF}} \propto &-\sum_{\alpha=1}^{N_V}\int_k \left(\check{\mathcal{P}}^\alpha_d\right)^\dagger K^{\mathsf{in}}_{\mathcal{X}}\left[\check{\mathcal{P}}^\alpha_a+\left(n_\beta+\frac{1}{2}\right)\check{\mathcal{P}}^\alpha_d\right]\\
&-\int_k \left(\check{\mathcal{V}}_d\right)^\dagger K^{\mathsf{in}}_{\mathsf{V}}\left[\check{\mathcal{V}}_a+\left(n_\beta+\frac{1}{2}\right)\check{\mathcal{V}}_d\right]\\
&-\int_k \left(\check{\mathcal{Z}}_d\right)^\dagger K^{\mathsf{in}}_{\mathsf{Z}}\left[\check{\mathcal{Z}}_a+\left(n_\beta+\frac{1}{2}\right)\check{\mathcal{Z}}_d\right]
\end{aligned}
$$

- $K^{\mathsf{in}}_{\mathcal{X}}$ and $K^{\mathsf{in}}_{\mathsf{V}}$ are similar to the $\mathcal{M} < -1$ result before.
- $K^{\mathsf{in}}_{\mathsf{Z}}$ is special:

$$
K^{\mathsf{in}}_{\mathsf{Z}} \propto K^{\mathsf{in}}_s = -\mathfrak{w}^2+\frac{\mathfrak{q}^2}{d-1}+\frac{\nu_s}{1+Q^2}\mathfrak{q}^2\Gamma_s(\mathfrak{w},\mathfrak{q})
$$

## Correlation functions

$$\langle T_{xy}(-w,-k)T_{xy}(w,k)\rangle^{\mathsf{Ret}} \sim i\mathcal{K}^{\mathsf{in}}_{d-1}(w,k)$$

$$\langle T_{vx}(-w,-k)T_{vx}(w,k)\rangle^{\mathsf{Ret}} \sim i\left[\frac{a_1}{\mathcal{K}^{\mathsf{in}}_{\mathcal{X}}(w,k)} + a_2\mathcal{K}^{\mathsf{in}}_{\mathcal{Y}}(w,k)\right]$$

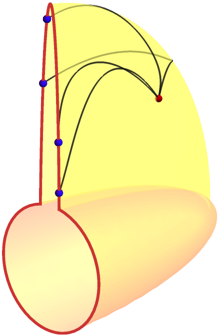
$$\langle J_{x}(-w,-k)J_{x}(w,k)\rangle^{\mathsf{Ret}} \sim i\left[\frac{b_1}{\mathcal{K}^{\mathsf{in}}_{\mathcal{X}}(w,k)} + b_2\mathcal{K}^{\mathsf{in}}_{\mathcal{Y}}(w,k)\right]$$

$$\langle T_{vv}(-w,-k)T_{vv}(w,k)\rangle^{\mathsf{Ret}} \sim i\left[\frac{c_1}{\mathcal{K}^{\mathsf{in}}_{s}(w,k)} + \frac{c_2}{\mathcal{K}^{\mathsf{in}}_{\mathsf{V}}(w,k)}\right]$$

$$\langle J_{v}(-w,-k)J_{v}(w,k)\rangle^{\mathsf{Ret}} \sim i\left[\frac{d_1}{\mathcal{K}^{\mathsf{in}}_{s}(w,k)} + \frac{d_2}{\mathcal{K}^{\mathsf{in}}_{\mathsf{V}}(w,k)}\right]$$

- **Main lesson:** The current correlators are encoded in terms of a few effective scalar operators: $\mathcal{O}_{\mathsf{X}}, \mathcal{O}_{\mathsf{V}}, \mathcal{O}_{\mathsf{Z}}$.
- We would like to take this to heart and treat $\hat{T}_{\mu\nu}, \hat{J}_\mu, \cdots$ are fully quantum operators, parameterized in terms of $\hat{\mathcal{O}}_{\mathsf{X}}, \hat{\mathcal{O}}_{\mathsf{V}}, \hat{\mathcal{O}}_{\mathsf{Z}}$.
- For this to be a meaningful statement, we need to compute the higher order correlators.

# Beyond the Gaussian level: Witten diagrams in SK geometry



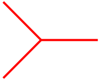
# The ingredients:

- Ingoing Bulk-to-Bdy Prop.: $G_{\text{in}}(\zeta, k)$.
- Outgoing Bulk-to-Bdy Prop.: $G_{\text{out}}(\zeta, k) = e^{-\beta\omega\zeta} G_{\text{in}}(\zeta, \bar{k})$
- Bulk-to-Bulk Prop:

  $G_{\text{bb}}(\zeta, \zeta'; k) = \mathcal{N}(k) e^{\beta\omega\zeta'} G_{\text{L}}(\zeta_{>}, k) G_{\text{R}}(\zeta'_{<}, k)$.

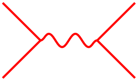
**Important:** $G_{\text{in}}(\zeta + 1, k) = G_{\text{in}}(\zeta, k)$.

- **Contact diagram:**



$$= \oint d\zeta \, \mathfrak{L}(\zeta) = \int_{r_{\mathsf{H}}}^{r_c} dr \left( \mathfrak{L}(\zeta(r)+1) - \mathfrak{L}(\zeta(r)) \right)$$

- **Exchange diagram**



$$\begin{aligned} &= \oint d\zeta \oint d\zeta' \left[ F_1(\zeta,\zeta')\Theta(\zeta-\zeta') + F_2(\zeta,\zeta')\Theta(\zeta'-\zeta) \right] \\ &= \int_{r_{\mathsf{H}}}^{r_c} dr \int_{r_{\mathsf{H}}}^{r_c} dr' \left[ \mathfrak{F}_1 \theta(r-r') + \mathfrak{F}_2 \theta(r'-r) \right] \end{aligned}$$

where

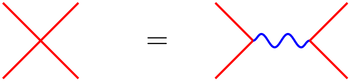
$$\mathfrak{F}_1 = F_1(\zeta,\zeta') - F_1(\zeta+1,\zeta') + F_2(\zeta+1,\zeta'+1) - F_2(\zeta,\zeta'+1) \,,$$

$$\mathfrak{F}_2 = F_1(\zeta+1,\zeta'+1) - F_1(\zeta+1,\zeta') + F_2(\zeta,\zeta') - F_2(\zeta,\zeta'+1) \,.$$

# Analytic structure of Thermal correlators

- Explicit, closed-form, calculations are only possible in $d = 2$ [Loganayagam, Rangamani, JV].
- General lesson: The analytic structure of higher order thermal correlators is determined by a single piece of data: $G_{\text{in}}(\zeta, k)$.
- The Ward identities due to unitarity and KMS follow from the grSK structure (bulk-to-bdy and bulk-to-bulk propagators).

- Long-lived modes and factorization channels:



- For $d > 2$, the Witten diagrams could be evaluated using the gradient expansion.
- An example where this can be done is the anomalous contributions for a fluid in $d = 4$ [Rangamani, JV, Zhou].

# Work in progress

- Higher order SK correlators for $d > 2$ [Ammon, Rangamani, Specht, JV].
- Connections to non-linear fluid actions [Rangamani, JV].
- Beyond the SK contour: OTOC [Ammon, Germerodt, Sieling, JV].

# Final musings

- Applications to more complicated thermal systems: rotation (Kerr black holes), superfluids (scalar backgrounds and phase transitions).
- While we focused on using grSK geometries to study fluids, we can learn many lessons about real time gravitational dynamics.
- Effective actions for chaotic systems [Blake, Lee, Liu][Haehl, Rozali]

- Beyond holography: gravitational SK contours for more general spacetimes (see recent work on dS correlators[Di Pietro, Gorbenko, Komatsu][Loganayagam, Shetye]).
- Is there something to be said in asymptotically flat spacetimes?

# Thank You!