

Pseudospectra of Holographic QNMs



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D. Arean, D. Fariña, K.L. arXiv:2307.08751 [hep-th]
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HoloTube, 30.01.2024

Pseudospectra of Holographic QNMs

- Quasinormal modes
- Gauge/Gravity Duality (vulgo Holography)
- QNMs in Holography
- Pseudospectra
- Pseudospectra of QNMs
- Summary and Outlook

Normal Modes

Eigen modes of string:

$$\frac{d^2\Phi(x)}{dx^2} + \lambda\Phi(x) = 0$$

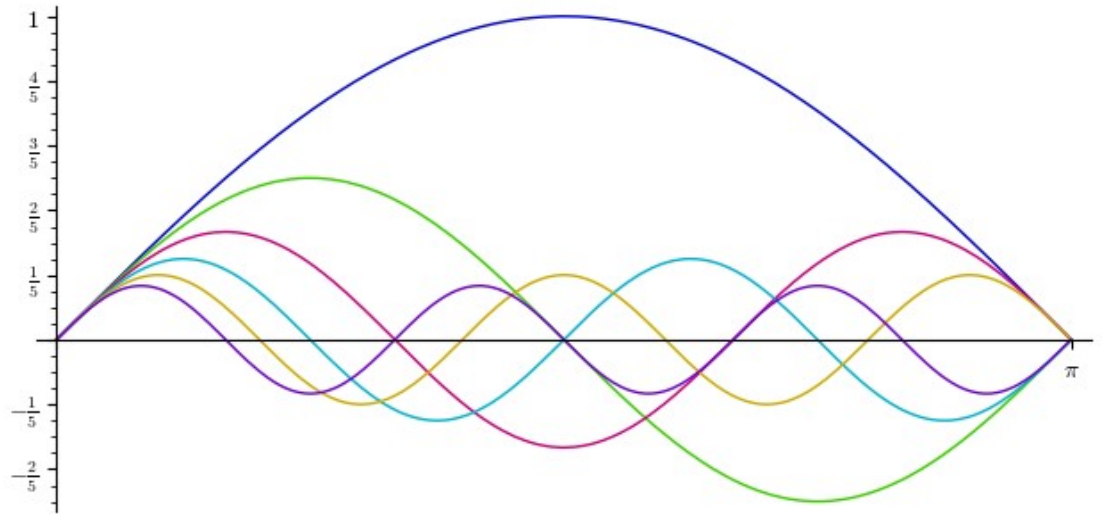
Boundary conditions:

$$\Phi(0) = \Phi(\pi) = 0$$

Hermitian operator:

$$\mathcal{L} = \frac{d^2}{dx^2}, \quad \mathcal{L}^\dagger = \mathcal{L}$$

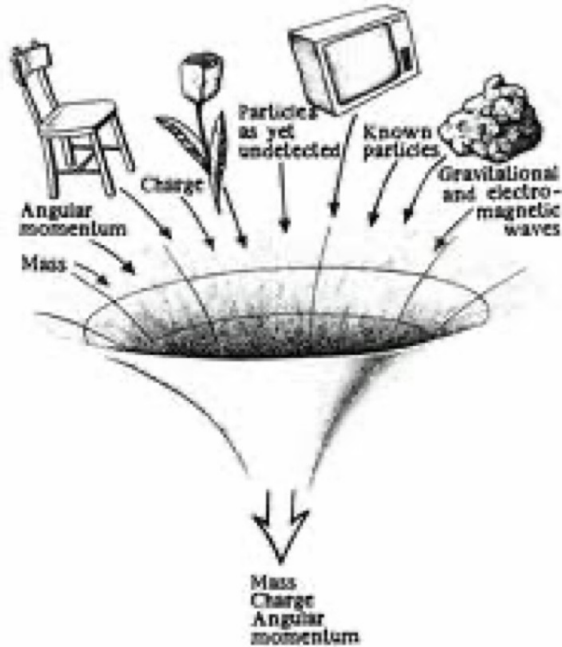
$$\langle \Psi, \Phi \rangle = \int_0^\pi dx \bar{\Psi} \Phi \quad \langle \Psi, \mathcal{L}\Phi \rangle = \langle \mathcal{L}\Psi, \Phi \rangle$$



$$\Phi_n(x) = \sin(nx)$$
$$\lambda = n^2$$

Quasi-Normal Modes

- Black Hole



→ Black Holes no Hair

→ Swallow everything

→ Fate of a perturbation

→ Either fall into BH

→ Or radiate off to infinity

→ Perturbation eventually dies off

Quasi-Normal Modes

How to compute Quasi Normal Modes:

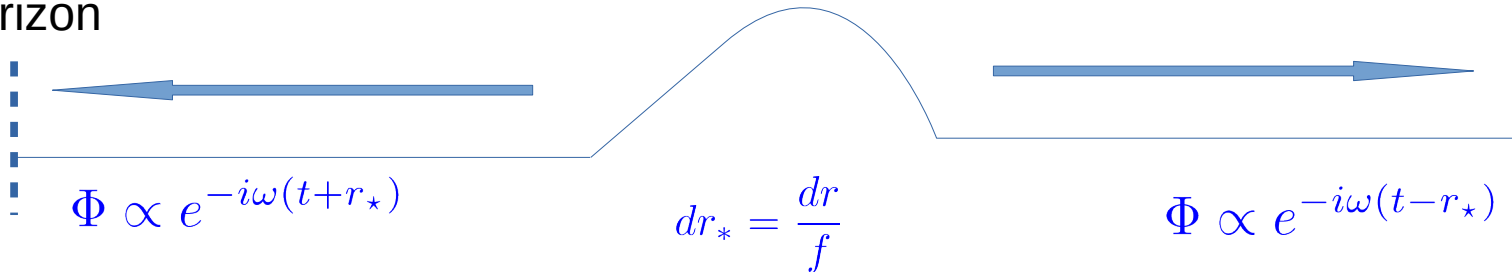
$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

$$\Phi = \phi(r)e^{-i\omega t}Y_{lm}(\Omega)$$

$$\phi'' + \frac{f'}{f}\phi + \left(\frac{\omega^2}{f^2} - \frac{l(l+1)}{r^2 f} + \frac{f'}{rf} \right) \phi = 0$$

“Outgoing” boundary conditions:

Horizon



Quasi-Normal Modes

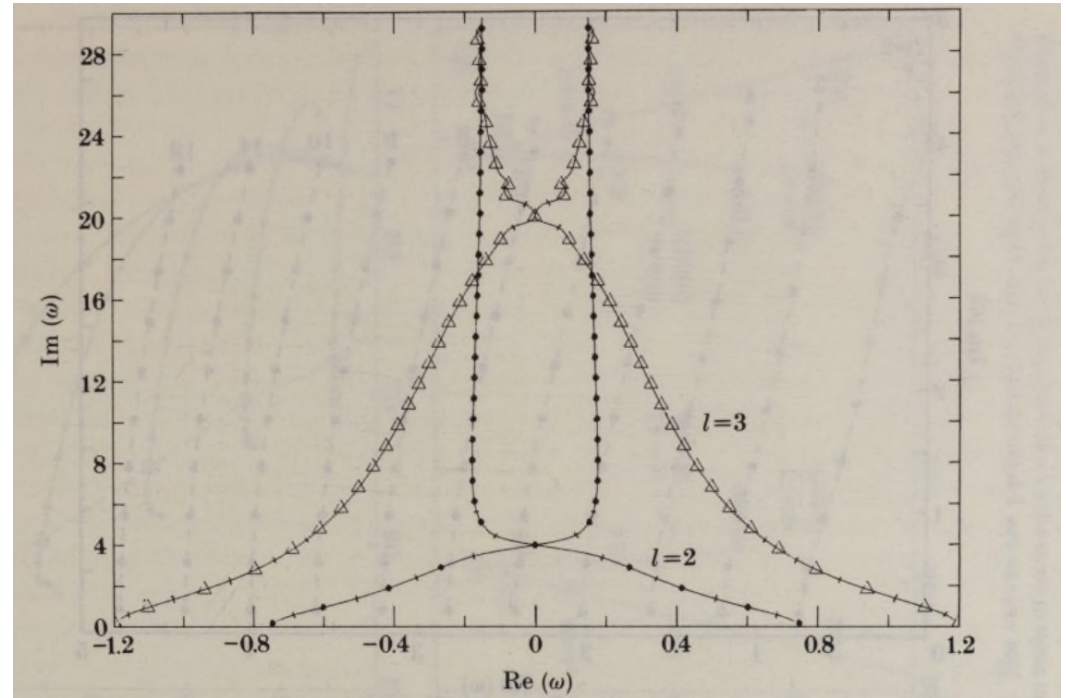
Leaky boundary conditions lead to complex frequencies

$$\omega_n = \Omega_n - i\Gamma_n$$

Oscillation Damping

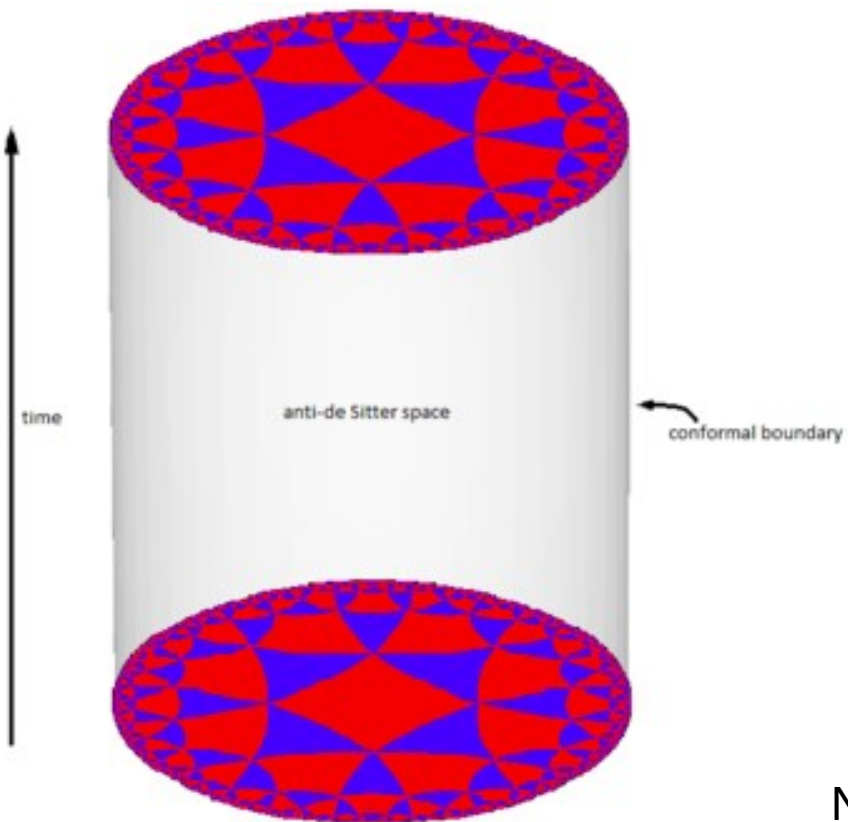
“Black Hole spectroscopy”

[Berti, Cardoso, Will, PRD 73 (2006)]



Holography

Gravity in asymptotically AdS = QFT



Holographic Dictionary

Gravity	Quantum Field Theory
<i>Metric</i>	<i>Energy Momentum Tensor</i>
<i>Gauge field</i>	<i>Conserved current</i>
<i>Scalar field</i>	<i>Scalar operator</i>

Notice: Symmetry in QFT = gauge principle in AdS

Holographic QNMs

Planar AdS Black hole: $ds^2 = \frac{r^2}{L^2} \left(- (1 - \pi^4 T^4 / r^4) dt^2 + dx^2 \right) + \frac{L^2 dr^2}{r^2 (1 - \frac{\pi^4 T^4}{r^4})}$

$$\Phi(r, t, \vec{x}) = e^{-i\omega t + i\vec{k} \cdot \vec{x}} \phi_{\omega, \vec{k}}(r)$$

Boundary condition Horizon: $\phi_{\omega, \vec{k}} \propto e^{-i\omega(t+r_*)}$

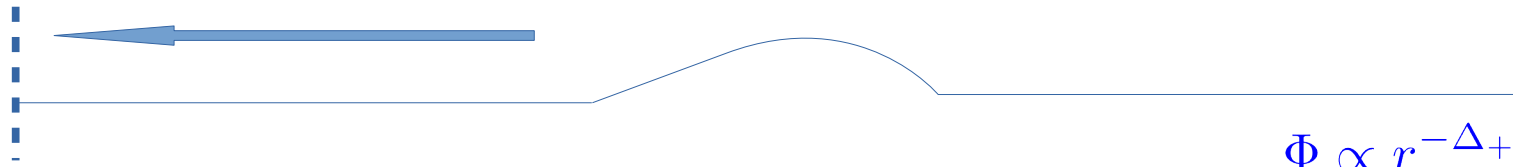
Boundary condition boundary: $\phi \approx A(\omega, \vec{k}) r^{-\Delta_-} (1 + \dots) + B(\omega, \vec{k}) r^{-\Delta_+} (1 + \dots)$

Retarded Green's function:

$$G_R(\omega, \vec{k}) = K \frac{B(\omega, \vec{k})}{A(\omega, \vec{k})}$$

[Horowitz, Hubeny],
[Birmingham, Sachs, Solodhukin]
[Kovtun, Son, Starinets]

Horizon



$$\Phi \propto r^{-\Delta_+}$$

Holographic QNMs

Example: Scalar field in BTZ black hole

$$G_R(\omega, k) = \frac{(\omega^2 - k^2)}{4\pi^2} \left[\psi \left(1 - i \frac{\omega - k}{4\pi T} \right) + \psi \left(1 - i \frac{\omega + k}{4\pi T} \right) \right]$$

$$\omega_n = \pm k - i4(n + 1)$$

Exact spectrum of QNMs !

In general no exact solution, e.g. scalar in AdS₅:

$$\phi'' + \left(\frac{5}{r} + \frac{f'}{f} \right) \phi' + \frac{\omega^2 - f^2 \vec{k}^2}{r^2 f} \phi = 0$$

“Christmas tree”

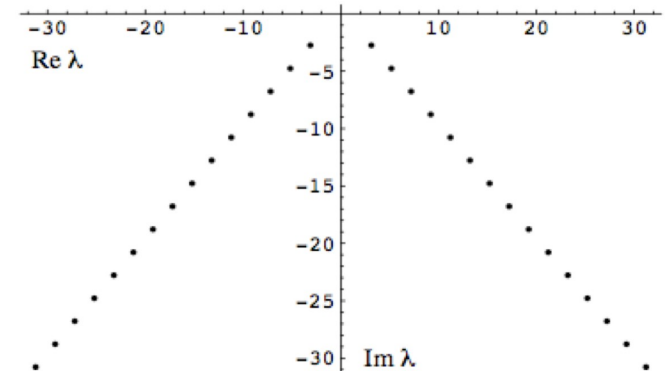


Fig. 1: The lowest 15 quasinormal frequencies in the complex λ -plane for $q = 0$.

[Starinets]

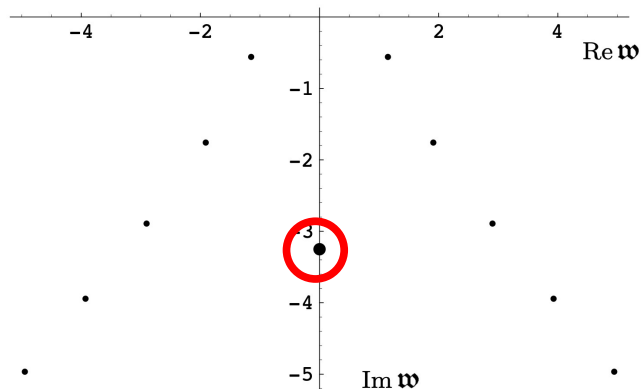
Holographic QNMs

Gauge fields: new ingredient gauge symmetry: conserved current $\partial_\mu J^\mu = 0$

$$\frac{d}{dt} Q = 0 \quad Q = \int d^3 x J^0$$

- 2 channels:
- Transverse is like scalar
 - Longitudinal new: diffusion

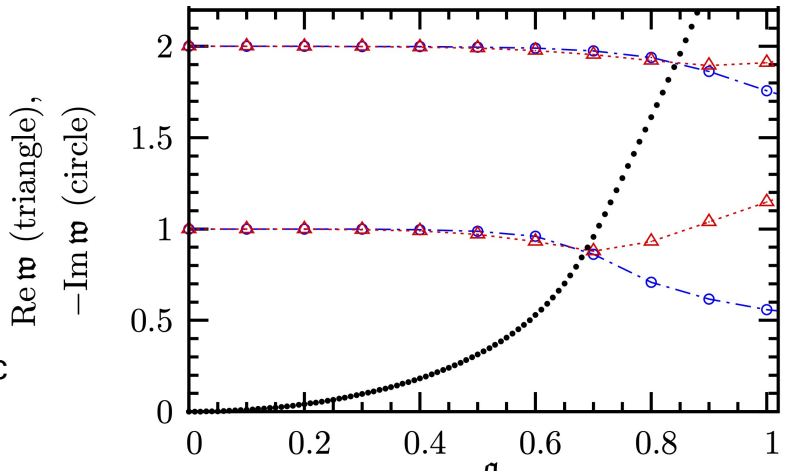
[Son, Starinets]



$$\omega_{\text{hydro}}(k) = -iDk^2$$

$$\omega = i \sum_{n=1}^{\infty} a_n k^{2n} \quad \text{Problematic Causality!}$$

[Amado, Hoyos, K.L., Montero]



Holographic QNMs

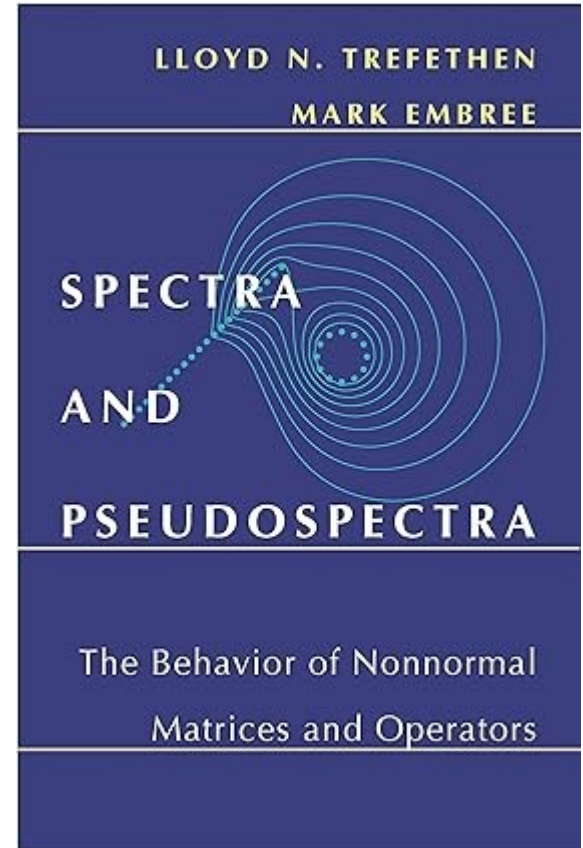
Quasinormal Modes are an essential ingredient of the Gauge/Gravity duality !

Pseudospectra

- QNMs are eigenvectors of non-Hermitian operators
- No spectral theorem $\mathcal{O} \neq \sum_n |n\rangle \lambda_n \langle n|$
- QNMs are not complete (only late time)
- Eigenfunctions are singular in Schwarzschild coords
- Robustness and physical significance in question

Near Horizon $\phi(r) = e^{-i\omega r_*} = e^{-i\Omega r_* - \Gamma r_*}$, $r_* \rightarrow -\infty$

No Hilbert space interpretation



Pseudospectra

Resolvent: $\mathcal{R}(\mathcal{L}, z) = (\mathcal{L} - z)^{-1}$

Spectrum: $\sigma(\mathcal{L}) = \{z \in \mathbb{C} : \|\mathcal{R}(\mathcal{L}, z)\| = \infty\}$

Eigenvalues: $\mathcal{L}u_n = \lambda_n u_n$

Operator norm: $\|\mathcal{L}\| = \sup_{u \in H} \frac{\|\mathcal{L}u\|}{\|u\|}$

Definitions of Pseudospectra:

- 1) Resolvent norm $\sigma_\epsilon(\mathcal{L}) = \{z \in \mathbb{C} : \|\mathcal{R}(\mathcal{L}, z)\| > 1/\epsilon\}$
- 2) Perturbation $\sigma_\epsilon(\mathcal{L}) = \{z \in \mathbb{C}, \exists \delta \mathcal{L}, \|\delta \mathcal{L}\| < \epsilon : z \in \sigma(\mathcal{L} + \delta \mathcal{L})\}$
- 3) Pseudo eigenvector $\sigma_\epsilon(\mathcal{L}) = \{z \in \mathbb{C}, \exists u^\epsilon : \|(\mathcal{L} - z)u^\epsilon\| < \epsilon \|u^\epsilon\|\}$

Theorem: The 3 definitions are equivalent

Pseudospectra

Condition number:

$$\kappa_i = \frac{\|v_i\| \|u_i\|}{|\langle v_i, u_i \rangle|}$$

Right eigenvector:

$$\mathcal{L}u_i = \lambda_i u_i$$

Left eigenvector: $\mathcal{L}^\dagger v_i = \lambda_i^* v_i$

Perturbation:

$$\|\delta\mathcal{L}\| = \epsilon$$

Perturbed eigenvalue: $(\mathcal{L} + \delta\mathcal{L})u_i(\epsilon) = \lambda(\epsilon)u_i(\epsilon)$

$$|\lambda(\epsilon)_i - \lambda_i| \leq \epsilon \kappa_i$$

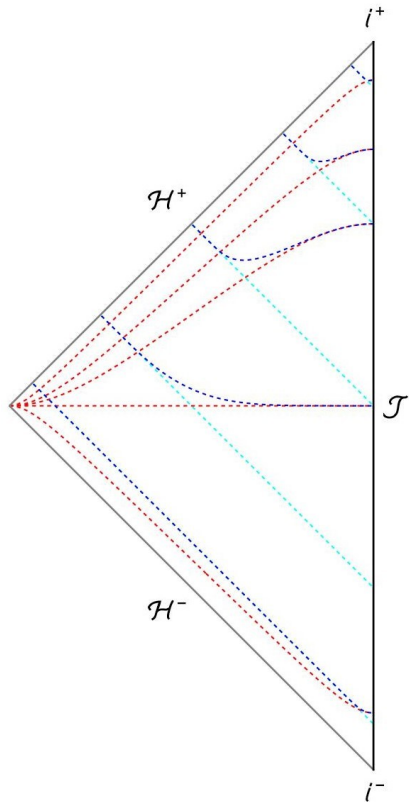
Def “small”: Let d_{\min} be the minimal distance between disconnected regions in the spectrum.

$\delta\mathcal{L}$, $\|\delta\mathcal{L}\| = \epsilon$ is small if

$$\frac{\epsilon}{d_{\min}} \ll 1$$

Pseudospectra

How to deal with the QNM problems: chose better coordinates!



----- Schwarzschild coordinates (worst)

$$ds^2 = r^2 [-f(r)dt^2 + d\vec{x}^2] + \frac{dr^2}{r^2 f(r)}$$

$$\phi(r) \propto (r - r_h)^{-i\omega/2}$$

----- Infalling Eddington-Finkelstein

$$dv = dt + \frac{dr}{r^2 f(r)}$$

$$ds^2 = r^2 [-f(r)dv^2 + d\vec{x}^2] + 2dvdr$$

----- "Regular"

$$\tau = v - (1 - r_h/r)$$

$$ds^2 = r^2 [-f d\tau^2 + d\vec{x}^2] + 2(1 - f)d\tau dr + (2 - f)\frac{dr^2}{r^2}$$

→ $\phi(r)$ regular at $r = r_h$

Pseudospectra

We need a physically motivated norm: **Energy** !

Energycurrent: $J = t^\mu T_{\mu\nu} dx^\mu$ $E[\Phi] = \int_{\Sigma_t} \star J$

Schwarzschild:	$E = \frac{1}{2} \int d^3x dr \left[r^2 f(\Phi')^2 + \frac{(\partial_t \Phi)^2}{r^2 f} + m^2 \Phi^2 + \frac{(\vec{\partial} \Phi)^2}{r^2} \right]$	$\frac{dE}{dt} = 0$
Infalling EF:	$E = \frac{1}{2} \int d^3x dr r^3 \left[r^2 f(\Phi')^2 + m^2 \Phi^2 + \frac{(\vec{\partial} \Phi)^2}{r^2} \right]$	$\frac{dE}{dv} = -r_h^3 \int_{r=r_h} (\partial_v \Phi)^2$
Regular:	$E = \frac{1}{2} \int d^3x dr r^3 \left[r^2 f(\Phi')^2 + m^2 \Phi^2 + \frac{(\vec{\partial} \Phi)^2}{r^2} + (2-f)(\partial_\tau \Phi)^2 \right]$	$\frac{dE}{d\tau} = -r_h^3 \int_{r=r_h} (\partial_\tau \Phi)^2$

Pseudospectra

- We want to re-write the wave equation as a standard eigenvalue problem
- Only possible in regular coordinates

$$\psi = \partial_\tau \Phi \quad \Psi = \begin{pmatrix} \Phi \\ \psi \end{pmatrix} \quad \mathcal{L} = i \begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix} \quad \longrightarrow \quad \mathcal{L}\Psi = \omega\Psi$$

- Compactify radial coordinate $\rho = 1 - \frac{r_h}{r}$

$$L_1 = [f(\rho) - 2]^{-1} \left[\frac{m^2 l^2}{(1-\rho)^2} + \mathbf{q}^2 - (1-\rho)^3 \left(\frac{f(\rho)}{(1-\rho)^3} \right)' \partial_\rho - f(\rho) \partial_\rho^2 \right]$$

$$L_2 = [f(\rho) - 2]^{-1} \left[(1-\rho)^3 \left(\frac{f(\rho) - 1}{(1-\rho)^3} \right)' + 2(f(\rho) - 1) \partial_\rho \right]$$

- Adjoint operator in energy norm $\mathcal{L}^\dagger = \mathcal{L} + \begin{pmatrix} 0 & 0 \\ 0 & -i\delta(\rho) \end{pmatrix} \quad \frac{d}{d\tau} E = -\bar{\psi}\psi|_{\rho=0}$

Pseudospectra

- No exact solutions → numerical methods
- Pseudospectral methods
- Chebyshev polynomials for interpolation

$$F(\rho) \approx \sum_{n=0}^N c_n T_n(\rho)$$

$$F(\rho_j) = \sum_{n=0}^N c_n T_n(\rho_j) \quad , \quad \rho_j = \frac{1}{2} \left(1 - \cos \left(\frac{j\pi}{N} \right) \right) \quad , \quad j = 0 \dots N$$

- Differential operator becomes a $(N+1) \times (N+1)$ matrix D $F'(\rho_j) = D_{jk} F(\rho_j)$
- Boundary conditions: delete rows and columns corresponding to $\rho = 1$
regularity corresponds to no boundary condition at $\rho = 0$
- Resolvent norm becomes maximal svd $\|\mathcal{L} - \omega \mathbf{1}\| \approx \inf(\text{svd})$
- Energy norm becomes a metric $2N \times 2N$ matrix $E \approx \bar{u}_k^* G_E^{km} u_m$, $u \approx (\phi(\rho_j), \psi(\rho_j))^T$

Pseudospectra

A toy example: $A = \begin{pmatrix} -1 & 0 \\ -50 & -2 \end{pmatrix}$ Eigenvalues: $\lambda_1 = -1$, $\lambda_2 = -2$

1. ℓ_2 norm: $\|u\| = [\bar{u}.u]^{1/2}$

$$\kappa_1 = \kappa_2 = \sqrt{2501} \approx 50$$

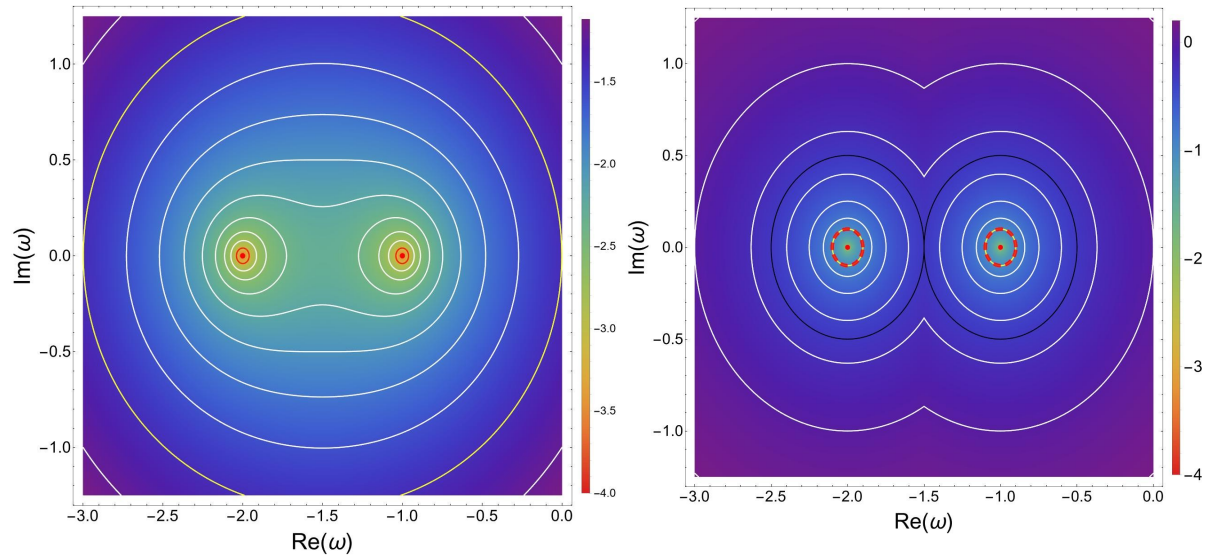
1. G-norm: $\|u\|_G = [\bar{u}.G.u]^{1/2}$

$$A^\dagger = [(G.A.G^{-1})^T]^* = A$$

$$G = \begin{pmatrix} 20000 & 50 \\ 50 & 1 \end{pmatrix}$$

$$\kappa_1 = \kappa_2 = 1$$

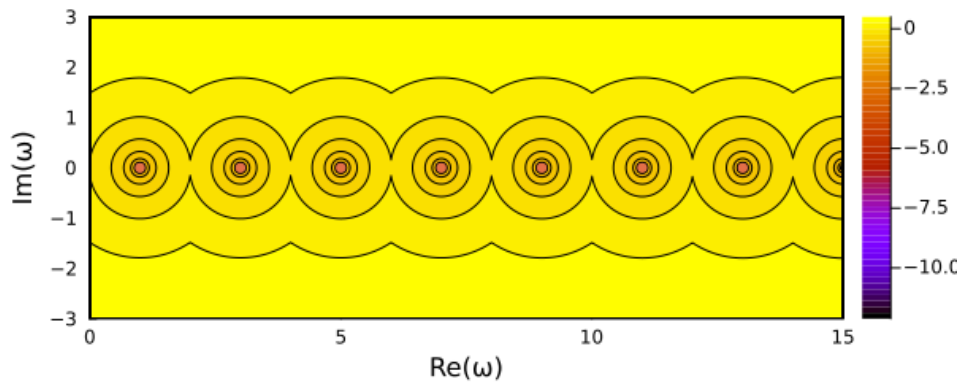
Contourmaps of $\log \|A - \omega \mathbf{1}\|$



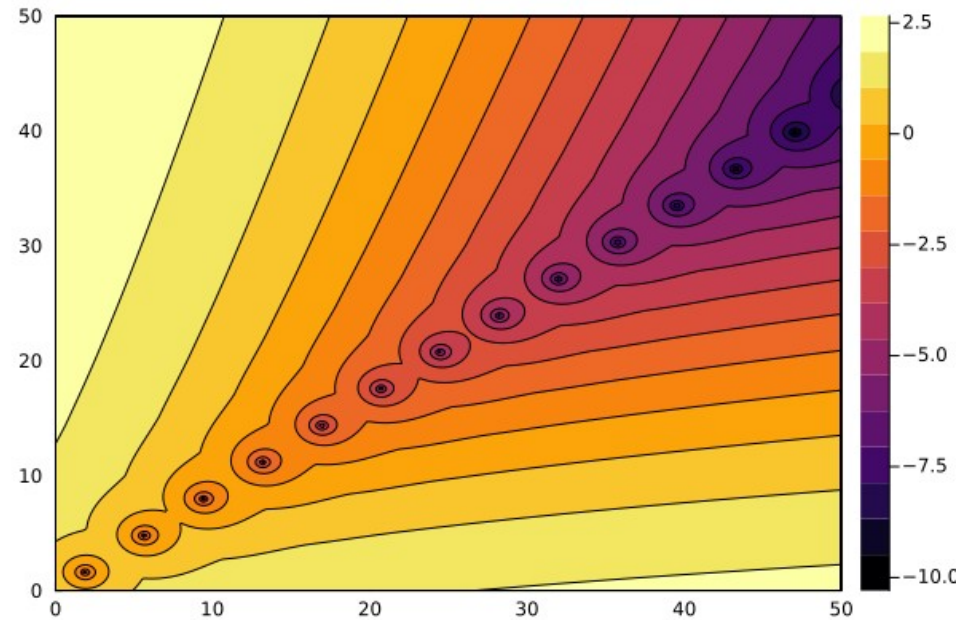
Pseudospectra

Harmonic Oscillator: $-\frac{d^2\phi}{dx^2} + cx^2\phi = \omega\phi$

$c=1$

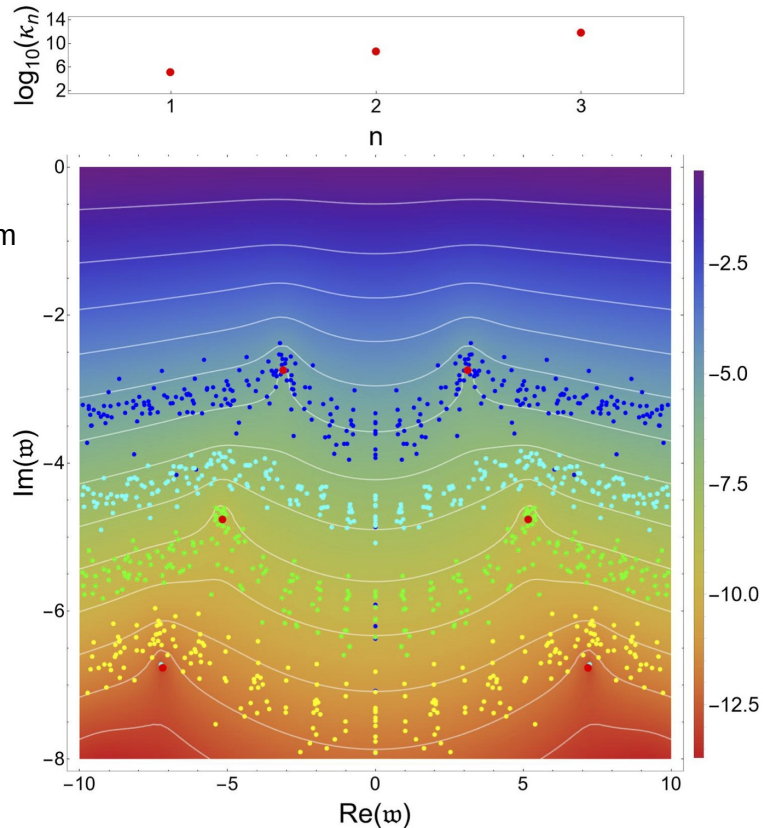


$c=1+3i$

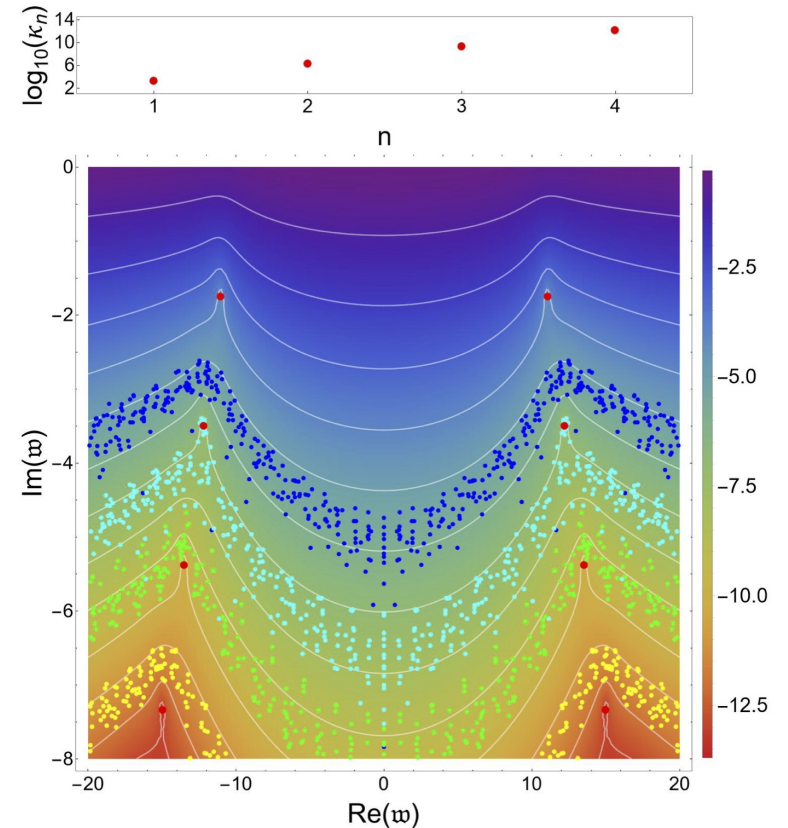


Pseudospectra

Pseudospectra of massless scalar in AdS₅:



(a) $m^2 l^2 = 0$, $q = 0$.



(b) $m^2 l^2 = 0$, $q = 10$.

“Selective” Pseudospectrum
Local random
Potential perturbations

$\|V_{\text{rand}}\|$

- 10^{-1}
- 10^{-3}
- 10^{-5}
- 10^{-7}

Pseudospectra

Zoom into first QNM:

Restricted
spectral instability

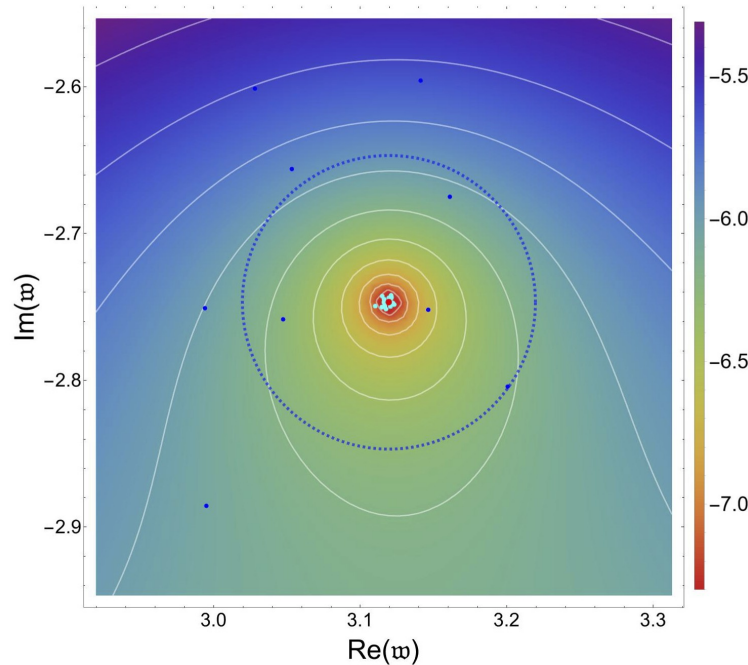
Restricted
spectral stability

Circle of stability

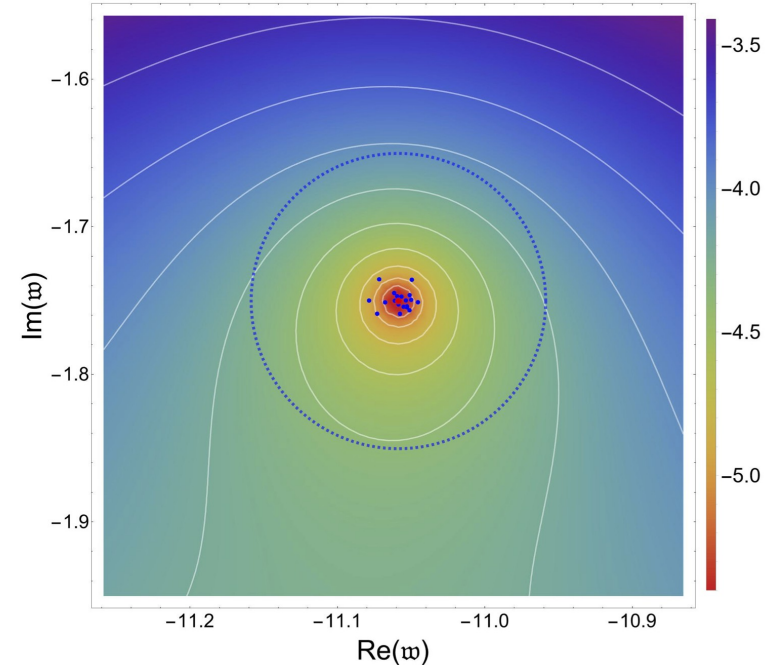
$$|\lambda(\epsilon)_i - \lambda_i| \leq \epsilon \kappa_i$$

● 10^{-1}

● 10^{-3}

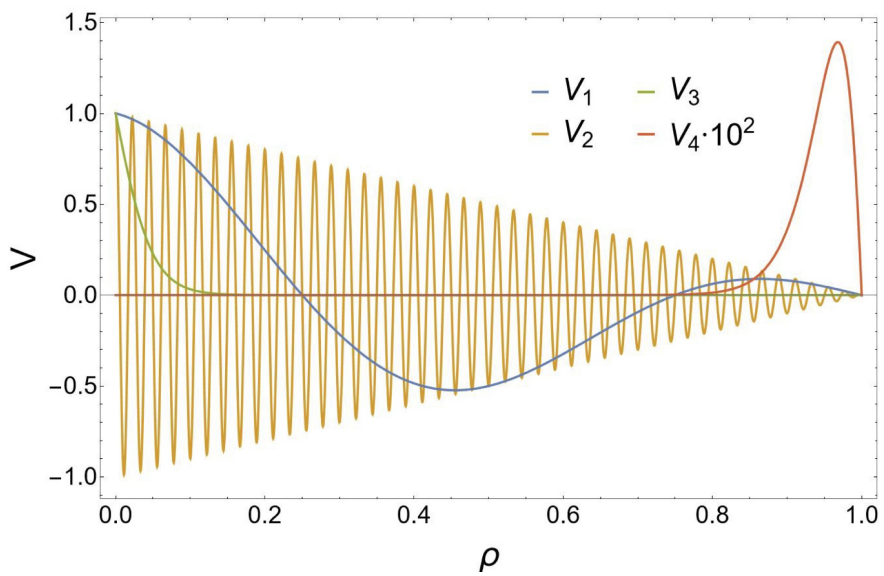


(a) $m^2 l^2 = 0$, $q = 0$.



(b) $m^2 l^2 = 0$, $q = 10$.

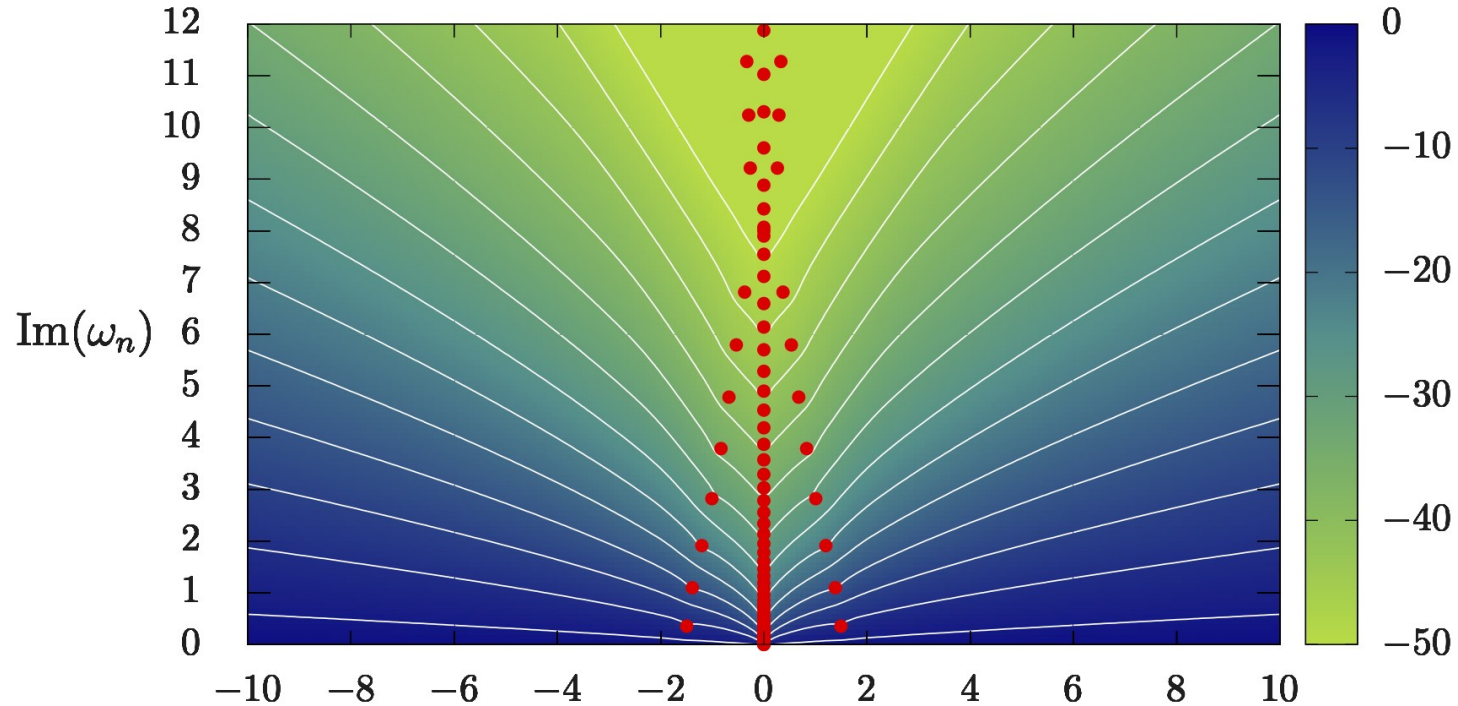
Pseudospectra



Pseudospectra

In asymptotically flat space:

[Jaramillo, Macedo, Al Sheik, PRX 11 (2021) 3, 031003 • e-Print: 2004.06434]



Review: [Destounis, Duque] e-Print: 2308.16227

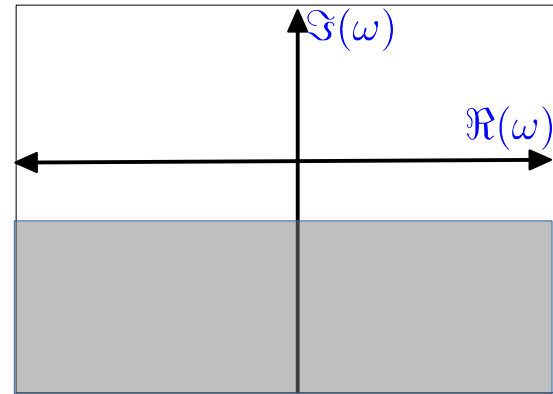
Pseudospectra

Caveats: Ingoing modes are integrable in the energy norm:

$$\phi_{\text{in}} \approx \rho^{i\omega} = \rho^{i\Omega} \rho^\Gamma$$

$$|\phi'_{\text{in}}|^2 \approx \rho^{2\Gamma-1} \quad \Gamma = -\Im(\omega) > \frac{1}{2}$$

- Hilbert space of square integrable functions with energy norm
In this sense the ingoing solutions belong to the spectrum
- Pseudospectrum in IEF $N \rightarrow$ infinity limit converges to this limit
- Math: Sobolev norm, Hilbert space $H^{(k)}$
Physics: higher derivative theories!
- In regular coords resolvent seems to be divergent in general
- Finite N provides a natural cutoff \rightarrow what is the optimal N ?
e.g stretched Horizon, Planck length away from horizon
- Universality? Holographic interpretation?



$$\|\Phi\|^2 = \int \sum_{m=0}^k |D^m \Phi|^2$$

[Warnick: CMP. 333 (2015) 2, 959-1035 •
e-Print: 1306.5760 [gr-qc]]

[Boyanov, Cardoso, Destounis, Jaramillo, Macedo]
e-Print: 2312.11998 [gr-qc]]

Summary

- Quasinormal Modes are central in gauge/gravity duality
- Subject to spectral instability
- Choice of norm is important → Energy norm
- Energy norm “ $H^{(1)}$ ” Hilbert space
- Hydrodynamic modes seem safe ? $\lim_{k \rightarrow 0} \omega_{\text{hydro}}(k) = 0$
- Physical significance of higher modes seems less clear
- Optimal lattice N ?
- Regular boundary conditions: beyond two derivative theories?



Thank you!