Relativistic Hydrodynamics under Rotation with Holography

Markus Amano

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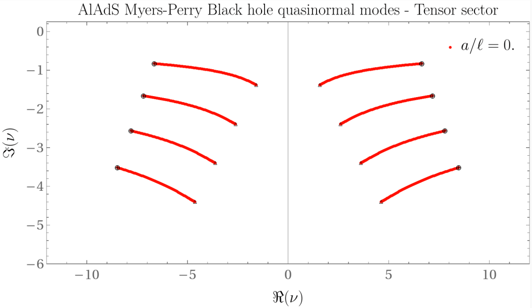
Yamagata University

Overview & Methods

Overview & Methods

Amano, Kaminski et al. 2023 an overview of recent developments arising from the study of simply spinning Myers-Perry black holes



Study of. . .

- aspects of (Topo. S^3) Rotating Black Holes in AdS and
- the dual rotating conformal plasma (model highly vortical heavy ion collisions) (STAR Collaboration 2017, Bantilan et al).

“Gravity Side”

$$S = \frac{1}{16\pi G_5} \int \sqrt{-g}(R - 2\Lambda) + S_{ct} \tag{1}$$

where $\Lambda = -6/\ell^2$.

- Rotating AdS blackhole $\longleftrightarrow$ Rotating strongly-coupled fluid

Rotating black holes ($R^{1,1} \times S^3$)

$S^3 \rightsquigarrow S^1 \times S^1$ (invariant under $SO(4) \rightsquigarrow SO(2) \times SO(2)$ rotations)

$S^1 \times S^1 \implies$ two axial angular momenta.

5D AdS Myers-Perry Black Hole (Hawking & Reall 1998)

$$
\begin{aligned}
ds^2 = {} & \frac{(1 + r_H^2 \ell^{-2})}{\rho^2 r_H^2} \left(ab\mathrm{d}t_H - \frac{b\,(a^2 + r_H^2)\sin^2(\theta_H)}{\Xi_a}\mathrm{d}\phi_H - \frac{a\,(b^2 + r_H^2)\cos^2(\theta_H)}{\Xi_b}\mathrm{d}\psi_H \right)^2 \\
& - \frac{\Delta_r}{\rho^2} \left(\mathrm{d}t_H - \frac{a\sin^2(\theta_H)}{\Xi_a}\mathrm{d}\phi_H - \frac{b\cos^2(\theta_H)}{\Xi_b}\mathrm{d}\psi_H \right)^2 + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta_H^2 + \frac{\rho^2}{\Delta_r}\mathrm{d}r_H^2 \\
& + \frac{\Delta_\theta \sin^2(\theta_H)}{\rho^2} \left(a\mathrm{d}t_H - \frac{a^2 + r_H^2}{\Xi_a}\mathrm{d}\phi_H \right)^2 + \frac{\Delta_\theta \cos^2(\theta_H)}{\rho^2} \left(b\mathrm{d}t_H - \frac{b^2 + r_H^2}{\Xi_b}\mathrm{d}\psi_H \right)^2
\end{aligned}
\tag{2}
$$

- Coordinates $(t_H, \theta_H, \phi_H, \psi_H, r_H)$
- Angular momentum parameters of a and b.

$$
\begin{aligned}
\Delta_r &= \frac{1}{r_H^2}(r_H^2 + a^2)(r_H^2 + b^2)(\frac{r_H^2}{\ell^2} + 1) - 2M\,, \\
\Delta_\theta &= 1 - \frac{a^2}{\ell^2}\cos(\theta_H)^2 - \frac{b^2}{\ell^2}\sin(\theta_H)^2\,, \\
\rho^2 &= r_H^2 + a^2\cos(\theta_H)^2 + b^2\sin(\theta_H)^2\,, \\
\Xi_a &= 1 - \frac{a^2}{\ell^2}\,, \quad \Xi_b = 1 - \frac{b^2}{\ell^2}\,.
\end{aligned}
$$

Simply Spinning Enhance the symmetry with $a = b$.

$$\begin{aligned} ds^2 = \frac{dr^2}{G(r)} - dt^2\left(\frac{r^2}{\ell^2}+1\right) + \frac{1}{4}r^2\left((\sigma^1)^2+(\sigma^2)^2+(\sigma^3)^2\right) \\ + \frac{2\mu}{r^2}\left(\frac{a\sigma^3}{2}+dt\right)^2 \end{aligned} \tag{3}$$

- Cooridnates: $(t, \theta, \phi, \psi, r)$
- Angular momentum parameter of a
- Enhanced Symmetry $S^1 \times S^1 \nearrow S^2 \times S^1$
- σ's are the left-invariant forms of $SO(3)$ rotations.

Gravitational Perturbations

Perturbed metric to first order

$$g^{p}_{\mu\nu} dx^{\mu} dx^{\nu} = \Big(g_{\mu\nu} + \epsilon \; h_{\mu\nu} + O(\epsilon^2) \Big) \, dx^{\mu} dx^{\nu} \,, \tag{4}$$

The Einstein Field Equations at first order (Wald 1984) are linear PDEs.

$$-\frac{1}{2} \nabla_{\mu} \nabla_{\nu} h - \frac{1}{2} \nabla^{\lambda} \nabla_{\lambda} h_{\mu\nu} + \nabla^{\lambda} \nabla_{(\mu} h_{\nu)\lambda} = \frac{2\Lambda}{D-2} h_{\mu\nu} \,, \tag{5}$$

The enhanced symmetry can be used to reduce the perturbation equations to ODEs.

$$h_{\mu\nu} = \int d\omega e^{-i\omega t} \sum_{\mathcal{J}=0} \sum_{\mathcal{M}=-\mathcal{J}}^{\mathcal{J}} \sum_{\mathcal{K}'=-(\mathcal{J}+2)}^{\mathcal{J}+2} h_{ij}(r,\omega,\mathcal{J},\mathcal{M},\mathcal{K}')\overline{\sigma}^i_\mu \overline{\sigma}^j_\nu D^{\mathcal{J}}_{\mathcal{K}'-Q(\overline{\sigma}^i)-Q(\overline{\sigma}^j)\mathcal{M}} \tag{6}$$

Q is the, W_3, angular momentum charge of the the ith basis.

$$Q(\overline{\sigma}^i) = \lambda \ if \ \ i = r, t, 3; \quad 1 \ if \ \ i = +; \quad -1 \ if \ \ i = -$$

Plugging the decomposed perturbation in to its equations of motion:

- The perturbations of different $((\mathcal{J},\mathcal{M}),\mathcal{K}')$ decouple
- The angular momentum quantum parameter, $\mathcal{M}$, does not appear in the equations.
- Perturbations are non-trivially labeled by $(\mathcal{J},\mathcal{K}')$

Similar to plane waves of black brane perturbations, $D^{\mathcal{J}}_{\mathcal{K}M}$ for a complete set on S^3.

$$\begin{aligned} [L_a, L_a] &= i\epsilon_{abc} L_c \\ [W_a, W_b] &= -i\epsilon_{abc} W_c \\ [W_a, L_b] &= 0 \end{aligned} \qquad (7)$$

$$\begin{aligned} L^2 D^{\mathcal{J}}_{\mathcal{K}\mathcal{M}} &= \mathcal{J}(\mathcal{J}+1) D^{\mathcal{J}}_{\mathcal{K}\mathcal{M}} \\ L_3 D^{\mathcal{J}}_{\mathcal{K}\mathcal{M}} &= \mathcal{M} D^{\mathcal{J}}_{\mathcal{K}\mathcal{M}} \\ W_3 D^{\mathcal{J}}_{\mathcal{K}\mathcal{M}} &= \mathcal{K} D^{\mathcal{J}}_{\mathcal{K}\mathcal{M}} \end{aligned} \qquad (8)$$

One can use the raising and lowering operators (forms)

$L_\pm = L_1 \pm i L_2$ $(\sigma^\pm = \frac{1}{2}(\sigma^1 \mp i\sigma^2))$

$$\begin{aligned} \sigma^1 &= d\phi \sin(\theta)\cos(\psi) - d\theta \sin(\psi) \\ \sigma^2 &= d\theta \cos(\psi) + d\phi \sin(\theta)\sin(\psi) \\ \sigma^3 &= d\psi + d\phi \cos(\theta) \\ \overline{\sigma}^i &= (\mathrm{d}t, \sigma^+, \sigma^-, \sigma^3, \mathrm{d}r) \end{aligned} \qquad (9)$$

Tensor, Vector, and Scalar Sectors

Based on the "$(\mathcal{J}, \mathcal{K}')$" classification of sectors there are an infinite number of sectors.

Amano, Kaminski et al. 2023 takes a look at three of these sectors.

Tensor $\mathcal{K}' = \mathcal{J} + 2$; h_{++}

Vector $\mathcal{K}' = \mathcal{J} + 1$; h_{+r}, h_{+t}, h_{+3} (, and h_{++} if $\mathcal{J} \geq 1$)

Scalar $\mathcal{K}' = \mathcal{J}$, h_{+-}; h_{ab} where $a, b \in \{r, t, 3\}$
(, h_{+r}, h_{+t}, h_{+3} if $\mathcal{J} \geq 1$) (, and h_{++} if $\mathcal{J} \geq 2$)

Quasinormal Modes

The quasinormal modes[1] are defined as **non-trivial** solutions to linearized Einstein equations and obey the two boundary conditions.

BCs

- Ingoing at the horizon
- Sourceless at the AdS Boundary (a Dirichlet boundary condition)

Parameters

- $\mathcal{J} = 0, 1/2, 1, 3/2, 2, \ldots$ is the discrete (angular) momentum.
- $\omega_{\mathcal{J}}$ are discrete eigen-frequencies (QNMs) such that the BCs are fulfilled.

[1] They are dual to the poles of retarted Greens functions

$$h_{\mu\nu} \sim r^2 h^{(0)}_{\mu\nu} - h^{(1)}_{\mu\nu}/r^2 \tag{10}$$

- $h^{(0)}_{\mu\nu} \leftrightarrow$ source of δT.
- $h^{(1)}_{\mu\nu} \leftrightarrow$ VEV of $\delta T_{\mu\nu} \equiv \langle \delta T_{\mu\nu} \rangle$.

$h^{(0)}_{\mu\nu} = 0$ Sourceless boundary condition.

Conservation Equations $\nabla_\mu T^{\mu\nu} \overset{\text{flat}}{=} \partial_\mu T^{\mu\nu} \overset{\text{Fourier}}{=} ik_\mu T^{\mu\nu} = 0$

Constitutive Equations[2] $T_{\mu\nu} = \epsilon u^\mu u^\nu + P\Pi^{\mu\nu} + \pi^{\mu\nu} + \ldots = \left(T^{(0)}\right)^{\mu\nu} + \delta T^{\mu\nu}$

Hydro Variables u^μ and T

[2] u^μ are the hydrodynamic velocity variables that points in the forward time direction. $\Pi^{\mu\nu} := u^\mu u^\nu + \eta^{\mu\nu}$.

Order 0 (Ideal Hydro) $\partial_\mu \left(\epsilon u^\mu u^\nu + P\Pi^{\mu\nu}\right) = 0$ (where $u = (1, 0, 0, 0)$)
Order 1 (First Order Hydro) (Kovtun 2019) $\partial_\mu \pi^{\mu\nu} \overset{\text{Fourier}}{=} ik_\mu \pi^{\mu\nu} = 0$

The generalized eigenvalue problem implicitly defines ω as a function of $|k|$. [3]

$$\omega = \sum_{a=0} \Omega_a |k|^a \tag{11}$$

Hydrodynamic Modes $\omega\big|_{|k|=0} = 0$
Non-Hydrodynamic Modes $\omega\big|_{|k|=0} \neq 0$

[3]Assuming we can rotate globally to align k to any direction.

Review Summary

Study Flow

1. Pick the 5D Myers-Perry "Simply Spinning" solution (dual to some strongly rotating plasma)
2. Solve linearized Einstein equation for quasinormal modes.
3. Analyze the spectra and compare with hydrodynamics (and non-hydrodynamics?)

QNMs Method

1. Derive Linear Einstein Field Equations in EF ingoing coordinates with mentioned $h_{\mu\nu}$ decomposition
2. Impose radial gague $h_{\mu r} = 0$
3. Discretize with collocation method on a pseudospectral grid and enforce sourceless BC.
4. Linearize eigenvalue problem and solve with Mathematica's `Eigenvalues`.

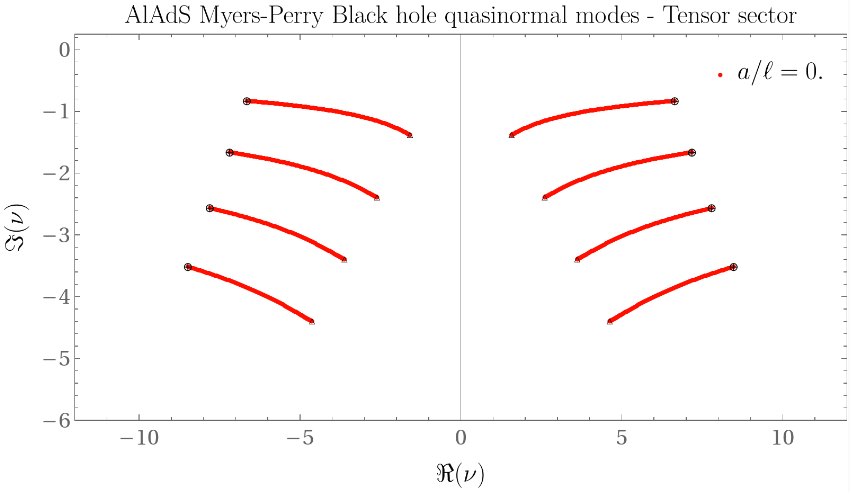
Results

Results

Amano, Kaminski et al.

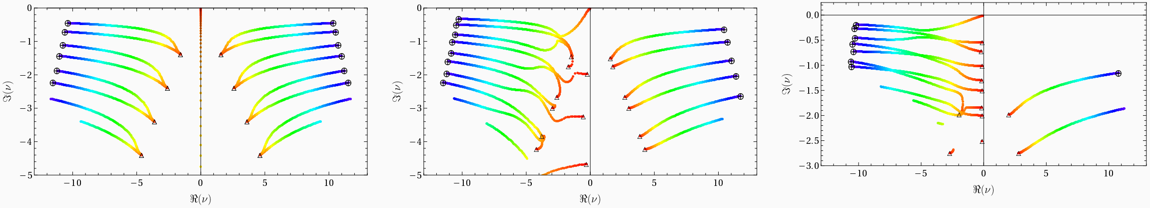
- Non-Hydrodynamic Modes and the effects of non-extremal rotation.
  - Tensor
  - Vector
  - Scalar
- Cross Spectrum Comparison
- The Emergence of Hydrodynamics
- Stability

$\mathcal{K}' = \mathcal{J} + 2$ Tensor Sector



$\mathcal{K}' = \mathcal{J} + 2;\ h_{++}$

$\mathcal{K}' = \mathcal{J} + 1$ Vector Fluctuations



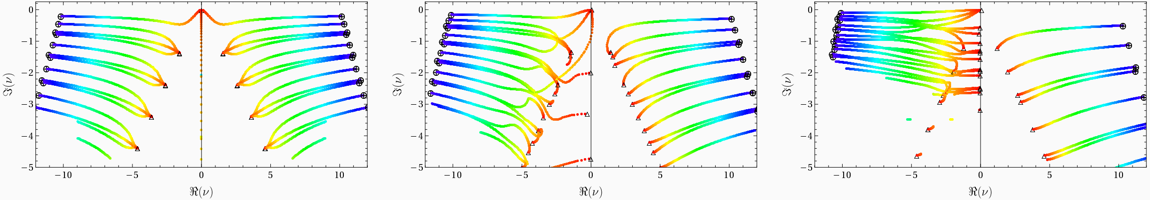
$a/\ell \in \{0, 1/2, 9/10\}$

$\mathcal{K}' = \mathcal{J} + 1$; h_{+r}, h_{+t}, h_{+3} (, and h_{++} if $\mathcal{J} \geq 1$)

$\mathcal{J} = 0, 1/2, 1, \ldots, 199/2, 100$, $r_+/\ell = 10$

$\triangle \equiv \mathcal{J} = 0$, $\bigoplus \equiv \mathcal{J} = 100$

$\mathcal{K}' = \mathcal{J}$ Scalar Fluctuations



$a/\ell \in \{0, 1/2, 9/10\}$

$\mathcal{K}' = \mathcal{J}$, h_{+-}; h_{ab} where $a, b \in \{r, t, 3\}$ (, h_{+r}, h_{+t}, h_{+3} if $\mathcal{J} \geq 1$) (, and h_{++} if $\mathcal{J} \geq 2$)

$\mathcal{J} = 0, 1/2, 1, \ldots, 199/2, 100$, $r_+/\ell = 10$

$\triangle \equiv \mathcal{J} = 0$, $\bigoplus \equiv \mathcal{J} = 100$

The Emergence of Hydrodynamics

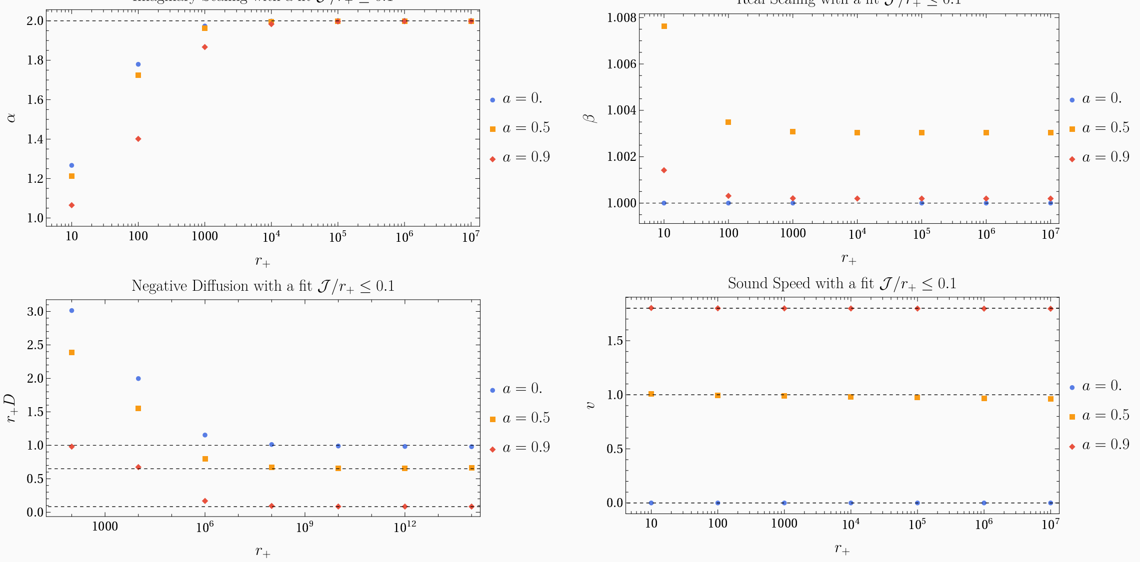
To study the dispersion relations of the lowest gapless modes, we looked at the momentum diffusion sector.

We solved spectra for $r_+ = \{10, 100, 1000, 10^4, 10^5, 10^6, 10^7\}$ and $\mathcal{J} = 0, 1/2, 1, \ldots, \mathcal{J}_{\max}$.[4]

To see the emergence of hydrodynamics, we fitted the data to the equation below with fitting parameters α, β, D, and v.

$$\omega = v\mathcal{J}^{\beta} - iD\mathcal{J}^{\alpha} \tag{12}$$

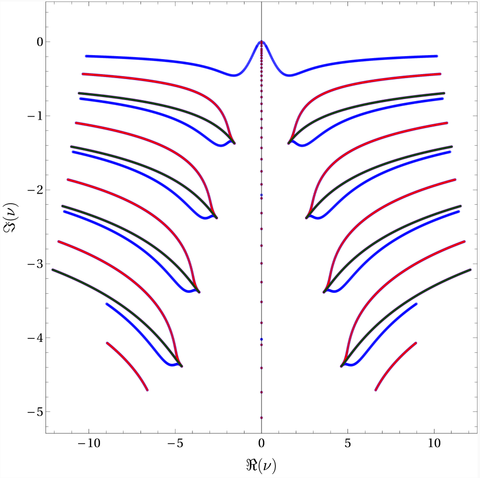
[4] $\mathcal{J}_{\max}/r_+ = j_{\max} = 0.1$

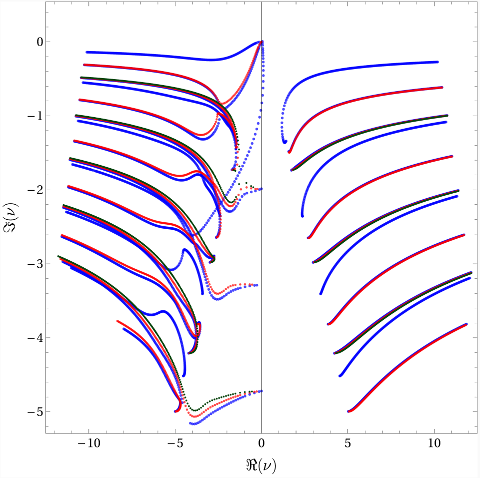


(Kovtun 2019)

$$\omega = v\mathcal{J}^{\beta} - iD\mathcal{J}^{\alpha} \tag{13}$$

Cross Spectrum Comparisons





- Tensor
- Vector
- Scalar

Stability

It's well known that the MPAdS5D solution suffers from superadiant instabilities (Murata 2009, Dias et al).

Study was done a regime that is stable for large enough r_+.

This instability is signaled by boundary turning from a timelike surface to a spacelike surface (for stationary observers).

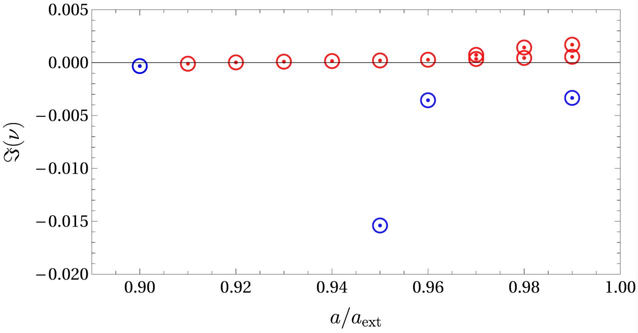


Figure 1: $\mathcal{J} = 5$ frequencies, zoomed in. Positive imaginary modes are unstable.

RFP (Read the Fantastic Papers!)

Amano et al. 2022

- Pole-Skipping
- Chaos
- Gravitational Shockwaves

Cartwright et al. 2023

- Critical Points

Conclusion

Summary and Outlook

Main Message

We find hydro applies in regimes where the dual fluid is not necessarily a homogeneously boasted fluid.

Outlook

- Look to calculate with a more general parameter space where $\mathcal{J}_\phi \neq \mathcal{J}_\psi$ ($a \neq b$).
  - No "axis of rotation" in current background.
  - Need PDE (Amado et al 2020, Amado et al. 2021)
- Different **sources** of rotation?
  - Vector graviton sourcing the rotation $\sim H_{\theta i} \sim \Omega_i r^2$
  - RN with magnetic field, $A_\theta \sim \Omega_i r^2$ Domenech et al. 2010
- Interpretation of Linear instability in the dual field theory?
- be used in "hydro codes"
- and more. . . RFP!

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- *Casey Cartwright*, *Matthias Kaminski*, *Jorge Noronha*, *Enrico Speranza* Cartwright et al. 2023