Relativistic Fluctuation Dynamics

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Applied Holography Webinar

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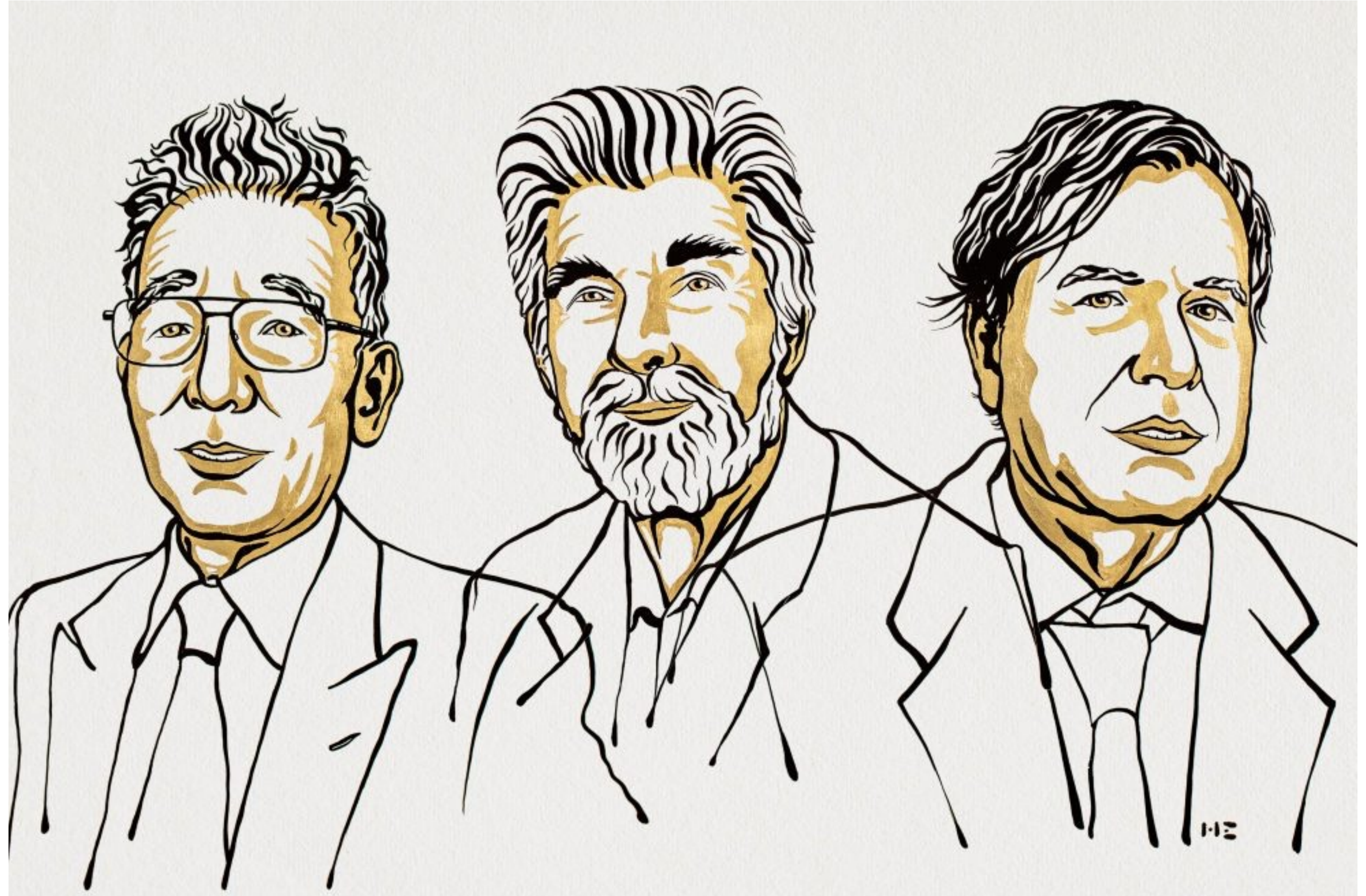
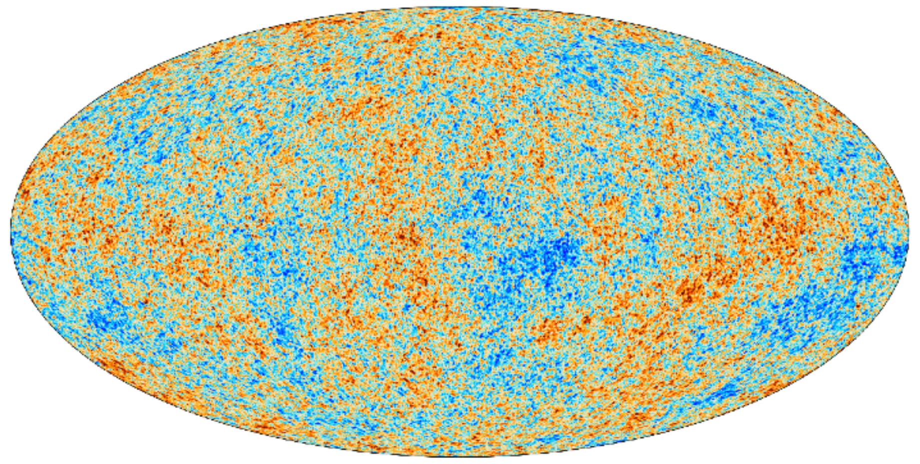
Motivation

Fluctuations on all length scales

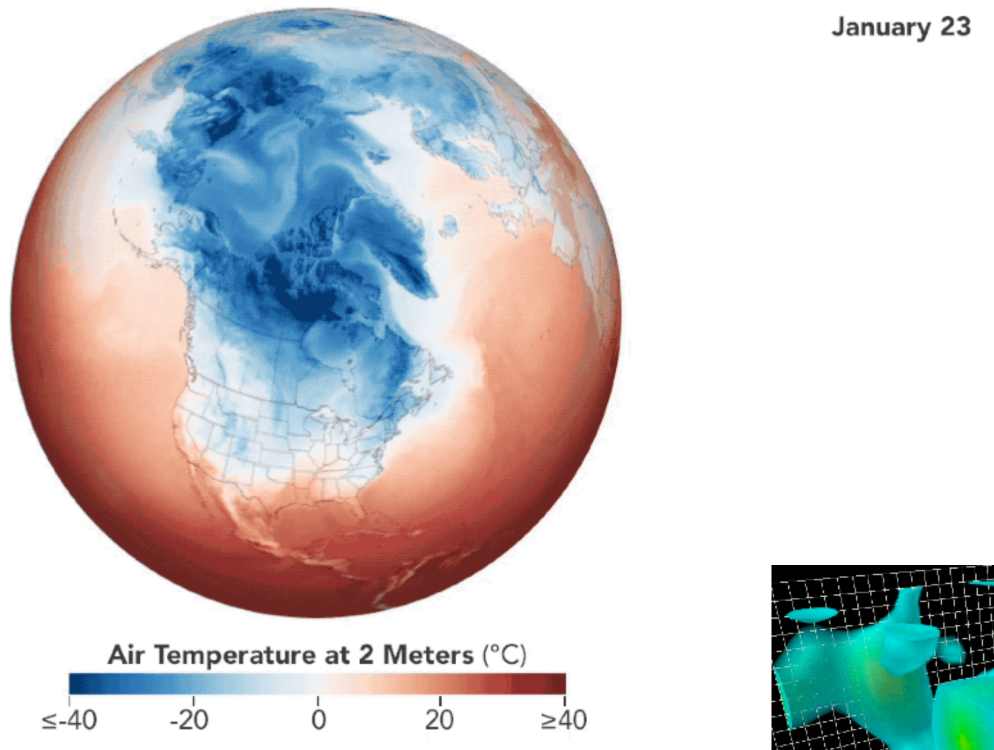
- Fluctuations are ubiquitous phenomena emerging on all length scales.



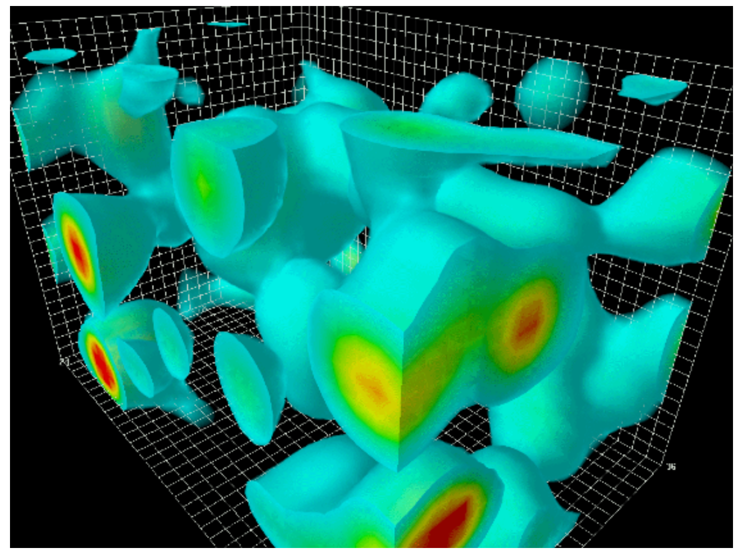
Nobel Prize in Physics 2021
S. Manabe, K. Hasselmann, G. Parisi



CMB



Atmosphere



Quantum fluctuations

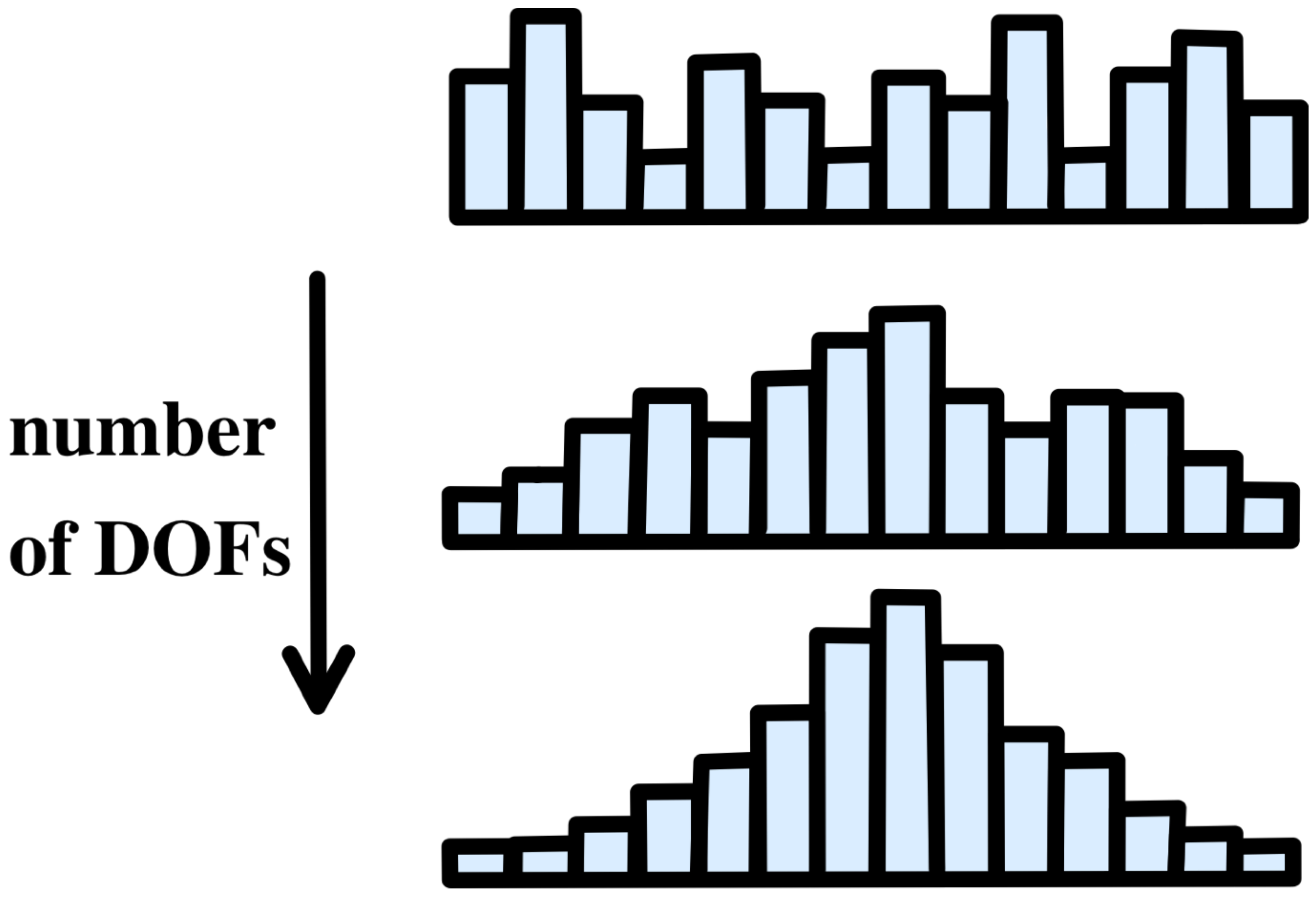
Fluctuations in equilibrium

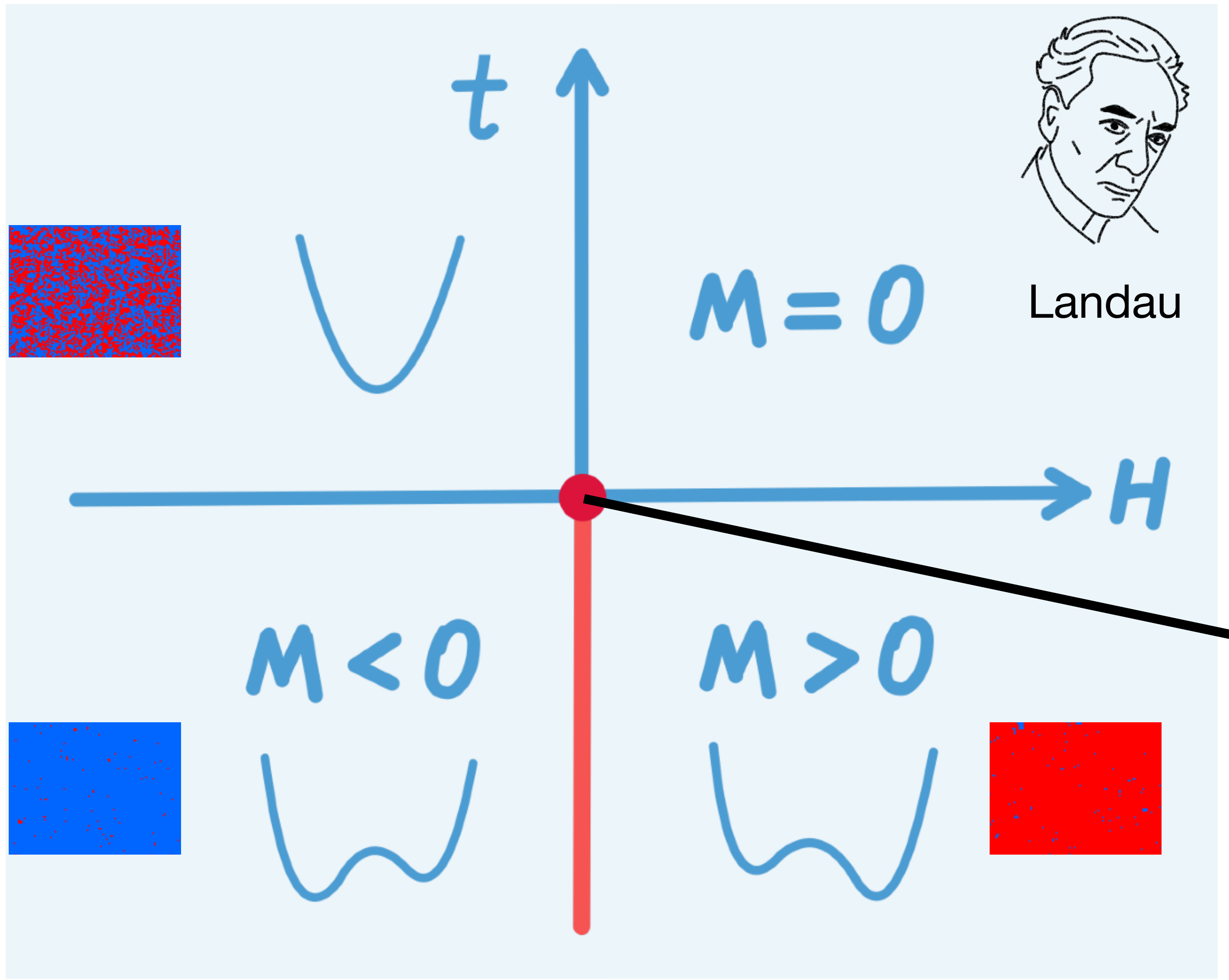
- Thermal fluctuations: systems possess *large* number of DOFs; *small* deviation from Gaussian due to the central limit theorem.

Susskind

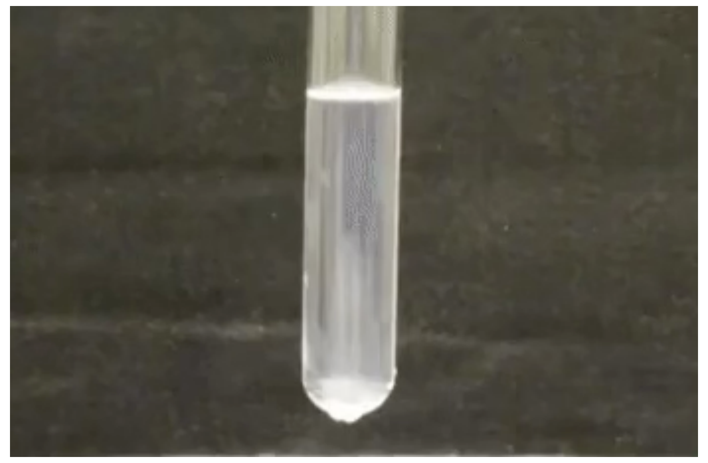
Thermal equilibrium is extremely boring.

- Non-Gaussian fluctuations become more important when systems possess *smaller* number of DOFs (e.g., *closer* to the critical point).

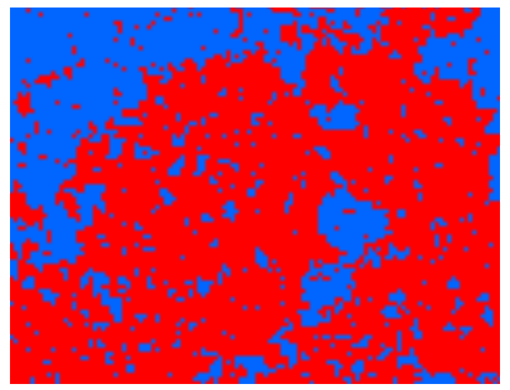




Ising phase diagram



$\xi \leftrightarrow \lambda_{\text{light}}$



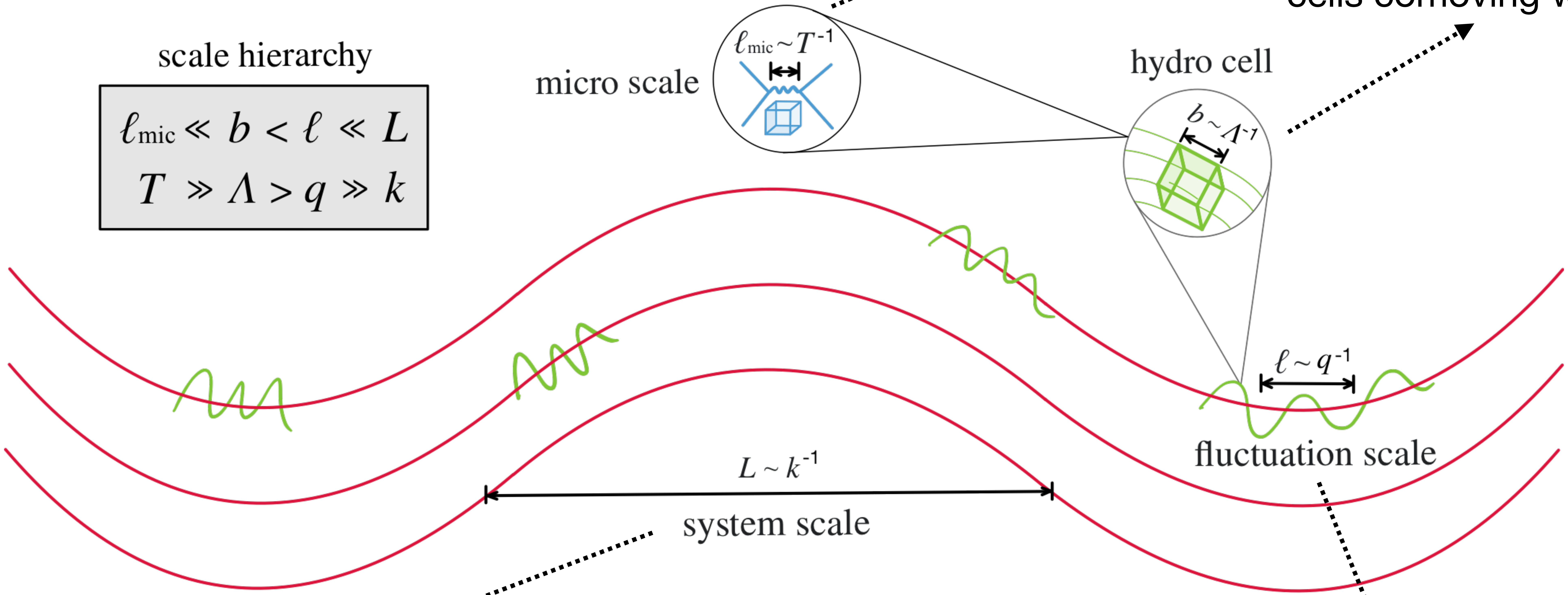
$\xi \to \infty$

Fluctuations out of equilibrium

- Hydrodynamic fluctuations:

noise

large number of *locally* thermalized cells comoving with fluid



fluctuations *not* equilibrated at *large* scales

evolution described by a set of conservation equations

$$\partial_t \psi = \nabla \cdot (\text{flux}\,[\psi]) \quad \text{where} \quad \psi = \begin{pmatrix} n \\ \epsilon \\ \pi_i \end{pmatrix} \begin{matrix} \rightarrow \text{charge} \\ \rightarrow \text{energy} \\ \rightarrow \text{momentum} \end{matrix}$$

The importance of hydrodynamic fluctuations

- Einstein's formula for diffusion coefficient: Einstein, 1905

$$D = \lim_{t\to\infty} \frac{1}{2t}\langle \Delta x^2(t)\rangle = \int_0^\infty d\tau \langle v(\tau)v(0)\rangle$$

- Long-time behavior:

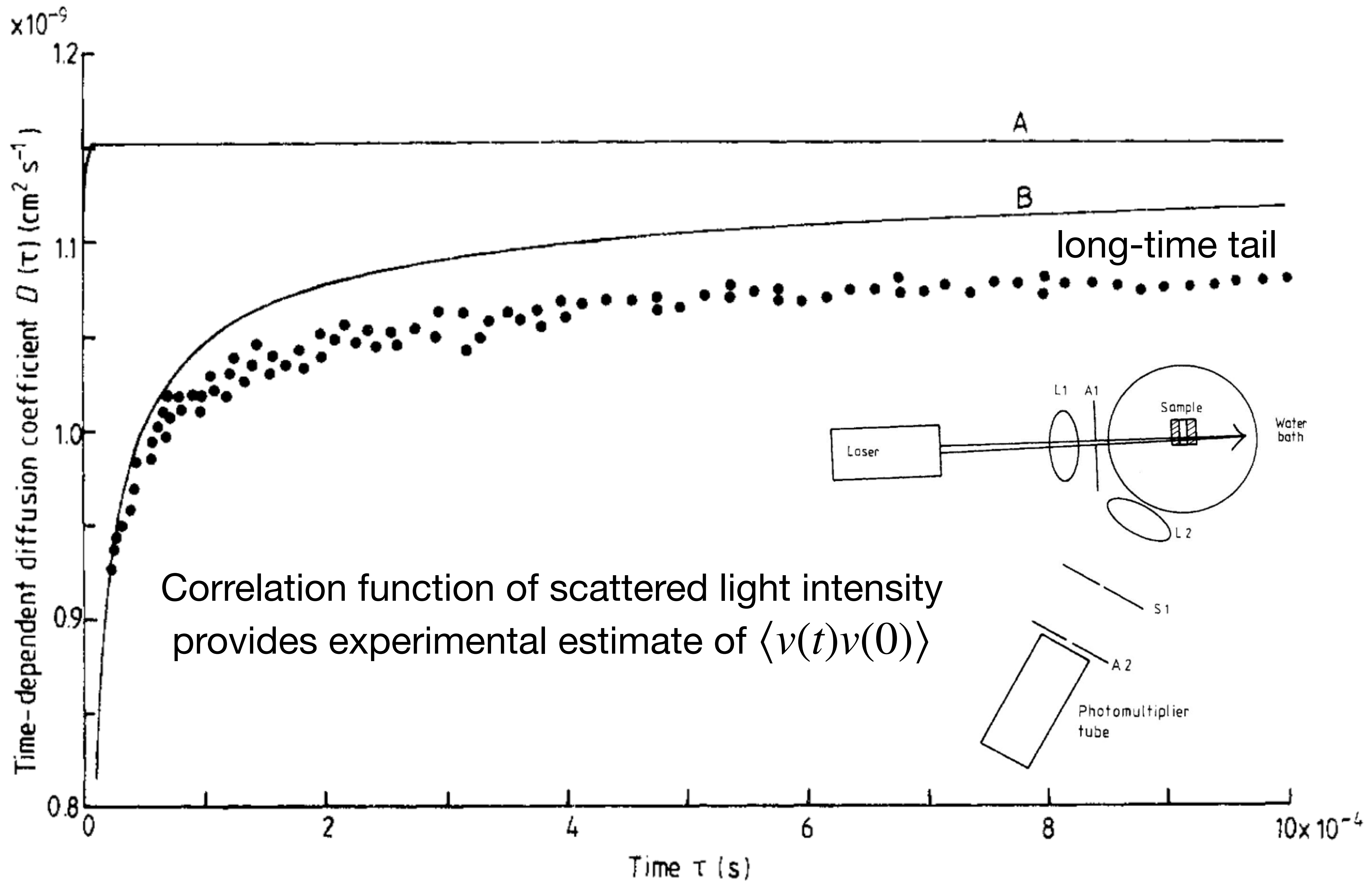
$$\langle v(t)v(0)\rangle \sim e^{-\mu t} \quad \rightarrow \quad D \sim \mu^{-1}$$

With only dissipation

$$\langle v(t)v(0)\rangle \sim t^{-3/2} \quad \rightarrow \quad D \sim t^{-1/2}$$

With also fluctuations

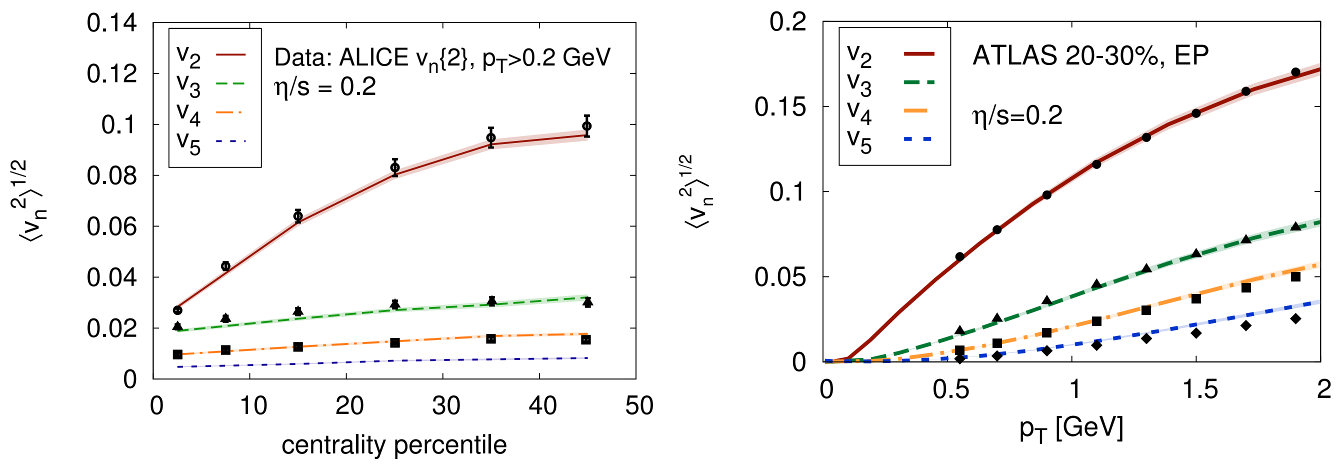
Paul et al, 1981, J. Phys. A: Math. Gen. 14 3301



Why fluctuating hydro works in QGP?

- Hydrodynamics works because:

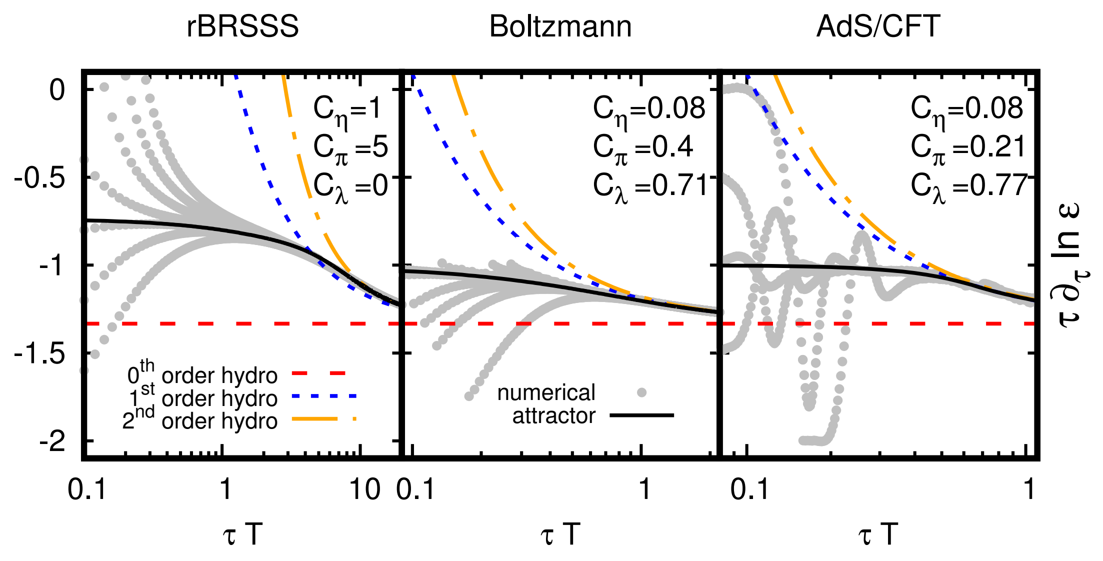
Particle **number** $\sim 10^2 - 10^4$: *large* enough



Flow collectivity manifests QGP as a *perfect fluid*

Gale et al, 1301.5893

Hydrodynamization **time** ~ 1 fm: *fast* enough

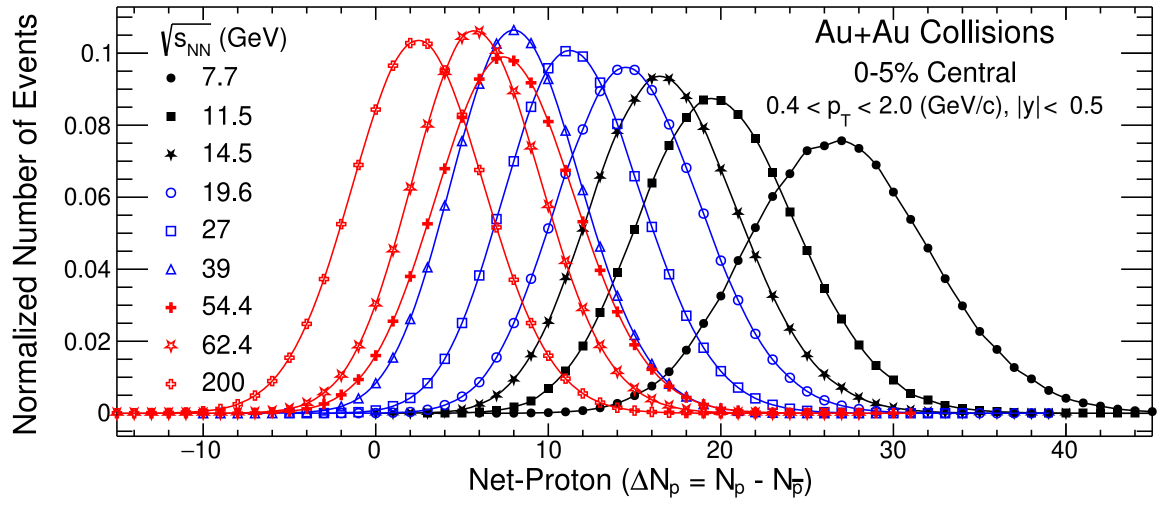


Hydrodynamic *attractor* far from equilibrium

Florkowski et al, 1707.02282, Romatschke et al, 1712.05815

- Fluctuations are important because:

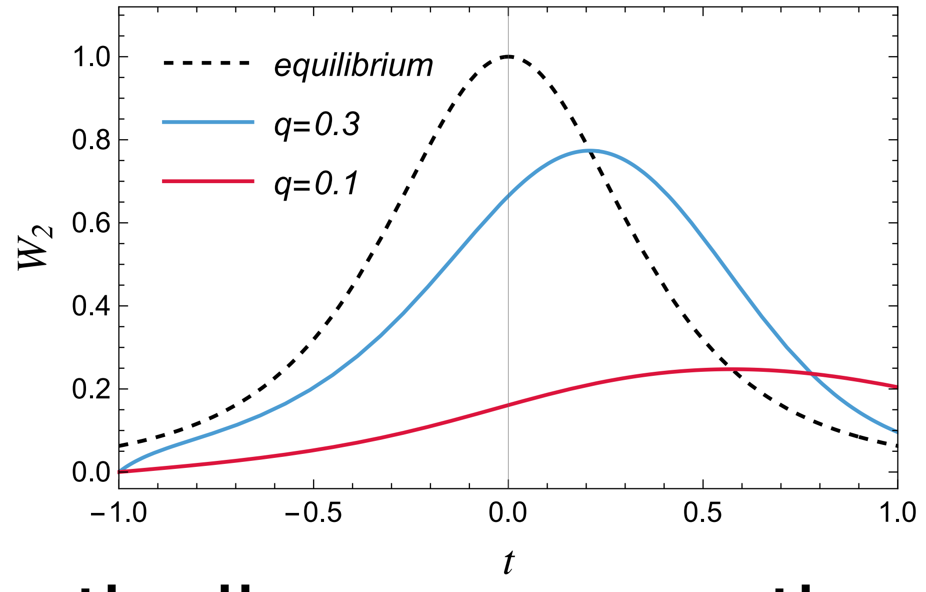
Fire ball **size** ~ 10 fm: *small* enough



Net-proton fluctuates event by event

Adam et al, 2001.02852

Correlation **length** $\sim 1 - 10$ fm: *large* enough

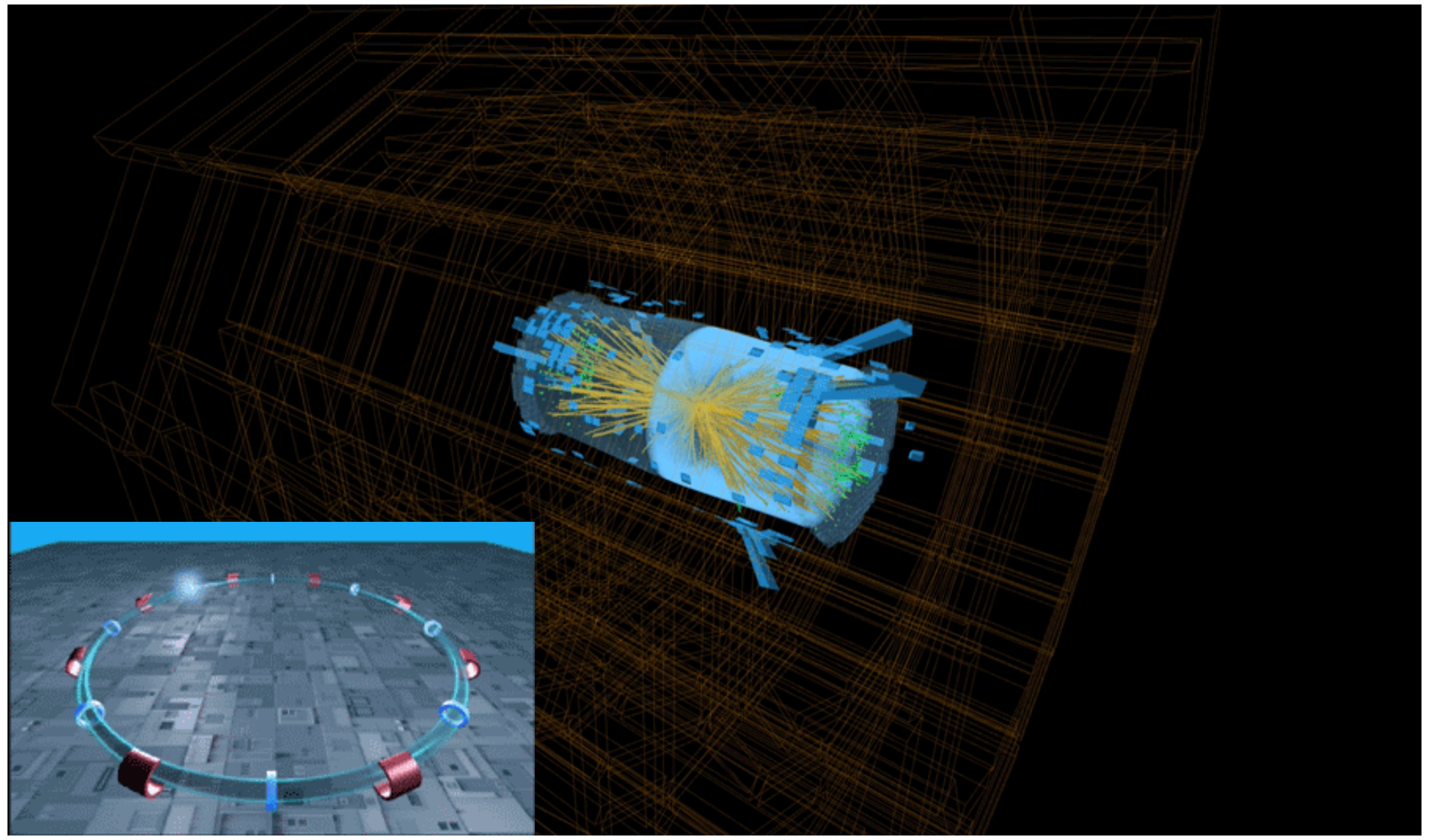


Correlation length diverges near the critical point

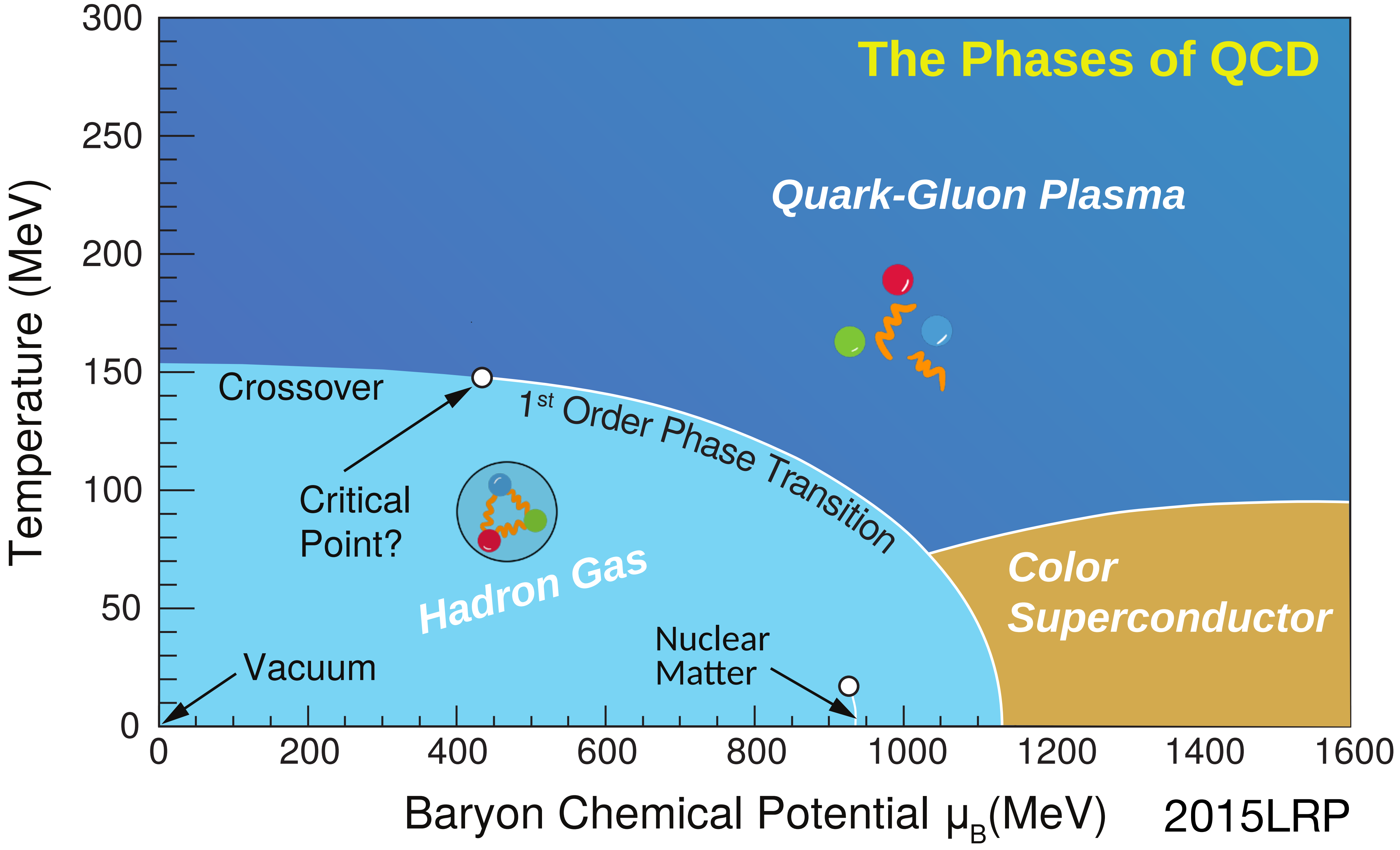
XA et al, 2009.10742

Experiment vs Theory

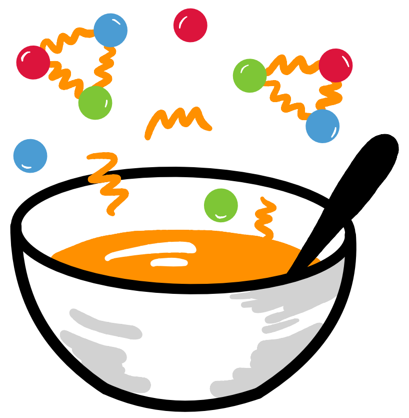
- Fluctuating hydrodynamics is a *non-equilibrium* approach to unraveling the *equilibrium* properties of QCD matters in different phases.



Collision event simulation at LHC (CERN)



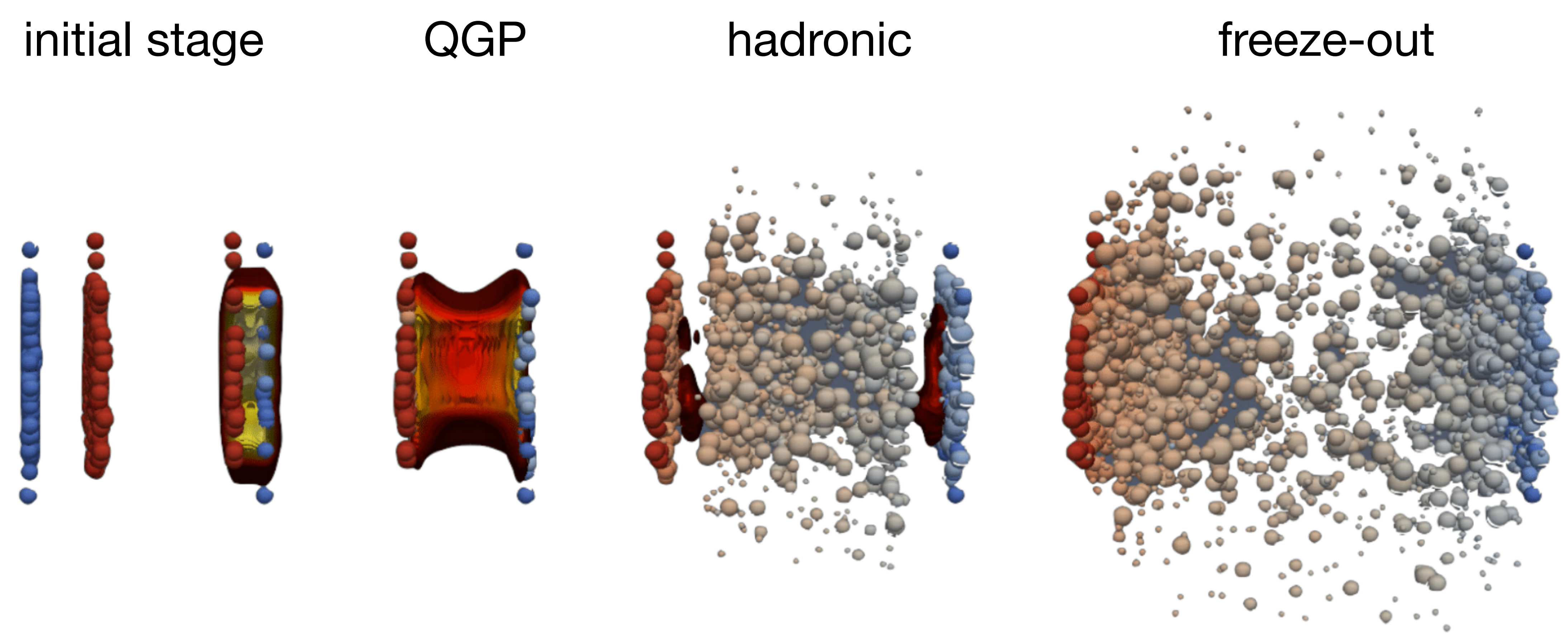
Out of equilibrium;
observables fluctuate
event-by-event



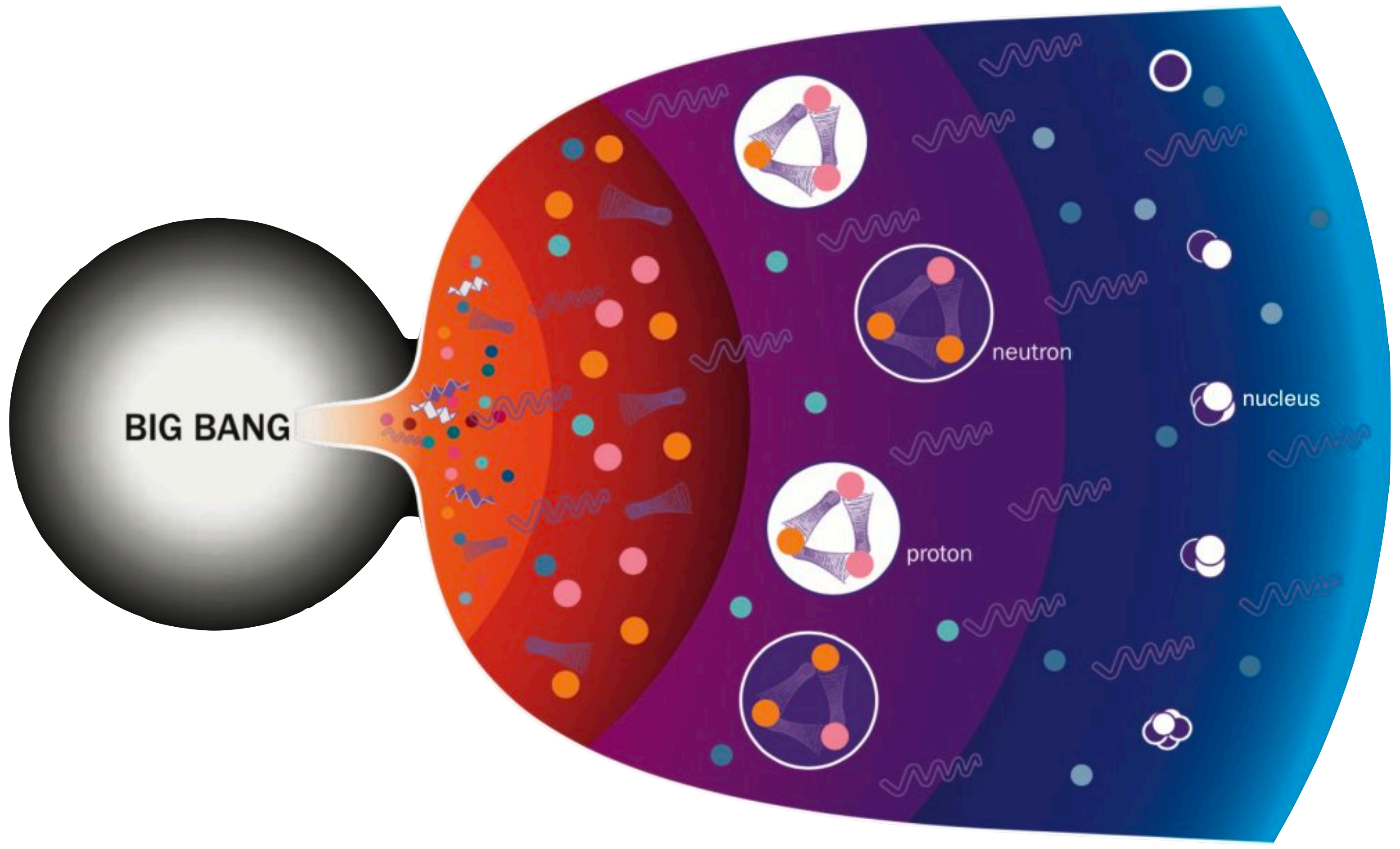
In equilibrium;
observables fluctuate
ensemble-by-ensemble

Small bang vs Big bang

- **Similarity**: extreme initial state; particle synthesis; system expands, cools followed by freezeout and thermalization.



History of a heavy-ion collision



History of Universe

- **Difference**:

Many events (HICs), high statistics measured in *momentum* coordinate

One event (CMB), cosmic variance measured in *space* coordinate

Theory for fluctuation dynamics

Theory

EFTs (top-down like)

Starting from effective action with first principles

e.g., Martin-Siggia-Rose (MSR), Schwinger-Keldysh (SK), Hohenberg-Halperin (HH), n-particle irreducible (nPI), etc.

Glorioso et al, 1805.09331
Jain et al, 2009.01356
Sogabe et al, 2111.14667
Chao et al, 2302.00720
…

EOMs (bottom-up like)

Starting from phenomenological equations with required properties

e.g., Langevin equations in *stochastic* description, Fokker-Planck (FP) equations in *deterministic* description.

Akamatsu et al, 1606.07742
Nahrgang et al, 1804.05728
Singh et al, 1807.05451
Chattopadhyay et al, 2304.07279
…

Two different EOMs

Stochastic

Langevin equation

Newton's equation + noise

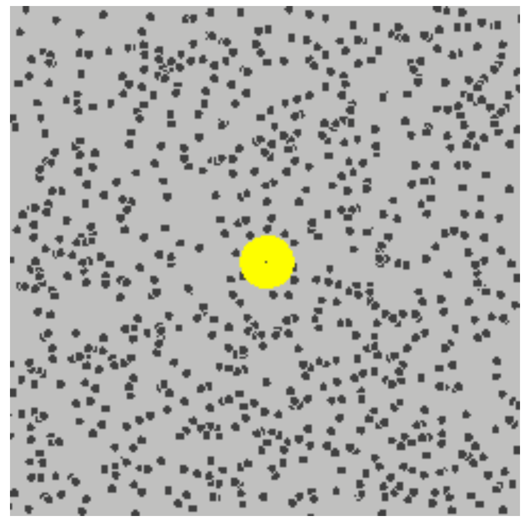
$$\partial_t \psi_i = F_i[\psi] + \eta_i$$

↓ drift

$$\langle \eta_i(x_1)\, \eta_j(x_2) \rangle = 2Q_{ij}\, \delta^{(4)}(x_1 - x_2)$$

↙ multiplicative noise ↘ infinite noise

Langevin Landau Lifshitz



Pros: *one* equation, albeit *millions* of samples

Cons: divergence due to infinite noise; ambiguity due to multiplicative noise

Deterministic

Fokker-Planck equation

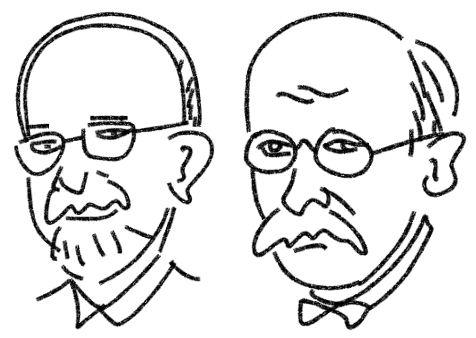
probability evolution equation

$$\partial_t P = \left(-F_i P + (M_{ij} P)_{,j}\right)_{,i}$$

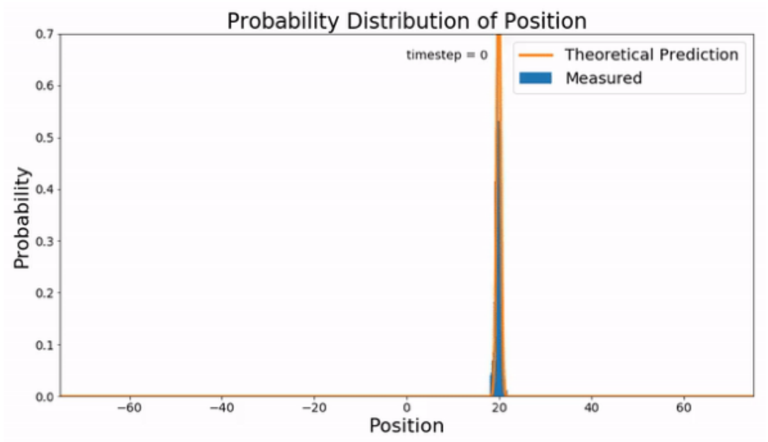
$$F_i = M_{ij} S_{,j} + M_{ij,j} \qquad M_{ij} = Q_{ij} + \Omega_{ij} \qquad P_{\mathrm{eq}} = e^S$$

Q_{ij} : Onsager matrix (symmetric)

Ω_{ij} : Poisson matrix (anti-symmetric)



Fokker Planck

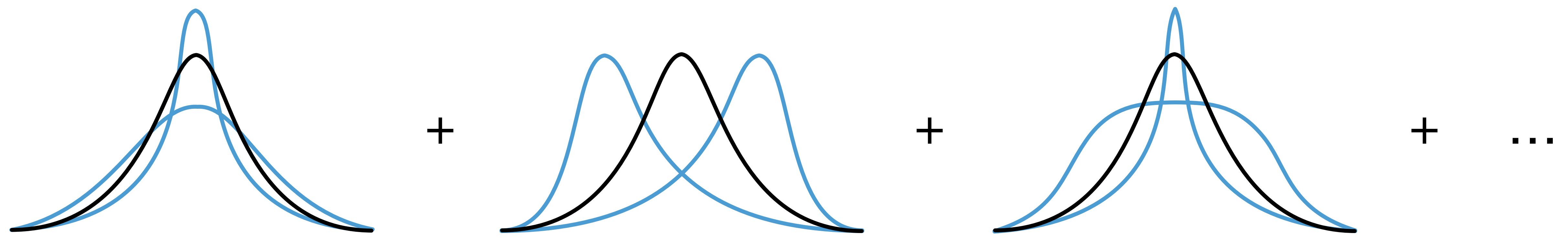


Pros: infinite noise regularized analytically; multiplicative noise well defined

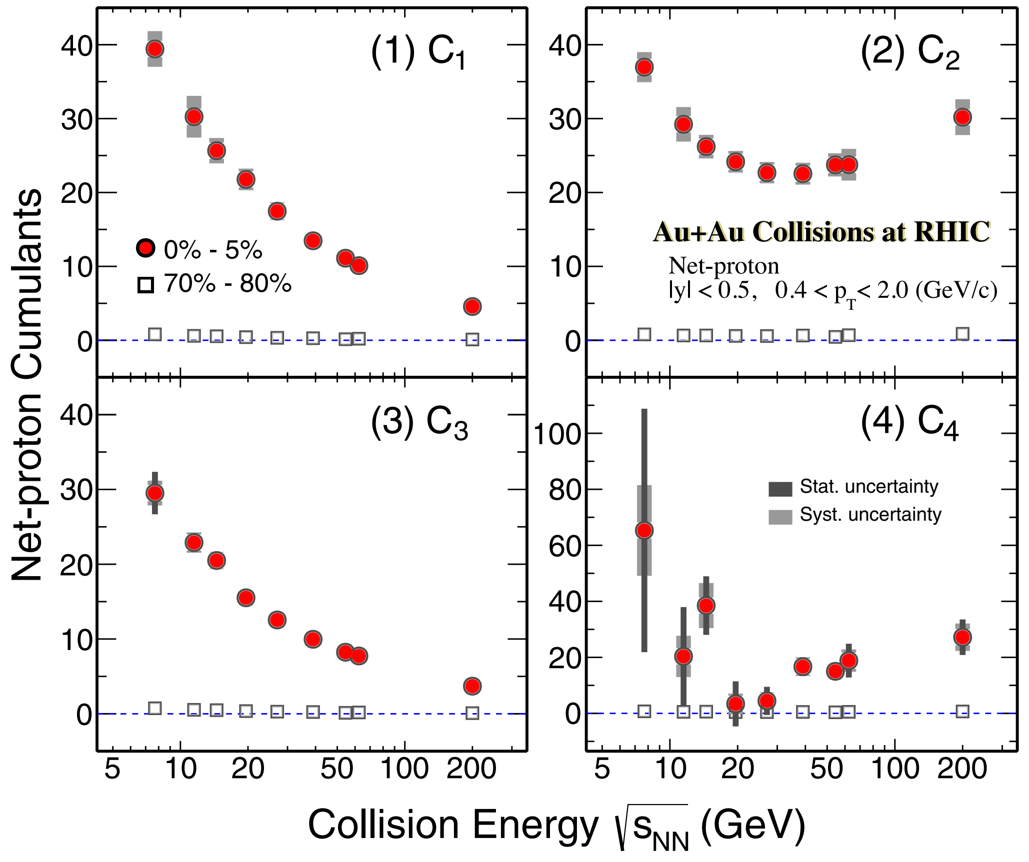
Cons: *millions* of equations, albeit *one* sample

Dynamics of correlators

- Both approaches consider n-pt correlators $G_n \equiv \langle \underbrace{\phi \ldots \phi}_{n} \rangle \equiv \int d\psi \, P[\psi] \, \underbrace{\phi \ldots \phi}_{n}$ where $\phi \equiv \psi - \langle \psi \rangle$.



variance ↓ width | skewness ↓ lopsidedness | kurtosis ↓ sharpness



cumulants measured in HIC

n-pt correlators are related to *cumulants* by integration

- Evolution equations for G_n :

XA et al, 2009.10742, 2212.14029

$$\partial_t P = (-F_i P + (M_{ij} P)_{,j})_{,i} \quad \longrightarrow \quad \partial_t G_n = \ldots$$

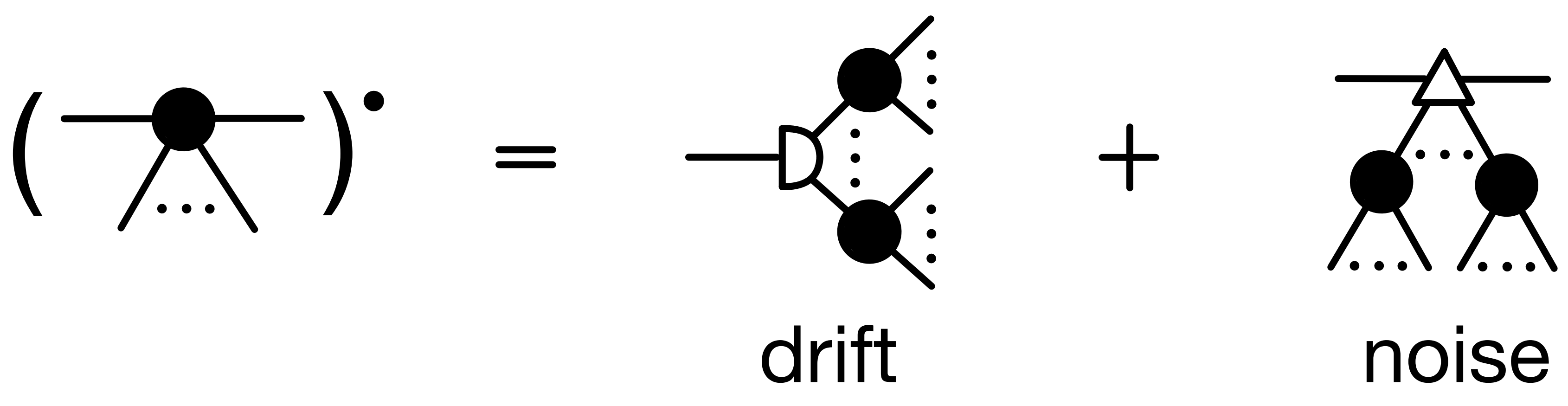
E.g., $\partial_t G_{ij} = F_{i,k} G_{kj} + F_{j,k} G_{ki} + 2M_{ij} + \frac{1}{2} F_{i,k\ell} G_{k\ell j} + \frac{1}{2} F_{j,k\ell} G_{k\ell i} + M_{ij,k\ell} G_{k\ell} + \ldots$

Diagrams and truncation

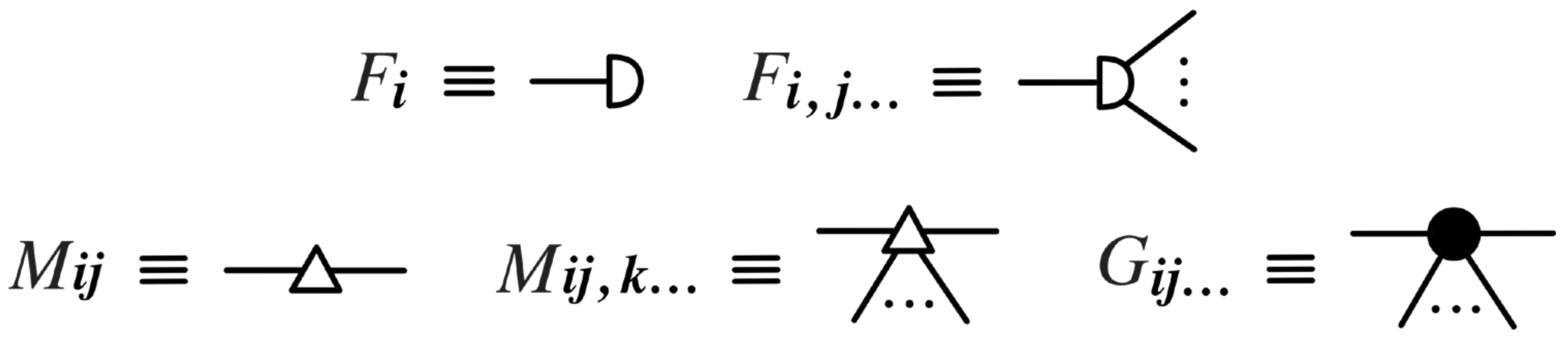
- Evolution equations for n-pt correlators (diagrams): XA et al, 2009.10742, 2212.14029

$$\partial_t G_n = \mathscr{F}\left[\langle\psi\rangle, G_2, G_3, \ldots, G_n, G_{n+1}, \ldots, G_\infty\right]$$

need ∞ equations to close the system!



all *combinatorial* configurations of trees



- Introducing the loop expansion parameters $\varepsilon \sim 1/$number of DOFs, the evolution equations can be systematically truncated and iteratively solved:

XA et al, 2009.10742

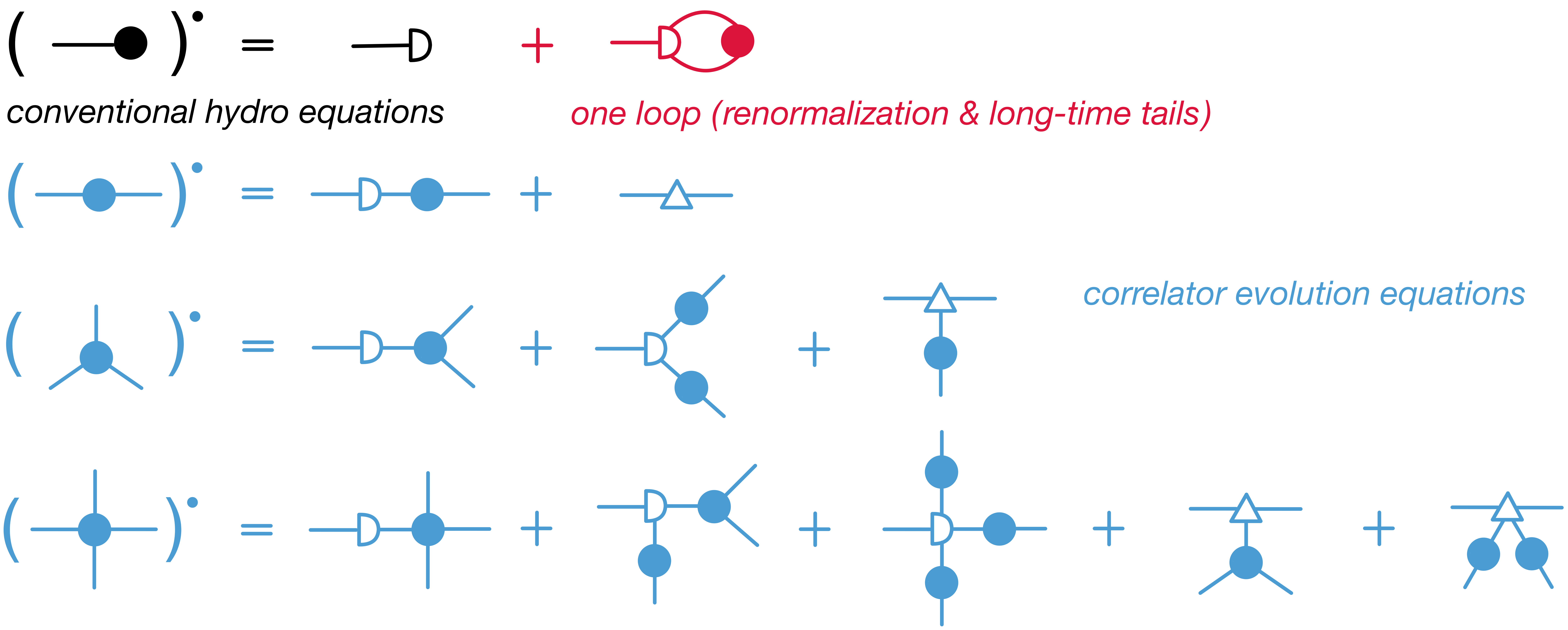
$$\partial_t G_n = \mathscr{F}\left[\langle\psi\rangle, G_2, G_3, \ldots, G_n\right] + \mathcal{O}(\varepsilon^n)$$ where $G_n \sim \varepsilon^{n-1}, \quad F_i \sim 1, \quad M_{ij} \sim \varepsilon\,.$

Hydrodynamics: $\varepsilon \sim (\xi/\ell)^3 \sim$ correlated volume / fluctuation volume

Holography: $\varepsilon \sim 1/N_c \sim$ 1 / number of colors

Truncated equations

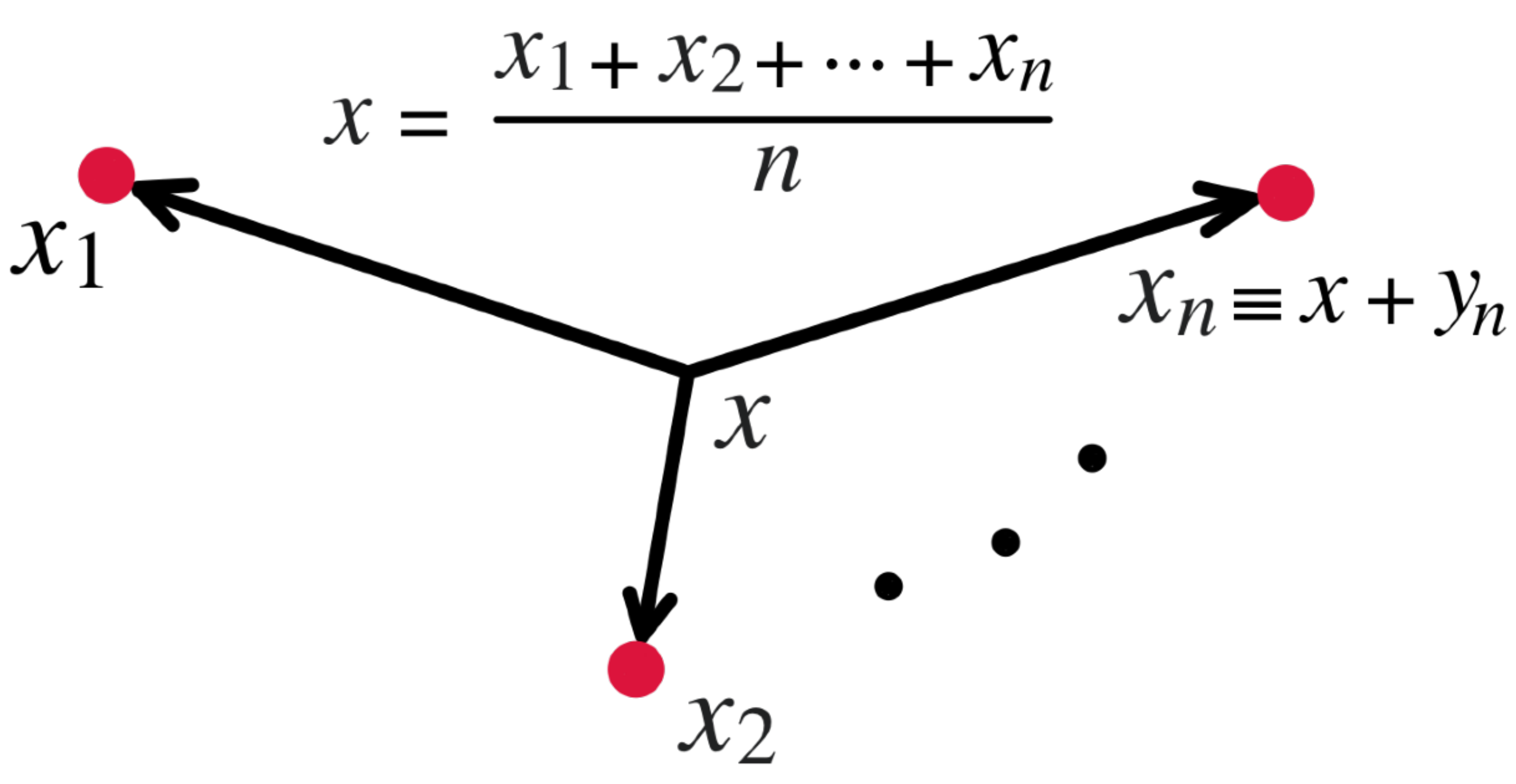
- First few truncated equations (diagrams): XA et al, 2009.10742, 2212.14029



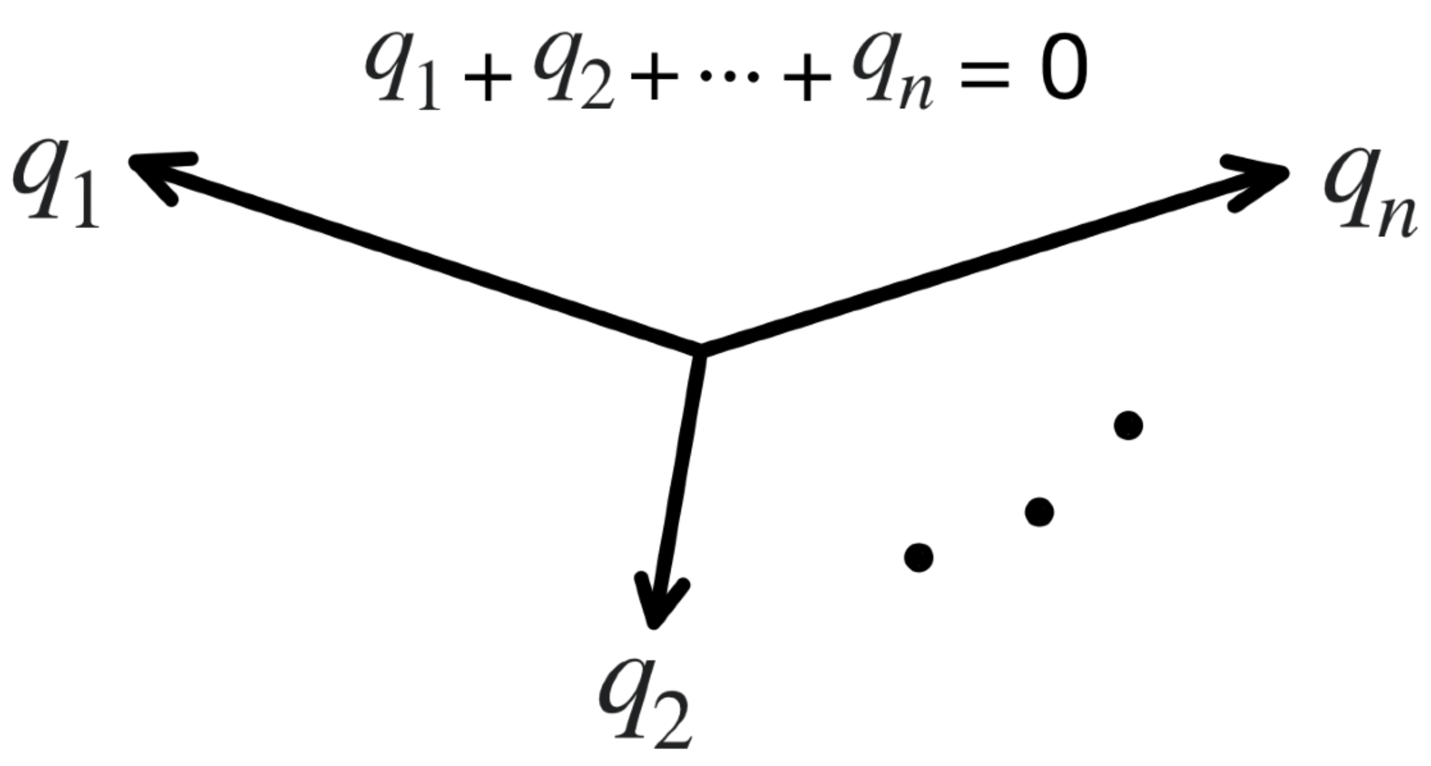
Multi-point Wigner function

- For fluctuation *fields*, we introduced the novel n-pt Wigner function XA et al, 2009.10742

$$W_n(x; q_1, \ldots, q_n) = \int d^3y_1 \ldots d^3y_n \, e^{-(iq_1y_1+\ldots+iq_ny_n)} \, \delta^{(3)}\left(\frac{y_1+\ldots+y_n}{n}\right) G_n(x; y_1, \ldots, y_n)$$



y-space



q-space

"While the bottom-up approach is useful in order to calculate two-point correlation functions, it is not immediately obvious how it should be generalized for the calculation of n-point correlation functions." Romatschke, 2019

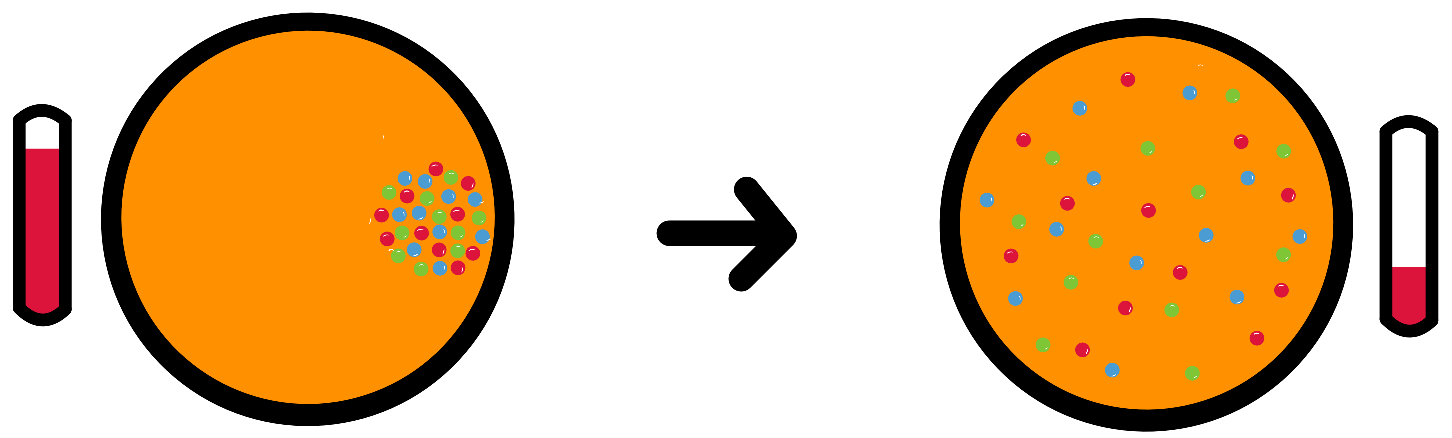
An example: charge diffusion near critical point

- Simple charge diffusion problem: XA et al, 2009.10742

$$\partial_t n = \nabla \lambda \nabla \alpha + \eta, \qquad \langle \eta(x)\eta(y)\rangle = 2\,\nabla^{(x)}\lambda\,\nabla^{(y)}\,\delta^{(3)}(x-y)$$

quantities	general	diffusive charge
variable	ψ_i	$n(\boldsymbol{x})$
variable index	i, j, k, etc.	$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, etc.
Onsager matrix	Q_{ij}	$\boldsymbol{\nabla}_{\boldsymbol{x}}\lambda\boldsymbol{\nabla}_{\boldsymbol{y}}\,\delta^{(3)}_{\boldsymbol{xy}}$
drift force	F_i	$\boldsymbol{\nabla}_{\boldsymbol{x}}\lambda\boldsymbol{\nabla}_{\boldsymbol{x}}\alpha$

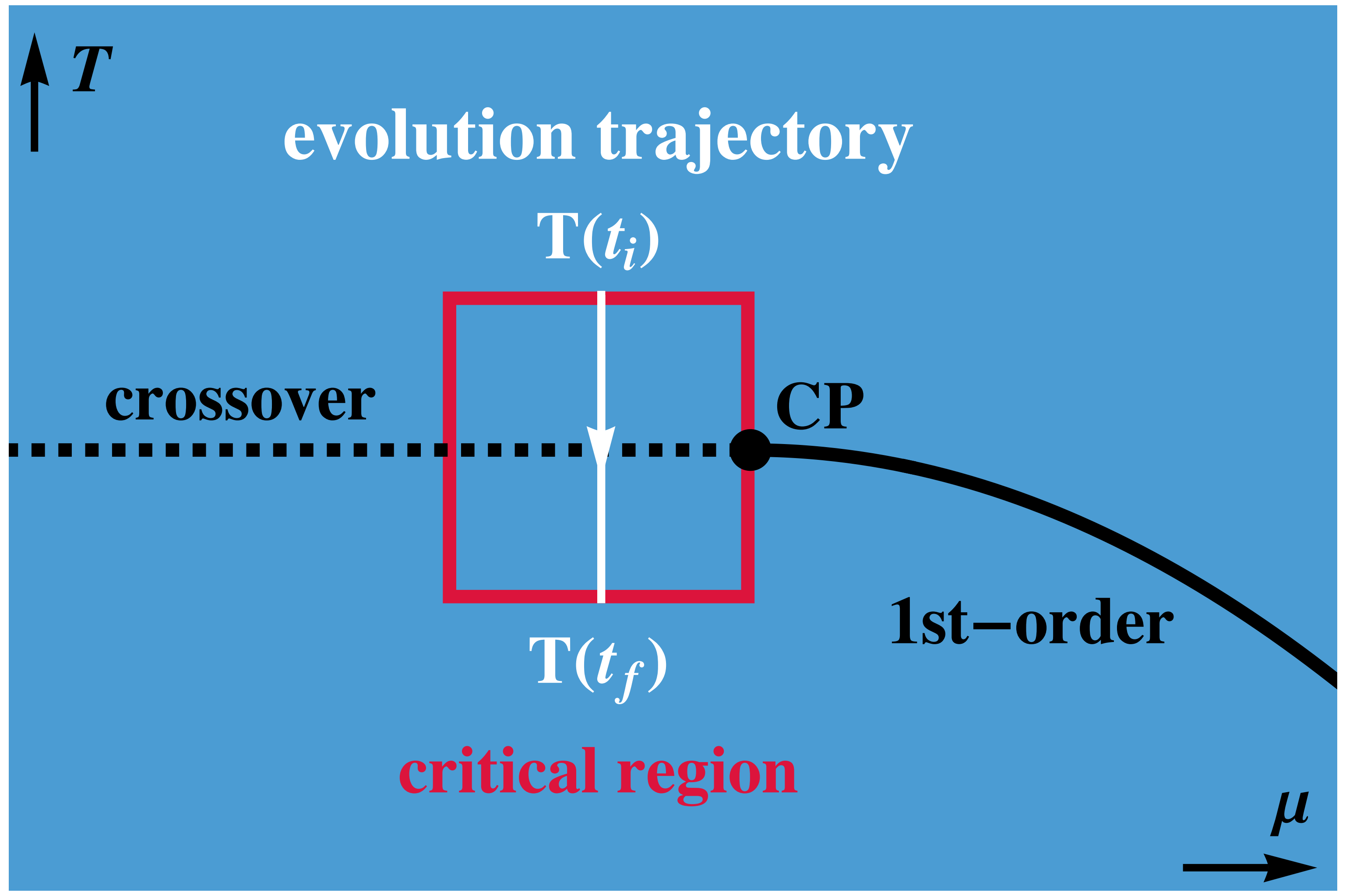
$n \equiv$ charge density; $\lambda \equiv$ conductivity; $\alpha \equiv$ chemical potential

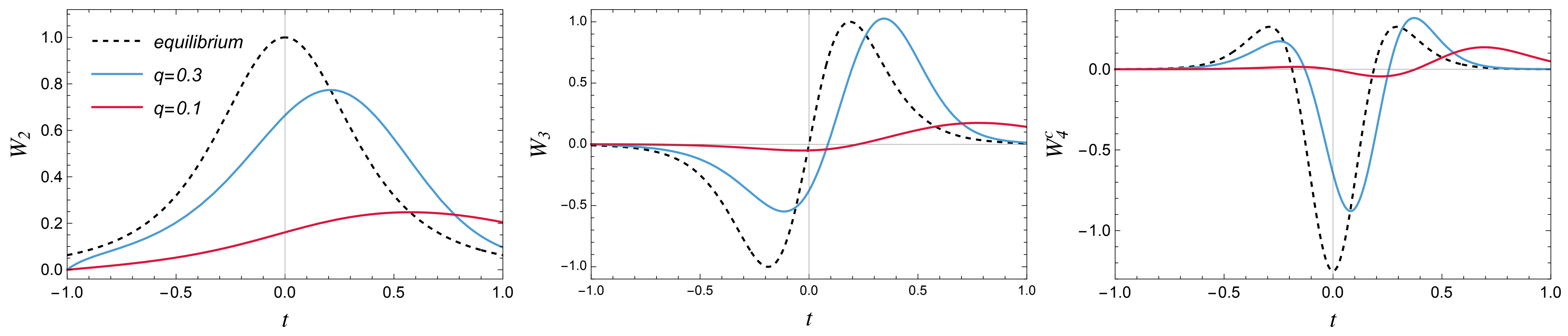


An example: charge diffusion near critical point

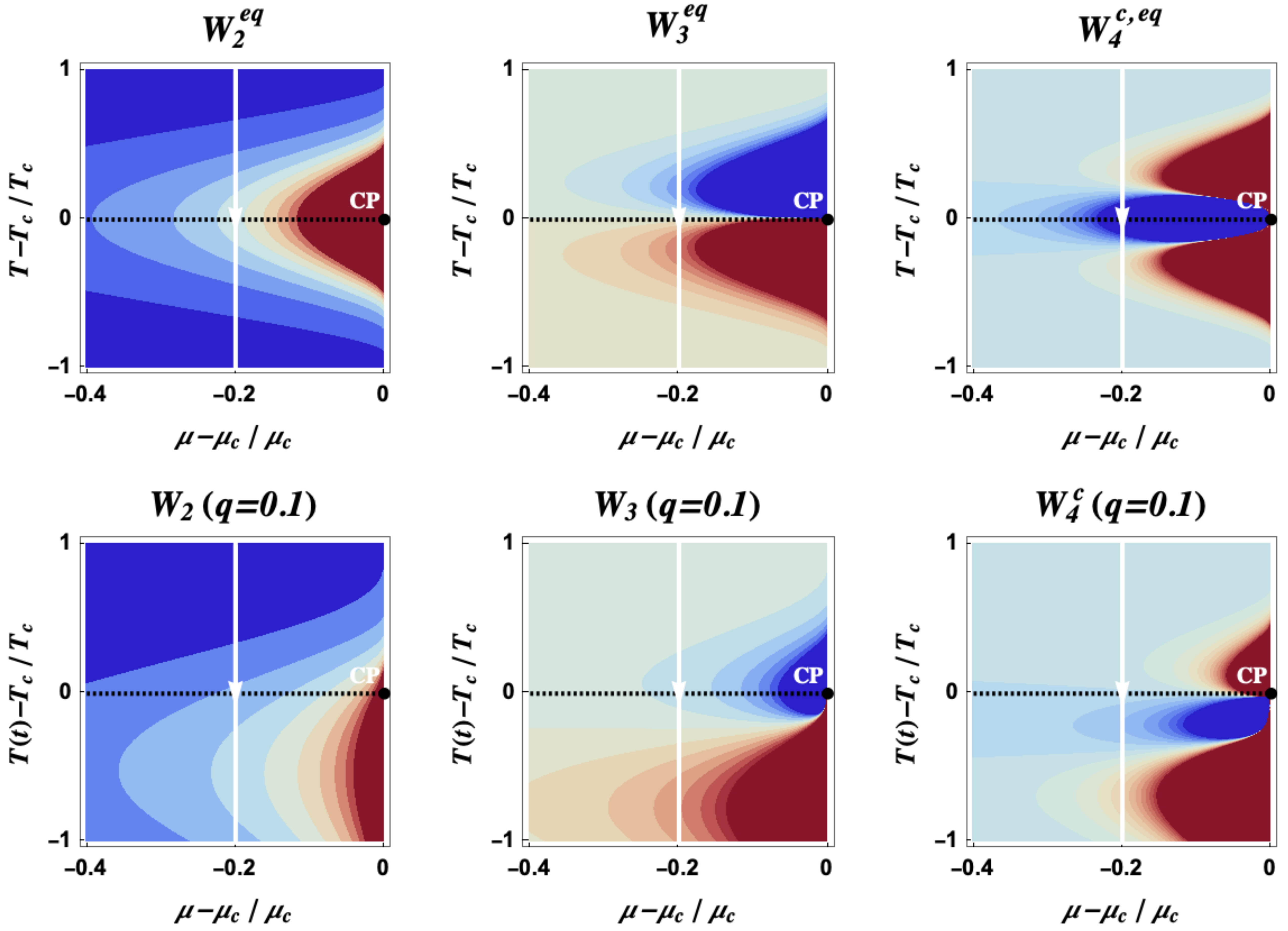
- Charge diffusion near QCD critical point: strong memory effect

XA et al, 2009.10742, 2209.15005





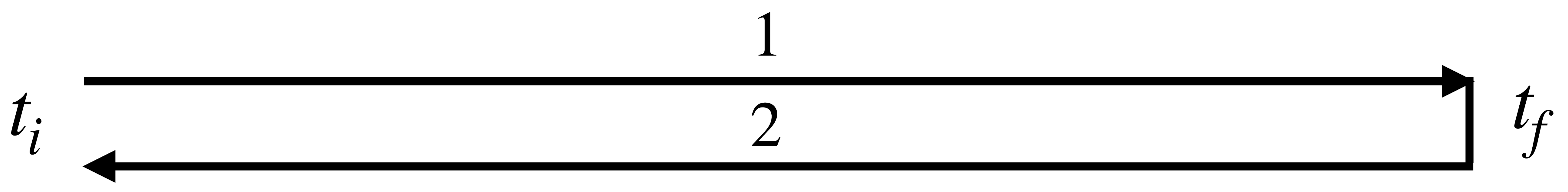
Evolution of correlators along a typical (white) trajectory



for recent numerical implementation with freeze-out procedure, see Pradeep et al, 2204.00639, 2211.09142

Connection to top-down approach

- Schwinger-Keldysh formalism



Schwinger Keldysh

$$Z = \int \mathscr{D}\psi_1 \mathscr{D}\psi_2 \mathscr{D}\chi_1 \mathscr{D}\chi_2 \, e^{i I_0(\psi_1, \chi_1) - i I_0(\psi_2, \chi_2)} = \int \mathscr{D}\psi_1 \mathscr{D}\psi_2 \, e^{i \int_\tau \mathscr{L}_{\text{EFT}}}$$

- The effective Lagrangian is constructed following *fundamental symmetries*:

Glorioso et al, 1805.09331; Jain et al, 2009.01356

$$\mathscr{L}_{\text{EFT}}(\psi_r, \psi_a) = \psi_{ai} Q_{ij}^{-1}(F_j - \dot{\psi}_{rj}) + i \psi_{ai} Q_{ij}^{-1} \psi_{aj} \quad \text{where} \quad \psi_r = \frac{1}{2}(\psi_1 + \psi_2), \quad \psi_a = \psi_1 - \psi_2$$

$$P[\psi] = \int_{\psi_r = \psi(t)} \mathscr{D}\psi_r \mathscr{D}\psi_a \, J(\psi_r) \, e^{i \int_{-\infty}^{t} d\tau \mathscr{L}_{\text{EFT}}} \longrightarrow \partial_t P = (-F_i P + (Q_{ij} P)_{,j})_{,i}$$

XA et al, in progress

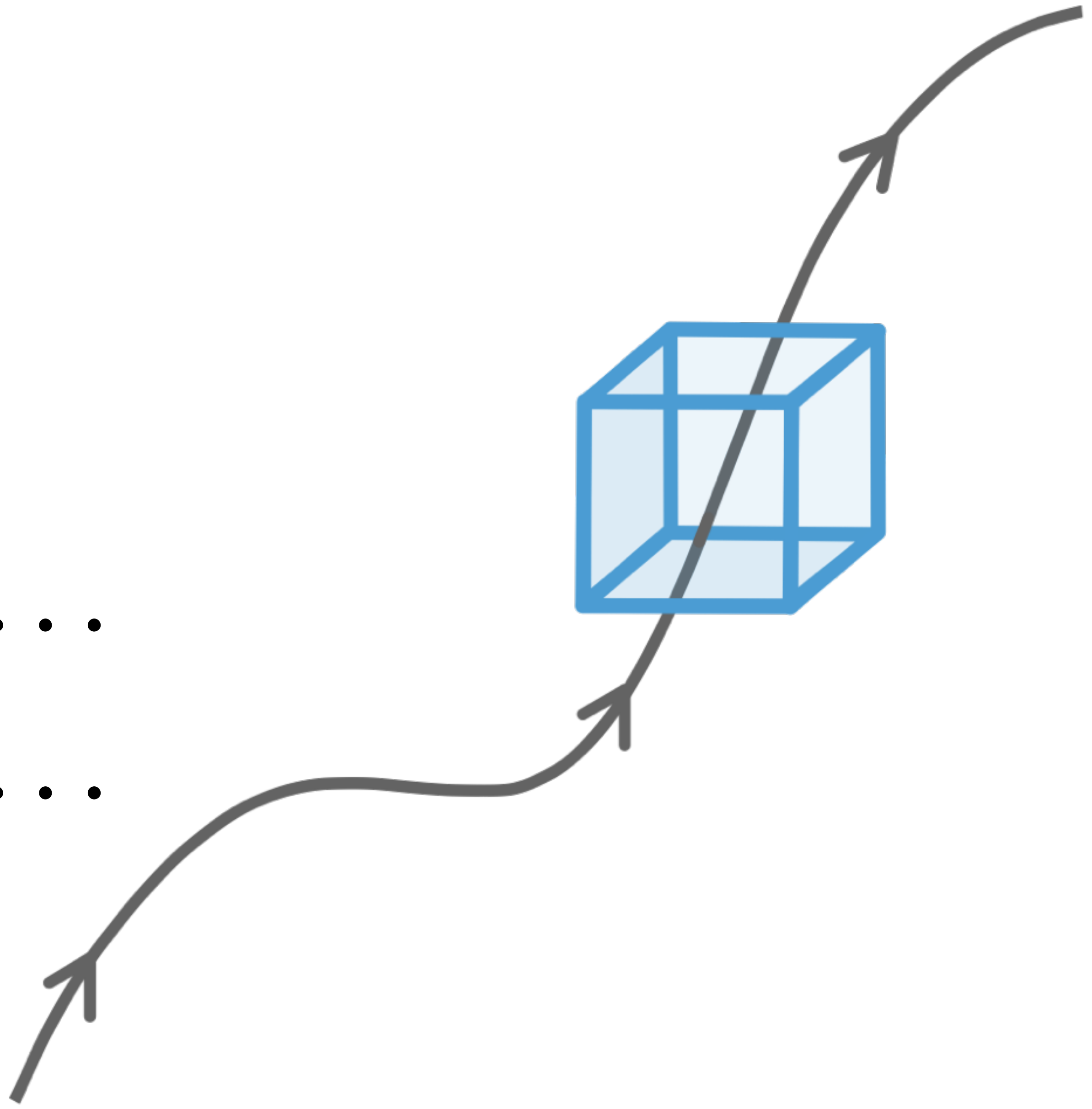
Fluctuation dynamics in relativistic fluids

Relativistic dynamics

Eulerian specification

more often used in non-relativistic theory

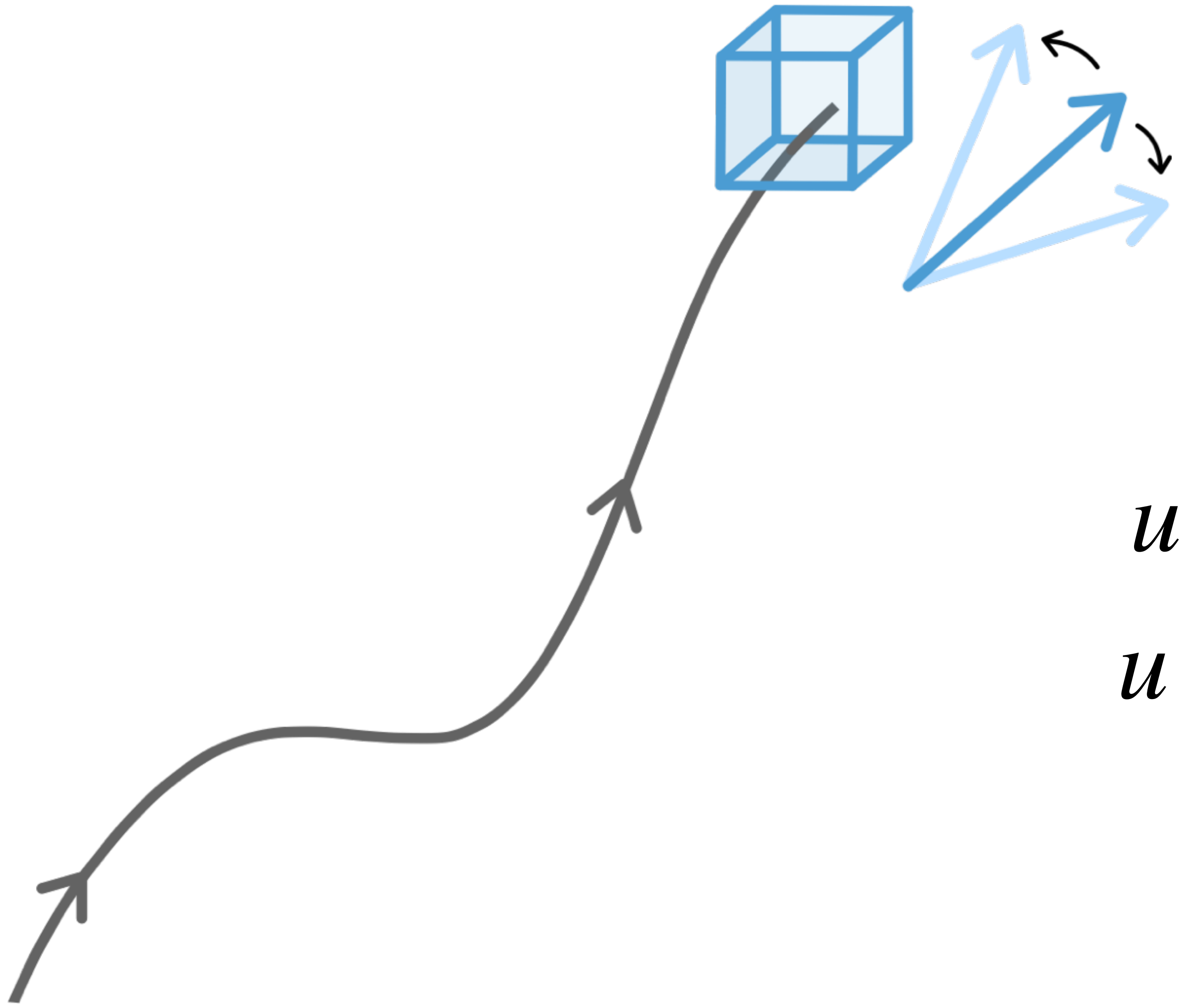
$$(\partial_t + v \cdot \nabla)\, \psi_i = \ldots$$
$$(\partial_t + v \cdot \nabla)\, G_n = \ldots$$



There is a global time for every observer. All correlators G_n can be measured at the same time in the same frame (lab).

Lagrangian specification

more convenient for relativistic theory



$$u = u(\psi)$$

$$u \cdot \partial\, \psi_i = \ldots$$
$$u \cdot \partial G_n = \ldots$$

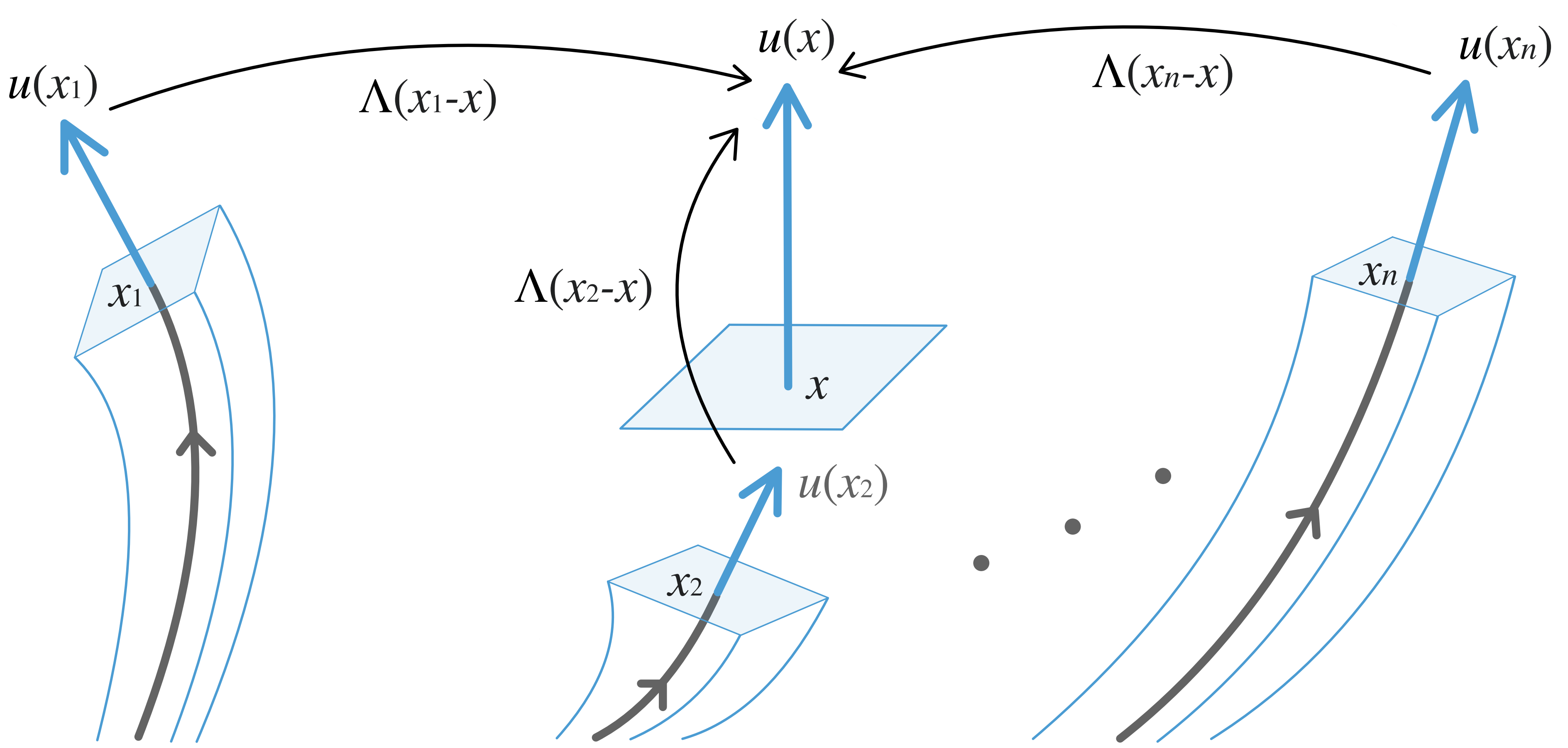
Each fluid cell has its own clock (proper time). How to define the analogous *equal-time* correlator G_n in relativistic theory?

Confluent formulation: correlator and derivative

- Confluent formulation: covariant description for the comoving fluctuations.

  See XA et al, 2212.14029 for more details

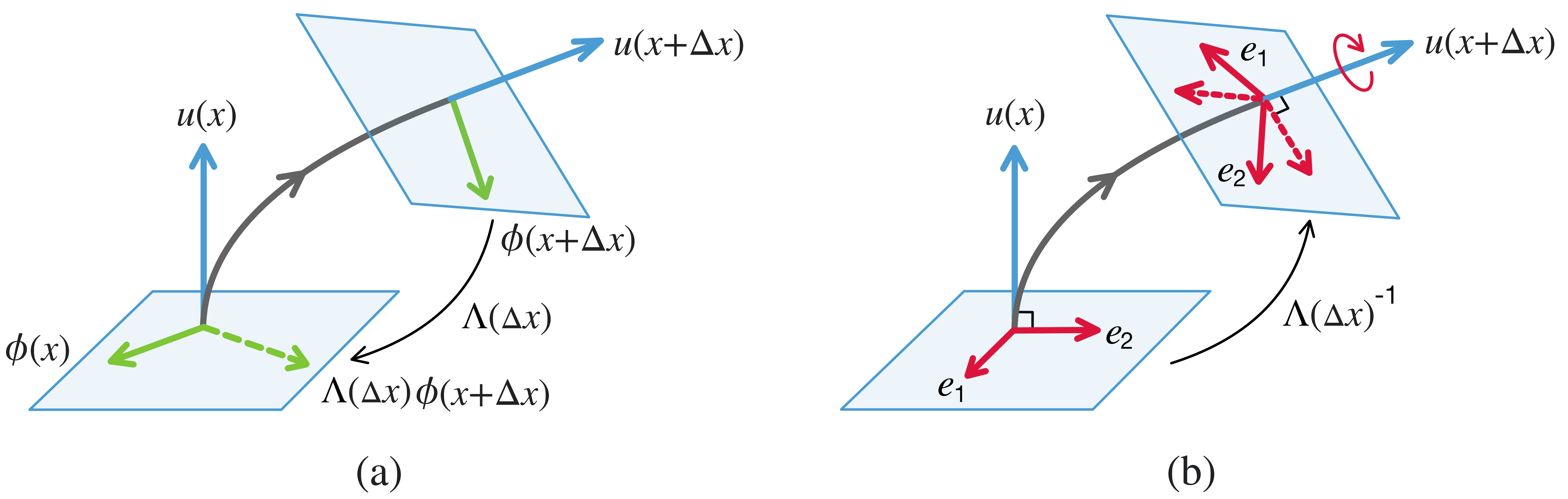
Confluent correlator $\bar{G}$



$$\bar{G}_{i_1\dots i_n} = \Lambda_{i_1}^{\ j_1}(x-x_1)\dots\Lambda_{i_n}^{\ j_n}(x-x_n)\bar{G}_{j_1\dots j_n}$$

boost all fields (measured at their own local rest frame) to one common frame (chosen at their midpoint)

Confluent derivative $\bar{\nabla}$



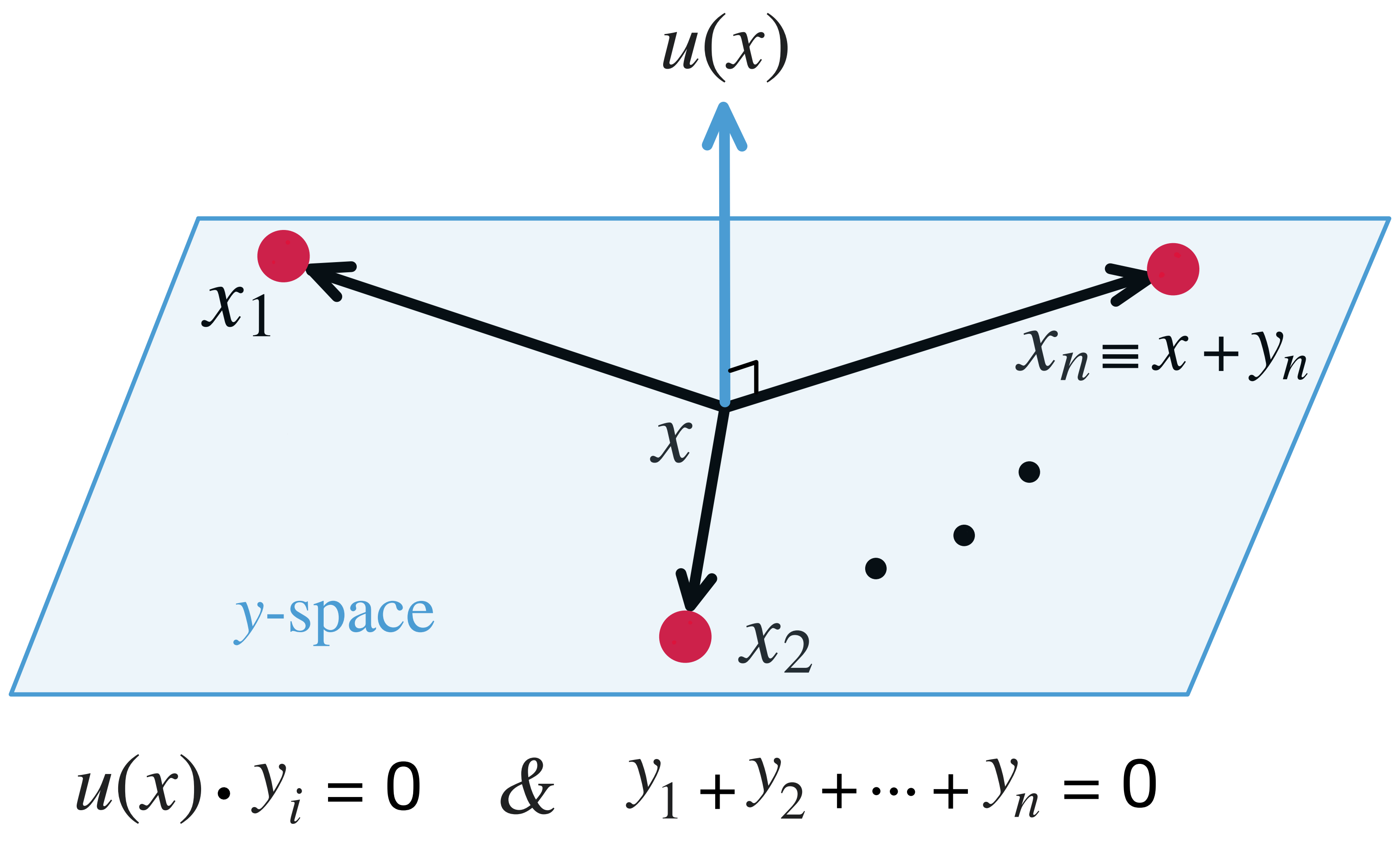
$$\bar{\nabla}_\mu \bar{G}_{i_1\dots i_n} = \partial_\mu \bar{G}_{i_1\dots i_n} - n\left(\mathring{\omega}^a_{\mu b} y_1^b \partial_a^{(y_1)} \bar{G}_{i_1\dots i_n} + \bar{\omega}^{j_1}_{\mu i_1} \bar{G}_{j_1\dots i_n}\right)_{\text{perm.}}$$

the frame at midpoint moves accordingly as the n points move, the difference of a given field before and after the movement is calculated in one same frame, with the equal-time constraint preserved by introducing the local triad e_a^μ with $a = 1,2,3$

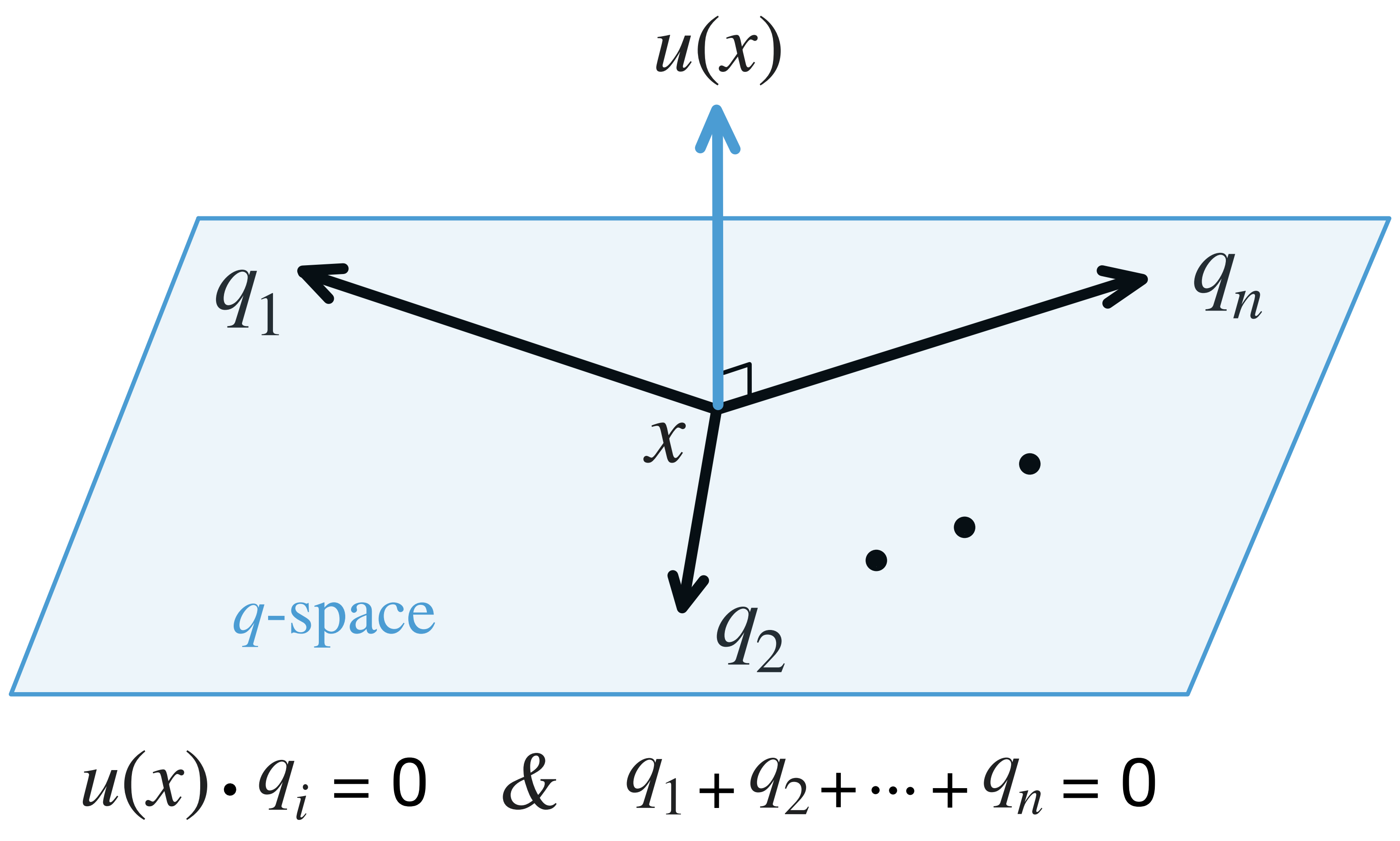
Confluent formulation: Wigner function

- The confluent n-pt Wigner transform from x-independent variable $y^a = e^a_\mu(x)\, y^\mu$ to q^a with $a = 1,2,3$. XA et al, 2212.14029

$$W_n(x; q_1^a, \ldots, q_n^a) = \int \prod_{i=1}^{n} \left(d^3 y_i^a\, e^{-i q_{ia} y_i^a}\right) \delta^{(3)}\left(\frac{1}{n}\sum_{i=1}^{n} y_i^a\right) \bar{G}_n(x + e_a y_1^a, \ldots, x + e_a y_n^a)$$



(a)



(b)

Confluent fluctuation evolution equations

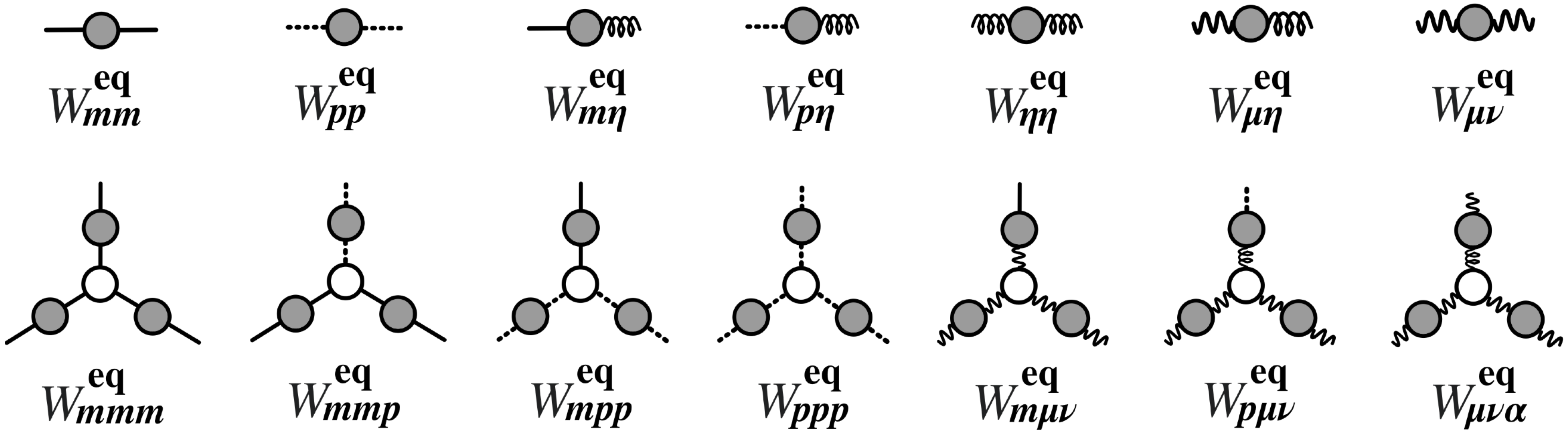
- Fluctuation evolution equations in the *impressionistic* form: XA et al, in progress

$$\mathscr{L} W_n = i c_s q (W_n - \ldots) - \gamma q^2 (W_n - \ldots) - \partial\psi W_n + \ldots \quad \text{where} \quad \mathscr{L} = u \cdot \bar{\nabla}_x + f \cdot \nabla_q$$

sound — dissipation — background gradient

of which the solutions match thermodynamics with entropy $S(m, p, u_\mu, \eta)$.

m: entropy per baryon; p: pressure; η: Lagrange multiplier for $u^2 = -1$.



Equilibrium solutions in diagrammatic representation

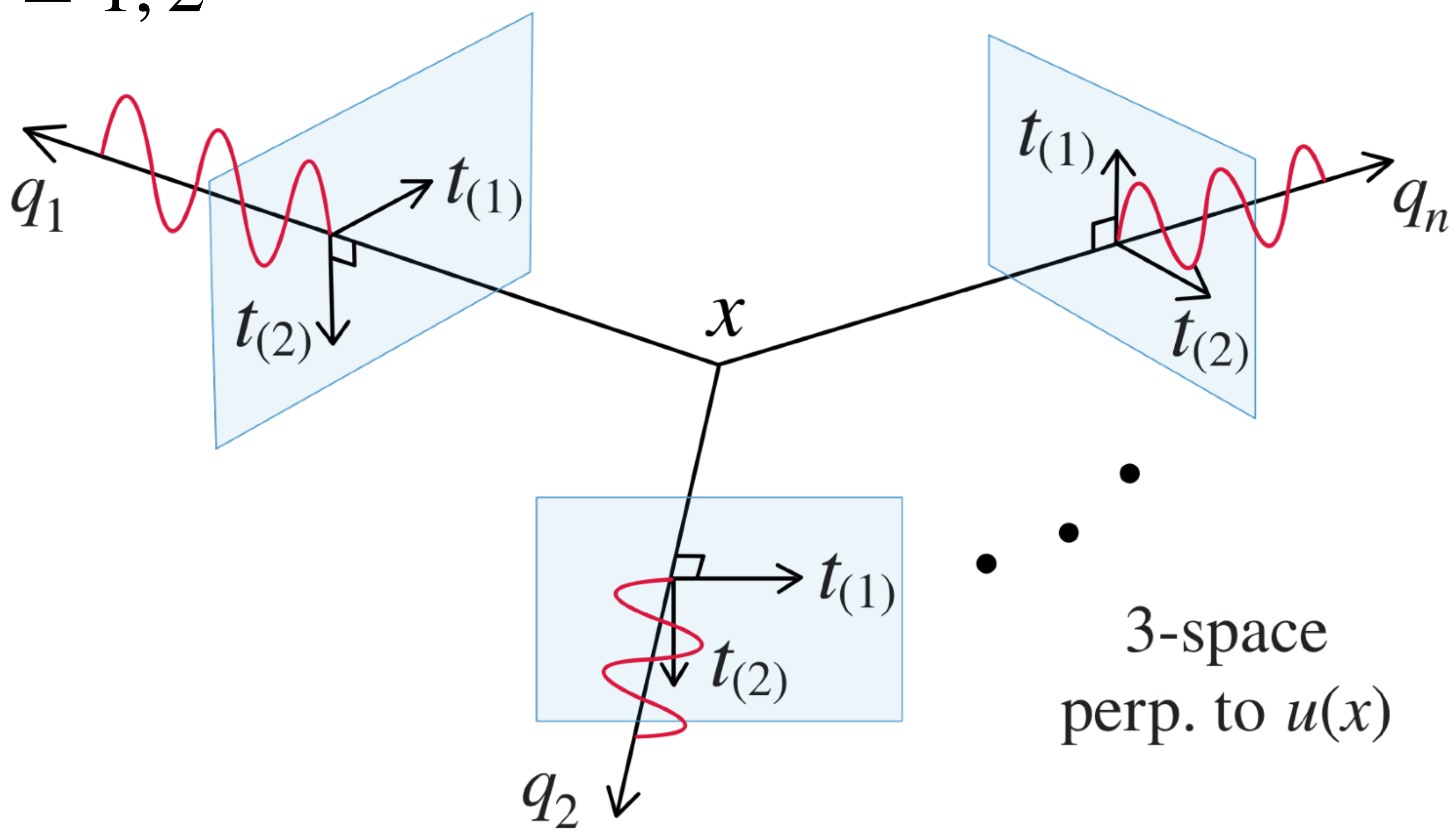
For $\phi = (\delta m, \delta p, \delta u_\mu)$, there are 21+56+126=**203** equations (for the 2-pt, 3-pt and 4-pt correlators) to solve——bite off more than one can chew!

Rotating phase approximation

- Step 1: choose a set of new bases in Fock space s.t. the ideal hydrodynamic equations are diagonalized with eigenvalues $\lambda_\pm(q) = \pm c_s \; q \;, \lambda_m(q) = \lambda_{(i)}(q) = 0.$

$$\phi = \begin{pmatrix} \phi_m \\ \phi_p \\ \phi_\mu \end{pmatrix} = \begin{pmatrix} \delta m \\ \delta p \\ \delta u_\mu \end{pmatrix} \longrightarrow \Phi = \begin{pmatrix} \Phi_m \\ \Phi_\pm \\ \Phi_{(i)} \end{pmatrix} \sim \begin{pmatrix} \delta m \\ \delta p \pm c_s w \hat{q} \cdot \delta u \\ t_{(i)} \cdot \delta u \end{pmatrix} \quad i = 1, 2$$

NB: n-pt correlators are analogous to n-particle quantum states lying in the Fock space.



- Step 2: for n-pt correlators $W_{\Phi_1 \ldots \Phi_n}(q_1, \ldots, q_n)$,

$$\text{if } \sum_{i=1}^{n} \lambda_{\Phi_i}(q_i) \begin{cases} = 0 & \longrightarrow \quad \text{slow mode (kept)} \\ \neq 0 & \longrightarrow \quad \text{fast mode (averaged out)} \end{cases}$$

E.g., $W_{+-}(q_1, q_2)$ is a slow mode since $\lambda_+(q_1) + \lambda_-(q_2) = c_s(\; q_1 \; - \; q_2 \;) = 0$;

$W_{+++}(q_1, q_2, q_3)$ is *not* a slow mode since $\lambda_+(q_1) + \lambda_+(q_2) + \lambda_+(q_3) = c_s(\; q_1 \; + \; q_2 \; + \; q_3 \;) \neq 0$.

As a result, we end up with 7+10+15=**32** equations to solve.

E.g., the 7 independent 2-pt slow modes are $W_{mm}, W_{m(i)}, W_{(i)(j)}, W_{+-}$.

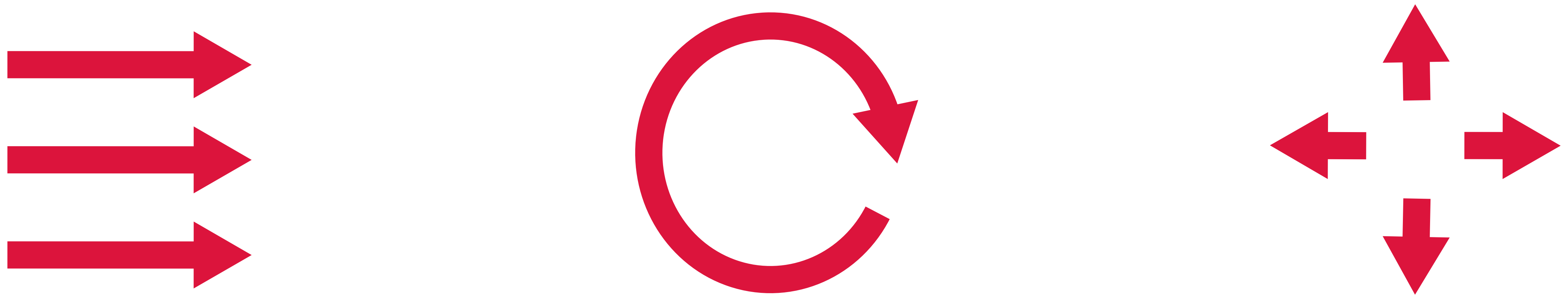
Hydro-kinetic equations

- The equation for W_{+-} has a kinetic interpretation:

$$\mathscr{L}W_{+-} \equiv \left((u + c_s\hat{q}) \cdot \nabla_x + f \cdot \nabla_q \right) W_{+-} = -\gamma q^2 \left(W_{+-} - \frac{T}{E} \right)$$

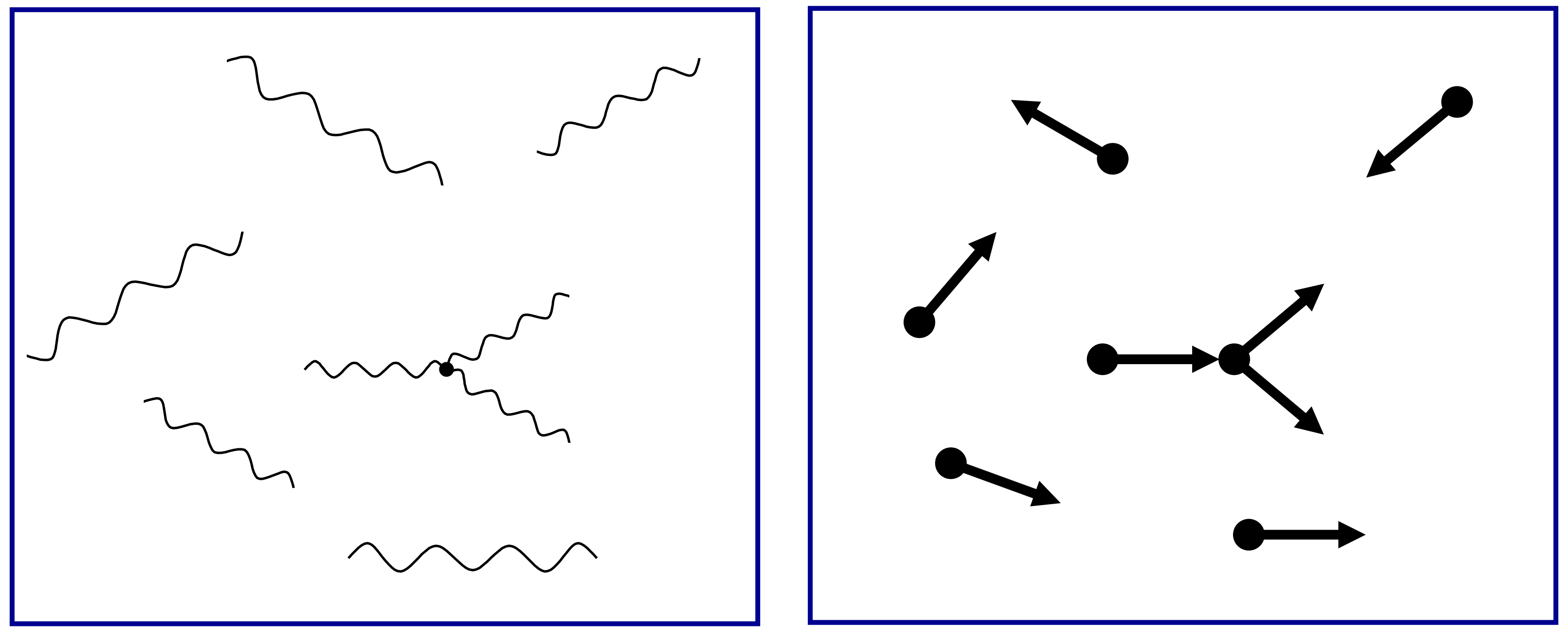
$$f_\mu = Ea_\mu + 2Ec_s\hat{q}^\nu\omega_{\nu\mu} + q_\nu \nabla^\perp_\mu u^\nu + \nabla_\mu E$$

Bose-Einstein (phonon) distribution $n = \dfrac{1}{e^{E/T} - 1}$ at high T



Inertial Coriolis "Hubble"

Phonons move on top of an arbitrary fluid with acceleration, rotation and expansion XA et al, 1902.09517



from Schafer

Fluctuation feedback

- Fluctuations give feedback to the bare quantities order by order in gradient expansion:

$$T_{\mu\nu}^{\text{physical}} = \underbrace{T_{\mu\nu}^{(0)} + T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(2)} + \ldots}_{\text{bare}} + \underbrace{\delta T_{\mu\nu}(\{G_n\})}_{\text{fluctuation}}$$

$$= \underbrace{T_{\mu\nu}^{R(0)} + T_{\mu\nu}^{R(1)} + T_{\mu\nu}^{R(2)}}_{\text{renormalized}} + \underbrace{\widetilde{T}_{\mu\nu}^{(3/2)} + \widetilde{T}_{\mu\nu}^{(3)} + \widetilde{T}_{\mu\nu}^{(9/2)} + \ldots}_{\text{long-time tails}}$$

where $G_n(x) \sim \int d^3q_1 \ldots d^3q_n \delta^{(3)}(q_1 + \ldots + q_n) W_n(x, q_1, \ldots, q_n)$

↑ need to find the solutions from equations for Wigner functions

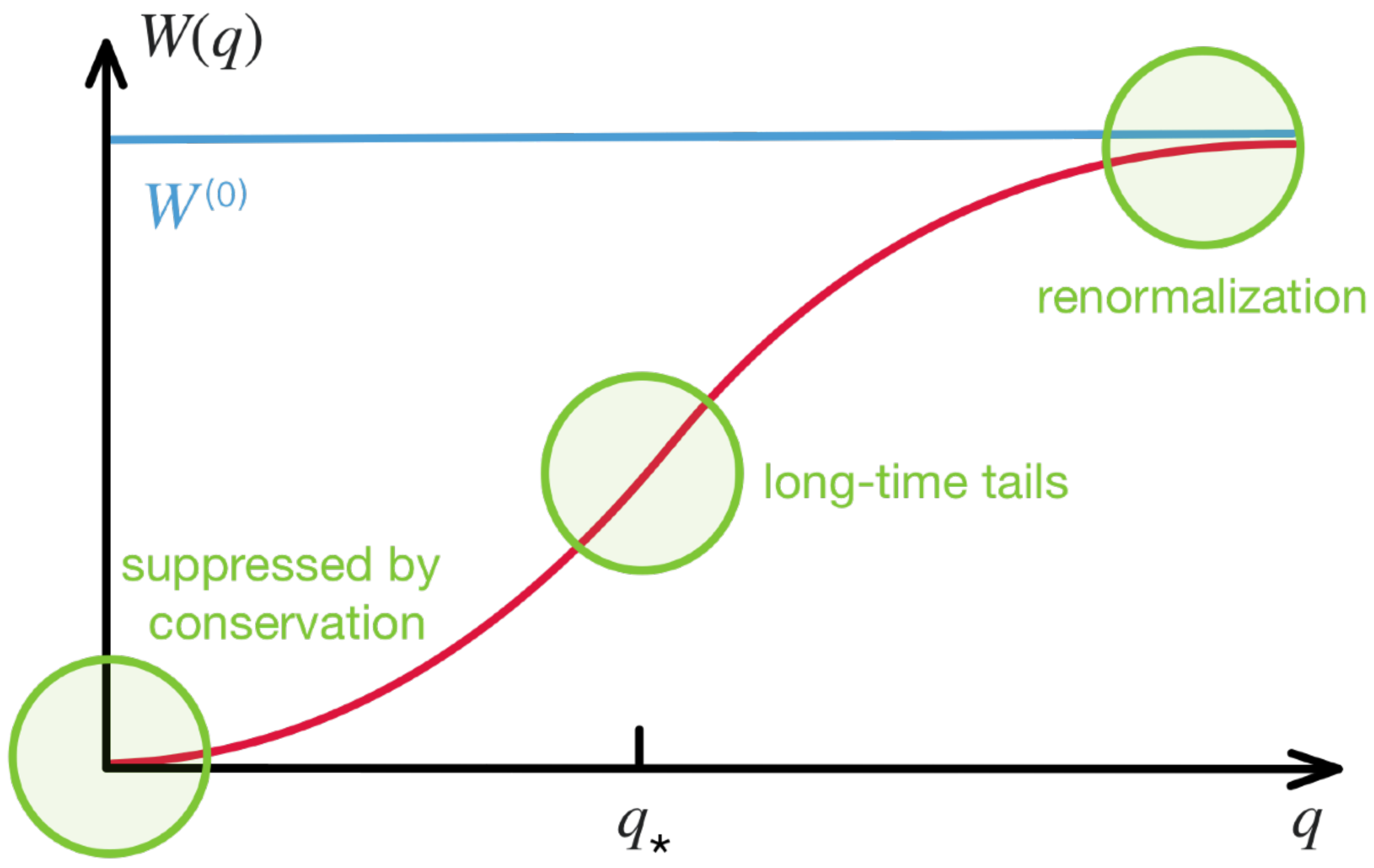
Renormalization

- Equation for 2-pt functions under RPA:

$$\mathscr{L}W(q) = -\gamma q^2(W(q) - W^{(0)}) - \partial\psi W(q)$$

with asymptotic solutions

$$W(q) = \frac{\gamma q^2 W^{(0)}}{-i\omega + \gamma q^2 + \partial\psi} = \begin{cases} W^{(0)}\left(1 - \dfrac{-i\omega + \partial\psi}{\gamma q^2} + \dots\right), & \gamma q^2 \gg \omega, \partial\psi \\ W^{(0)}\dfrac{\gamma q^2}{-i\omega + \partial\psi}\left(1 - \dfrac{\gamma q^2}{-i\omega + \partial\psi} + \dots\right), & \gamma q^2 \ll \omega, \partial\psi \end{cases}$$



- Perturbation analysis for $W = W^{(0)} + W^{(\text{neq})}$ where $W^{(\text{neq})} = W^{(1)} + \dots$ gives:

$$W^{(1)} \sim \frac{\partial\psi}{\gamma q^2} \implies G^{(1)} = \int_q W^{(1)} \sim \frac{\Lambda}{\gamma}\partial\psi \longrightarrow$$ renormalize transport coefficients

(regularize infinite noise analytically)

E.g.,

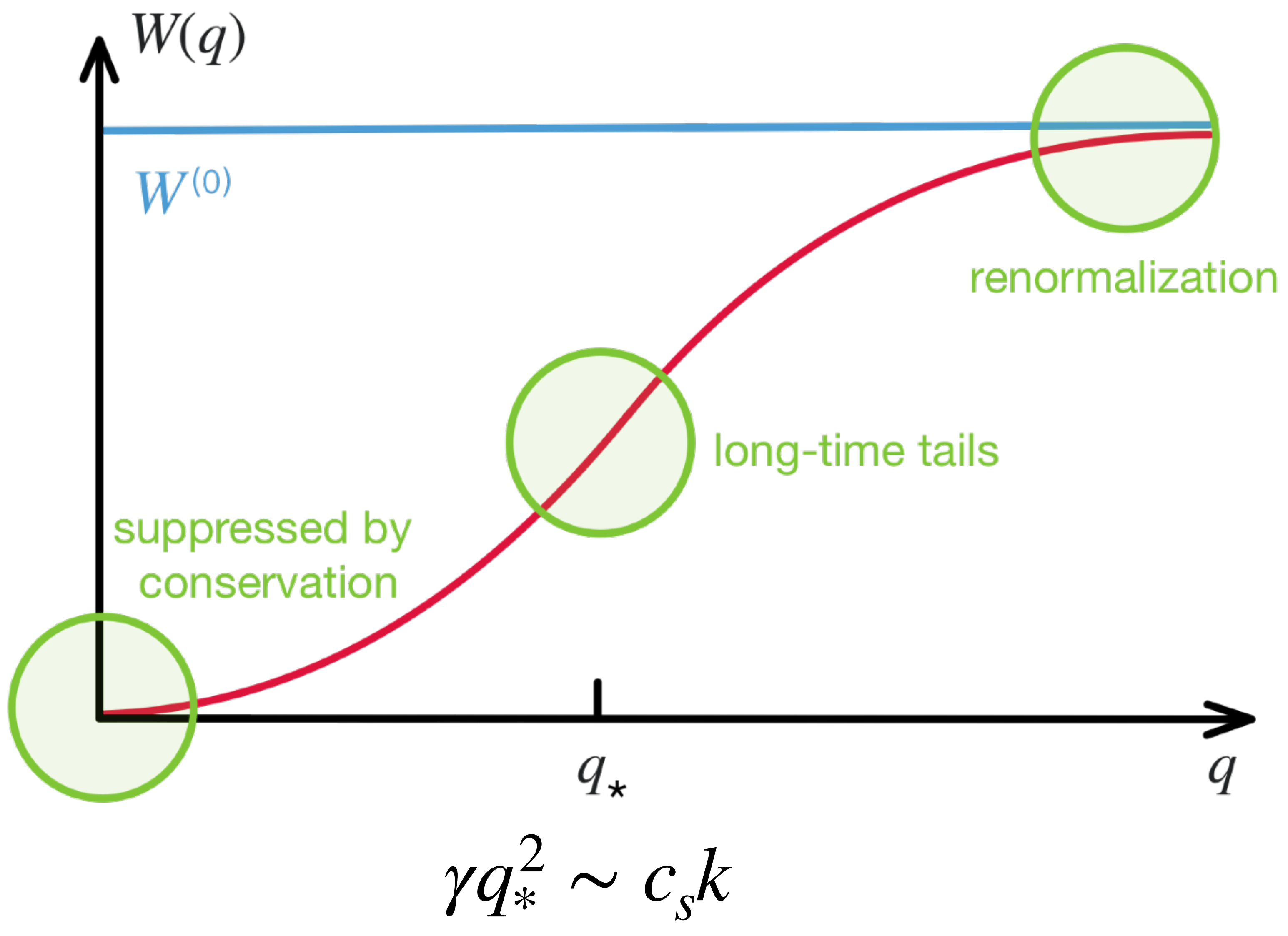
$$\eta_R = \eta + \frac{T\Lambda}{30\pi^2}\left(\frac{1}{\gamma_L} + \frac{7}{2\gamma_\eta}\right), \quad \zeta_R = \zeta + \frac{T\Lambda}{18\pi^2}\left(\frac{1}{\gamma_L}(1 - 3\dot{T} + 3\dot{c}_s)^2 + \frac{2}{\gamma_\eta}(1 - 3(\dot{T} + c_s^2)/2)^2 + \frac{9}{4\gamma_\lambda}(1 - \dot{c}_p)^2\right), \quad \lambda_R = \lambda + \frac{T^2 n^2 \Lambda}{3\pi^2 w^2}\left(\frac{c_p T}{(\gamma_\eta + \gamma_\lambda)w} + \frac{c_s^2}{2\gamma_L}\right)$$

Long-time tails

- The remaining non-equilibrium part of 2-pt function:

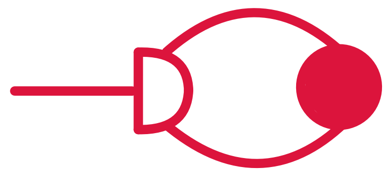
$$\widetilde{W} = W^{(\mathrm{neq})} - W^{(1)} \sim \underbrace{\frac{\partial\psi}{-i\omega + \gamma q^2 + \partial\psi} - \frac{\partial\psi}{\gamma q^2}}_{\text{subtracting local divergence}}$$

$$\Longrightarrow \quad \widetilde{G} = \int_q \widetilde{W} \sim \frac{\partial\psi}{\gamma^{3/2}}(i\omega + \partial\psi)^{1/2} \sim q_*^3 \sim k^{3/2}$$



- Generically, for arbitrary n,

$$\widetilde{G}_n(x) = \int \underbrace{d^3q_1 \dots d^3q_n \delta^{(3)}(q_1 + \dots + q_n)}_{n-1 \text{ independent } q \text{ integration}} \widetilde{W}_n(x, q_1, \dots, q_n) \sim \epsilon^{n-1} \sim q_*^{3(n-1)} \sim k^{3(n-1)/2}$$

the leading contribution ($k^{3/2} \sim t^{-3/2}$) results from 2-pt correlators via 

E.g., $\Pi(\omega) = \eta(\omega)\partial \cdot u \sim \xi^3 \left(1 - (\omega\xi^3)^{1/2}\right) \partial \cdot u$

Interplay with background in the critical regime

- Different slow modes may relax with different time scales near critical point due to *critical slowing down*. Stephanov, 1104.1627; Berdnikov et al, 9912274; XA, 2003.02828

- Hydro+/++: hydrodynamics with parametrically slow modes (e.g., $\Gamma(q) \sim \xi^{-3} \ll \omega$)

$$\begin{cases} \partial_\mu T^{\mu\nu}_{\text{physical}}(\psi_R, \widetilde{W}) = 0 \\ \mathscr{L}\widetilde{W}(q) = -\Gamma(q)\widetilde{W}(q) - \partial\psi_R \widetilde{W}(q) \end{cases}$$

- In the critical regime ($\Gamma_\Pi \sim \xi^{-3}$), Muller-Israel-Stewart theory is an example of the single-mode Hydro+, e.g., Stephanov et al, 1712.10305; Abbasi et al, 2112.14747

$$\begin{cases} \partial_\mu T^{\mu\nu}(\psi, \Pi) = 0 \\ \dot{\Pi} = -\Gamma_\Pi(\Pi - \Pi_{\text{NS}}) \end{cases}$$

Conclusion

Recap

- Various approaches for fluctuating hydro have been developed, each with its own pros and cons, and can be connected with others.
- For the first time we developed a covariant framework for fluctuation dynamics incorporating non-Gaussian hydrodynamic fluctuations.

Outlook

- Need efforts to simulate the fluctuation equations with background.
- Need freeze-out prescription for the connection to observables.
- More…