Non-linear Response in real-time Holography

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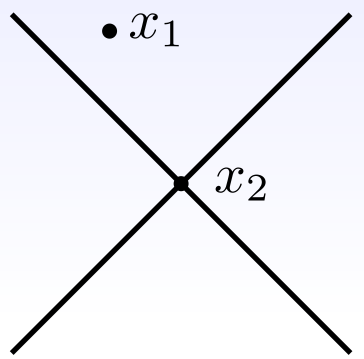
Introduction

- Interested in retarded response of strongly coupled QFTs at finite temperature.

- Simplest (linear) case: 2-point function.



$$G_{ra}(x_1, x_2) = -i\,\theta(x_1^0 - x_2^0)\langle[\mathcal{O}(x_1), \mathcal{O}(x_2)]\rangle$$

- Crucially depends on time-ordering. Need to go beyond standard holography: real-time holography.

- Standard holography is well-suited to Euclidean problems/ problems that do not involve a state/initial data.

$$Z_{QFT}[J] = \exp(-S_{grav}[J])$$

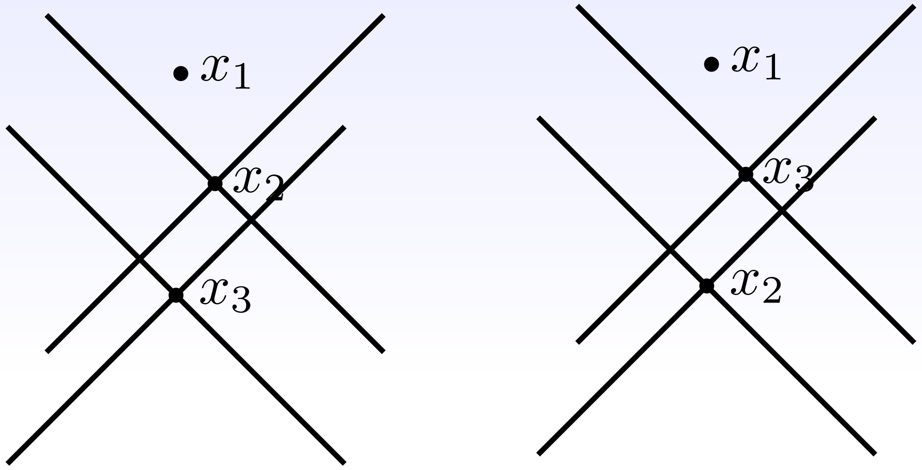
- However, 2-point functions extensively studied in the literature.

$$G_{ra}(x_1, x_2) = -i\,\theta(x_1^0 - x_2^0)\langle[\mathcal{O}(x_1), \mathcal{O}(x_2)]\rangle$$

- Proposed ingoing boundary conditions by [Son, Starinets '02] [Herzog, Son '02]. Found simple poles at QNM frequencies.
- Unsatisfactory: interpretation from a boundary perspective?
- Ingoing condition can be derived explicitly from a real-time holographic perspective [Skenderis, van Rees '08][van Rees '09]

- Simplest case involving non-linearities: 3-point function.

$$G_{raa}(x_1, x_2, x_3) = -\theta(x_1^0 - x_2^0)\theta(x_2^0 - x_3^0)\langle[[\mathcal{O}(x_1), \mathcal{O}(x_2)], \mathcal{O}(x_3)]\rangle + (2 \leftrightarrow 3)$$



- Non-linear problem in the bulk: consider interactions in bulk.

- **Today:** compute retarded scalar 3-point function in the real-time holography a la [Skenderis, van Rees '08], <u>without assuming ingoing boundary conditions at the horizon.</u>

Motivations…

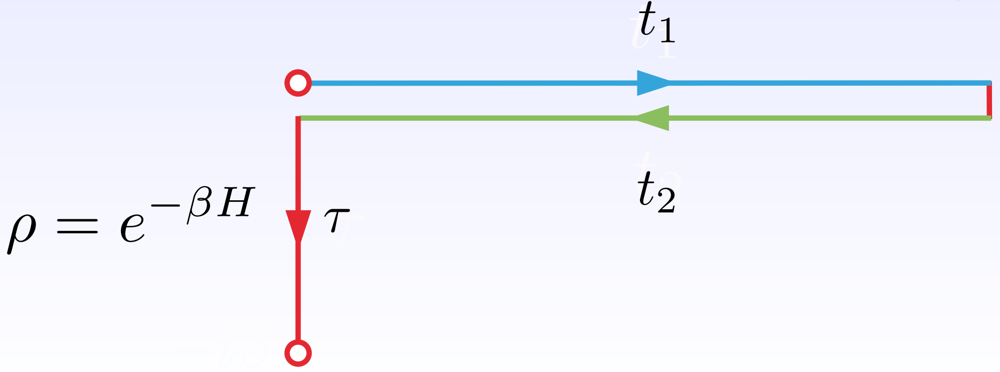
- Extend real-time holography.
- Is there a simple ingoing computation for higher point correlation functions?
- QNMs as poles of G_{ra}. Does this universality of BH ringdown extend to higher points?

Overarching line: non-linearities in black holes in AdS as well as in flat spacetime, and their implications.

1. Real-time Holography

- We need a QFT generating function for computing the correlators of interest.

- Consider a QFT on a closed Schwinger-Keldysh contour.



- Label operators/sources by position on the contour.

$$\mathcal{O}_1(x), \mathcal{O}_2(x) \quad \rightarrow \quad J_1(x), J_2(x)$$

- Switch to a convenient basis: a/r basis

$$J_a(x) = J_1(x) - J_2(x), \qquad J_r(x) = \frac{1}{2}\left(J_1(x) + J_2(x)\right) \ ,$$

$$\mathcal{O}_a(x) = \mathcal{O}_1(x) - \mathcal{O}_2(x), \qquad \mathcal{O}_r(x) = \frac{1}{2}\left(\mathcal{O}_1(x) + \mathcal{O}_2(x)\right) \ .$$

- QFT path integral on this complex-time contour generates contour-ordered correlation function.

$$G_{\alpha_1\ldots\alpha_n}(x_1, x_2 \ldots, x_n) = 2^{n_r-1} i \frac{\delta^n Z}{\delta J_{\bar{\alpha}_1} \ldots \delta J_{\bar{\alpha}_n}}\Bigg|_{J_r=J_a=0},$$

$$Z[J_a(x), J_r(x)] = \left\langle T_p e^{-i\int (J_a(y)\mathcal{O}_r(y)+J_r(y)\mathcal{O}_a(y))} \right\rangle,$$

Contour ordering

- One can check that the contour ordering indeed gives

Symmetric $G_{rr}(x_1, x_2) = -i\,\langle\{O(x_1), \mathcal{O}(x_2)\}\rangle\,,$

Retarded $G_{ra}(x_1, x_2) = -i\,\theta(x_1^0 - x_2^0)\langle[\mathcal{O}(x_1), \mathcal{O}(x_2)]\rangle\,,$

Advanced $G_{ar}(x_1, x_2) = i\,\theta(x_2^0 - x_1^0)\langle[\mathcal{O}(x_1), \mathcal{O}(x_2)]\rangle\,,$

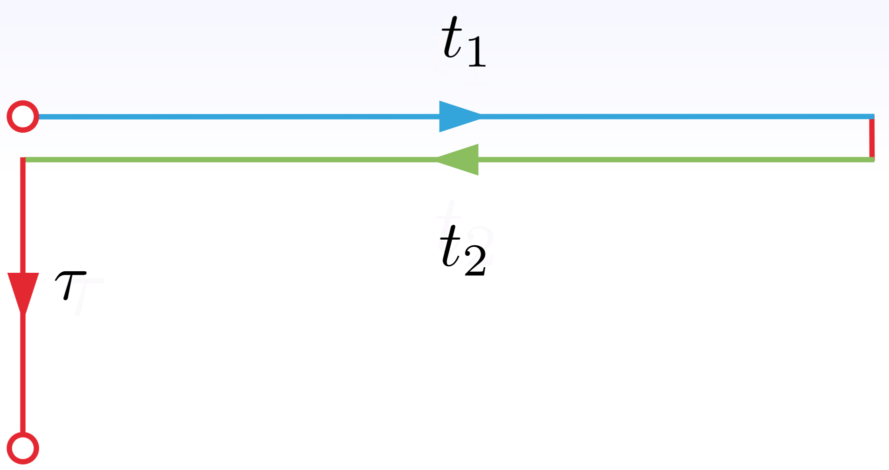
$G_{aa}(x_1, x_2) = 0\,,$

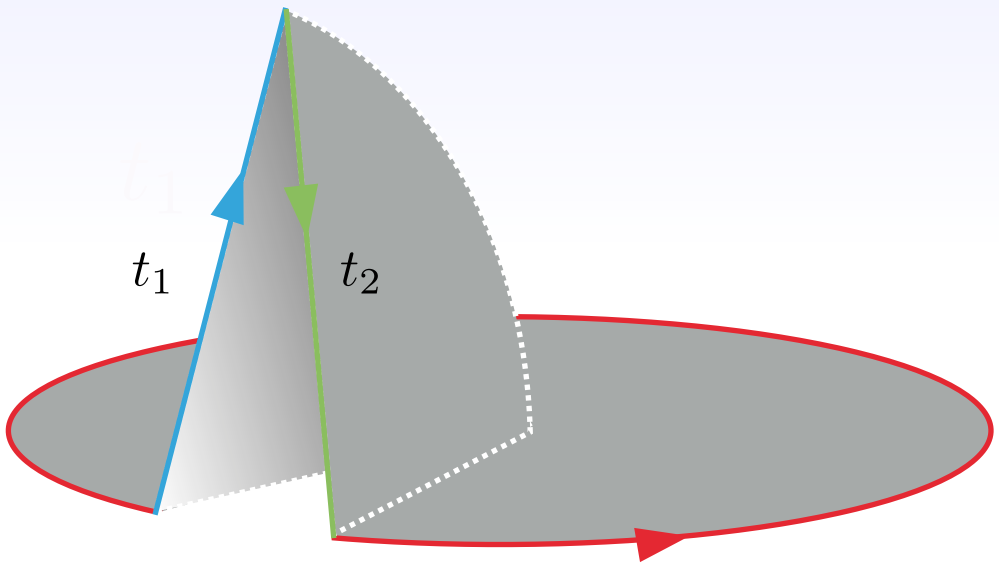
+ higher point functions…

(modulo Fluctuation-Dissipation theorems)

- Power of holography: gives $Z[J]$ from gravity.

- What is the bulk geometry dual to the complex SK contour?





- [Skenderis, van Rees '08]: Euclidean cigar, joint with two Lorentzian black hole segments.

- Demand fields to be piecewise-smooth and C^1 at the joints.

2. Holography & ingoing boundary conditions

- Consider a massive scalar field in the bulk, with $\lambda\phi^3$ interactions.

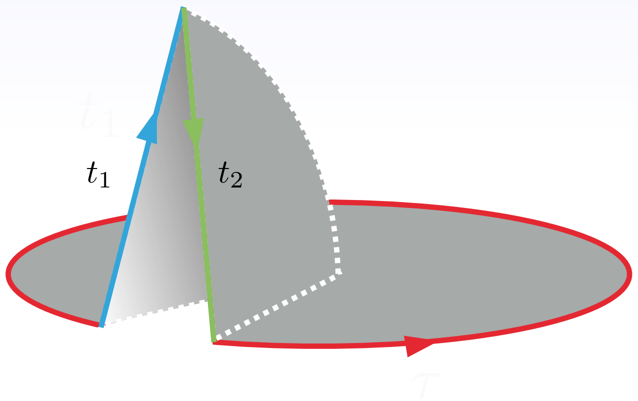
- With no sources, the thermal state has $\langle\mathcal{O}_r\rangle = 0$.

- Turn on perturbative sources $J_1 = J_2 = J_r \sim \epsilon$. Then

Control the 2- and 3-point function, piece-wise smooth

$$\phi = \psi\,\epsilon + \chi\,\epsilon^2 + O(\epsilon)^3,$$

$$g = \bar{g} + O(\epsilon)^2$$



- This gives the following BVPs:

$$(\Box_{\bar{g}_i} - \Delta(\Delta - d))\,\psi_i = 0\,,$$

$$\lim_{r\to\infty} r^{d-\Delta}\epsilon\psi_1 = J_1,$$

$$\lim_{r\to\infty} r^{d-\Delta}\epsilon\psi_2 = J_2,$$

$$\lim_{r\to\infty} r^{d-\Delta}\epsilon\psi_E = 0,$$

First order, BVP1

$$(\Box_{\bar{g}_i} - \Delta(\Delta - d))\,\chi_i = 3\lambda\psi_i^2\,,$$

$$\lim_{r\to\infty} r^{d-\Delta}\chi_i = 0$$

Second order, BVP2
Non-linear

+matching conditions between the segments

Solving the first BVP

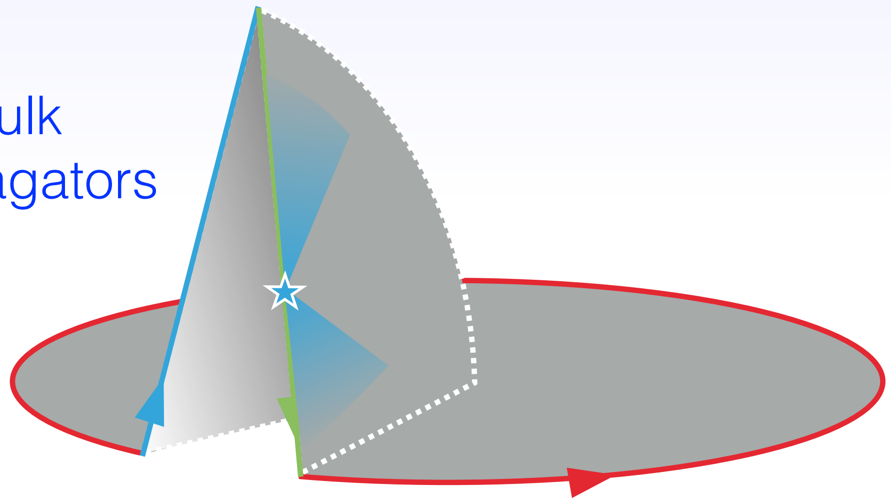
- Consider the influence of a single delta function source on $\mathcal{M}_1$ for the entire spacetime. The bulk-boundary propagators are

$$\tilde{\Delta}_{11} = \int \frac{d\omega}{2\pi}\, e^{i\omega t_1} \left(c^R_{11} G^R(r) + c^A_{11} G^A(r) \right) ,$$

$$\tilde{\Delta}_{21} = \int \frac{d\omega}{2\pi}\, e^{i\omega t_2} \left(c^R_{21} G^R(r) + c^A_{21} G^A(r) \right) ,$$

$$\tilde{\Delta}_{E1} = \int \frac{d\omega}{2\pi}\, e^{-\omega\tau} \left(c^R_{E1} G^R(r) + c^A_{E1} G^A(r) \right) .$$

bulk propagators



- Fix coefficients $c^{R,A}_{ij}$ by gluing and boundary values.

- Repeat for $\mathcal{M}_2$ to get $\tilde{\Delta}_{i2}$.

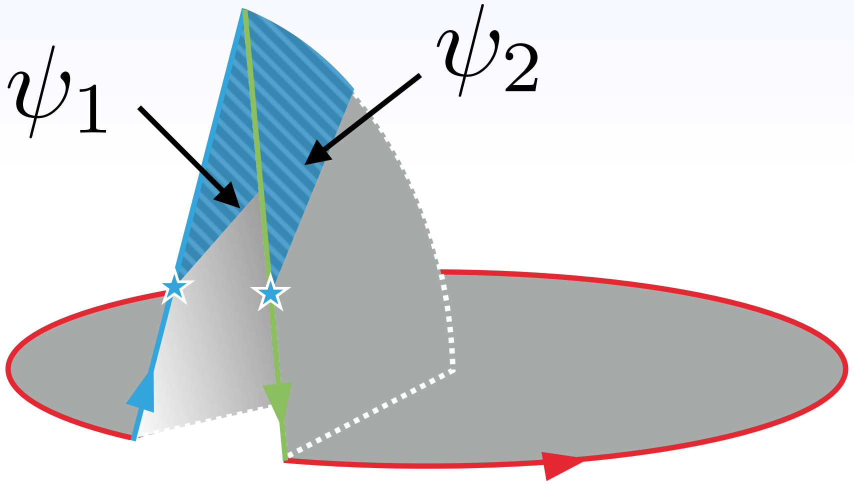
- Find total field on $\mathcal{M}_1$ given both sources by convolving with bulk-boundary propagator.

$$\psi_1(t,r) = \int dt_1' \left(\tilde{\Delta}_{11}(t,r;t_1') + \tilde{\Delta}_{12}(t,r;-t_1')\right) J_r(t_1') \,,$$

built entirely from G^R

$$\tilde{\Delta}_{1r} = \int \frac{d\omega}{2\pi} e^{i\omega t_1} G^R(r),$$

- Similarly for $\psi_2, \psi_E = 0$.



Conclusion:

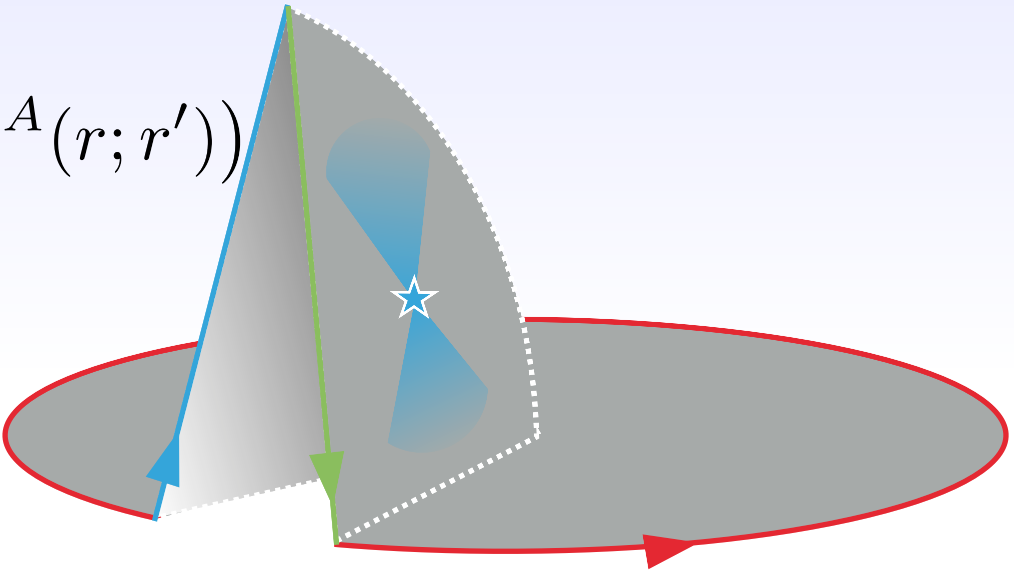
- Causal response to boundary sources.
- Equivalent to ingoing boundary conditions on a single Lorentzian piece: Son-Starinets prescription for 2-point functions.

Solving the second BVP

- Now we have sources in the bulk.

- We need the matrix of bulk-bulk propagators.

$$\Delta_{ij} = \int d\omega d^{d-1}k\, e^{-i\omega t_i + ikx} \left(c^R_{ij} G^R(r;r') + c^A_{ij} G^A(r;r')\right)$$



- Fix $c_{ij}^{R,A}$ by gluing, bulk delta functions.

- Finally, convolve with the source.

$$\begin{aligned}\chi_1(t,r) &= \int dt_1' dr' \Delta_{11}(t,r;t_1',r')\psi_1^2(t_1',r') + \int dt_2' dr' \Delta_{12}(t,r;t_2',r')\psi_2^2(t_2',r') \\ &= \int dt_1' dr' \Delta_{1r}(t,r;t_1',r')\psi_1^2(t_1',r')\end{aligned}$$

Conclusion:

- Causal response to ψ_i .
- Equivalent to ingoing boundary conditions on a single Lorentzian piece.

3. Numerical results for AdS_5/CFT_4

Computing G_{raa} numerically

- We now know how to compute: ingoing boundary conditions at the horizon of a (single) Lorentzian BH spacetime.

- Focus in momentum space.

$$r_3(p_1,p_2) \equiv \int d^dx_1\, d^dx_2 G_{raa}(0,x_1,x_2)e^{i(p_1\cdot x_1+p_2\cdot x_2)}$$

- Solve for ψ , χ subject to boundary conditions.

$$\psi = s_1\psi_1(r)e^{ip_1\cdot x} + s_2\psi_2(r)e^{ip_2\cdot x},$$

$$\chi = s_1^2\chi_{11}(r)e^{2ip_1\cdot x} + s_1s_2\chi_{12}(r)e^{i(p_1+p_2)\cdot x} + s_2^2\chi_{22}(r)e^{2ip_2\cdot x}.$$

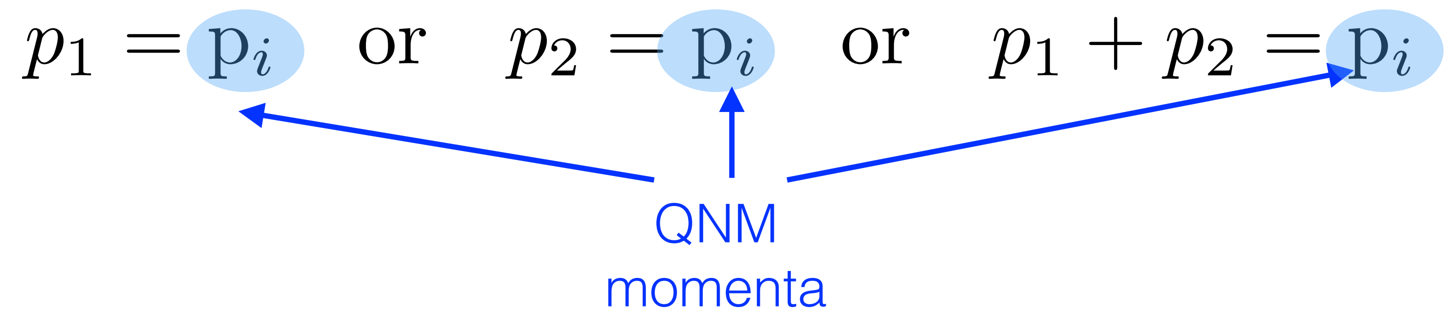
Do not contribute to the 3pt function of interest

- From boundary expansion, extract 2- and 3-point functions.

$$\psi_i = \frac{1}{r^{d-\Delta}} + \ldots + \frac{\psi_i^{(\Delta)}}{r^{\Delta}} + \ldots , \qquad \chi_{ij} = \frac{\chi_{ij}^{(\Delta)}}{r^{\Delta}} + \ldots . \quad \Longrightarrow \quad \begin{aligned} r_2(p_1) &= (2\Delta - d)\psi_1^{(\Delta)} , \\ r_3(p_1,p_2) &= (2\Delta - d)\chi_{12}^{(\Delta)} . \end{aligned}$$

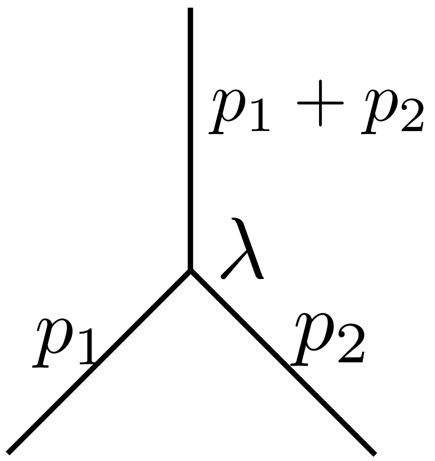
- Numerically focus on $d = 4, \Delta = 5/2, \lambda = 1$ at zero spatial momenta.

- Expect singularities when



$$p_1 = \mathrm{p}_i \quad \text{or} \quad p_2 = \mathrm{p}_i \quad \text{or} \quad p_1 + p_2 = \mathrm{p}_i$$

QNM
momenta



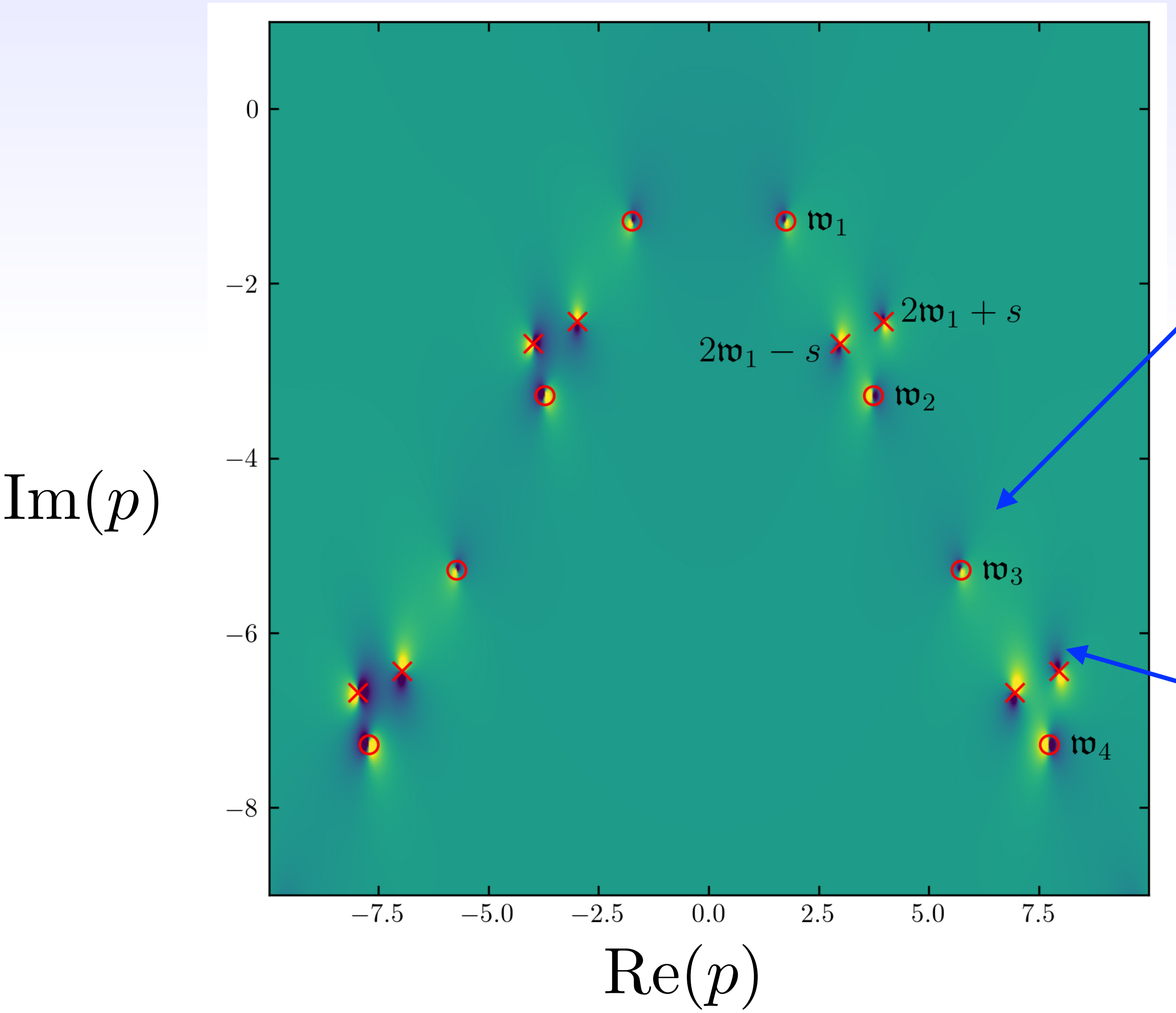
- Parametrise results:

$$p_1 = (p+s)/2$$
$$p_2 = (p-s)/2$$

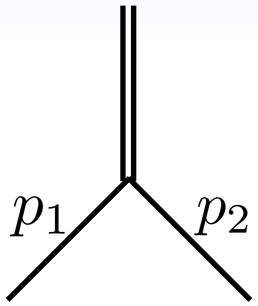
G^{OOO}_{raa} in momentum space

$$Re\left[\boxed{r_3}\left(\overbrace{\frac{p+s}{2}}^{p_1},\overbrace{\frac{p-s}{2}}^{p_2}\right)\right]_{\boxed{s=fixed}}$$

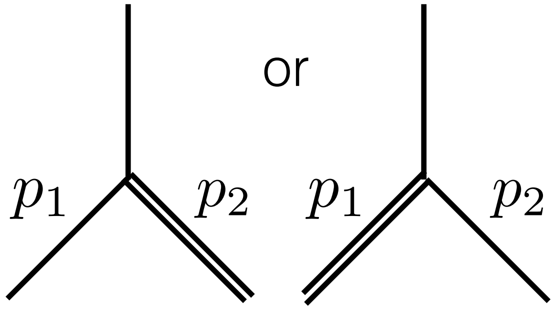
generic



Resonant QNM

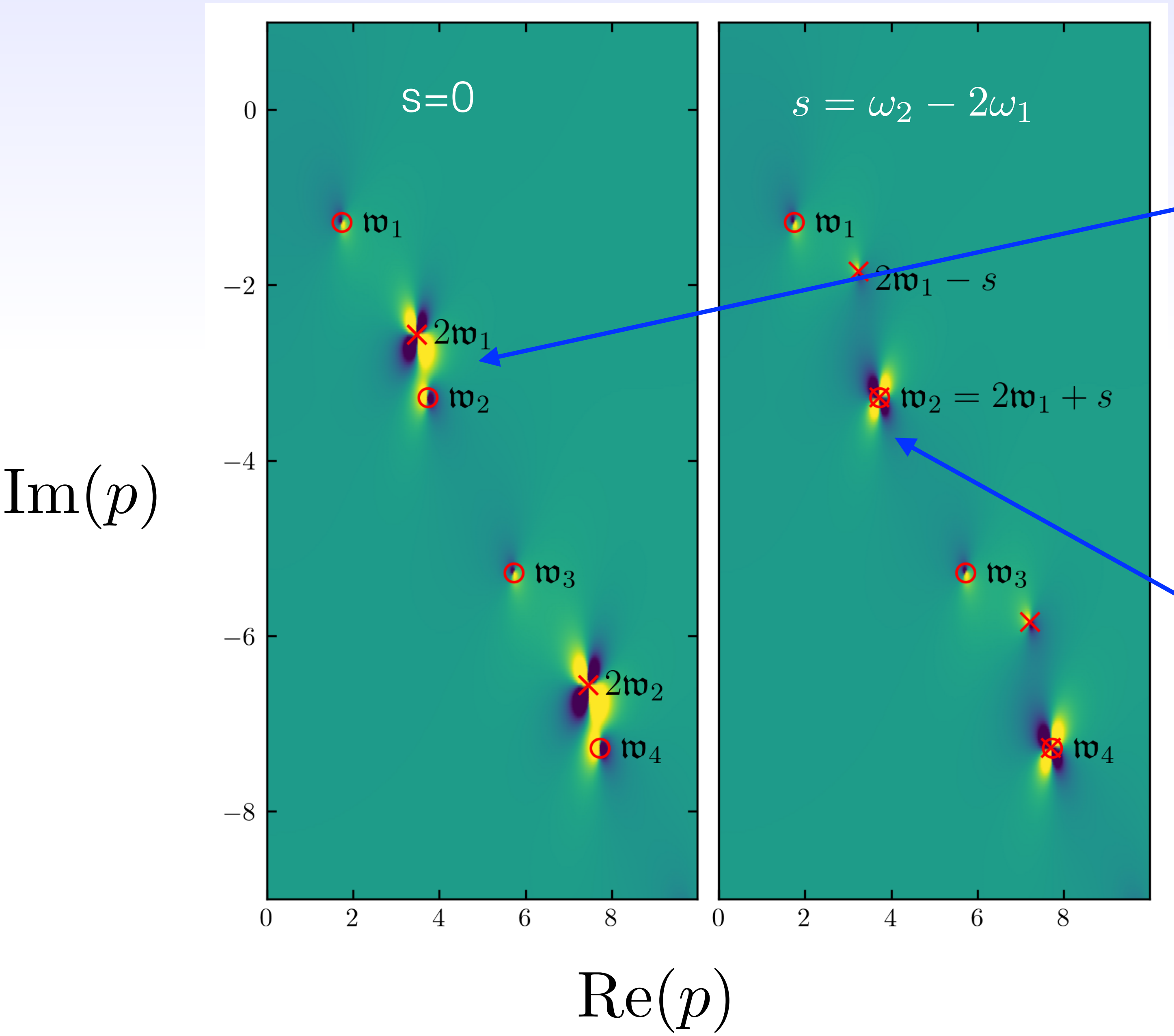


Directly driven QNM



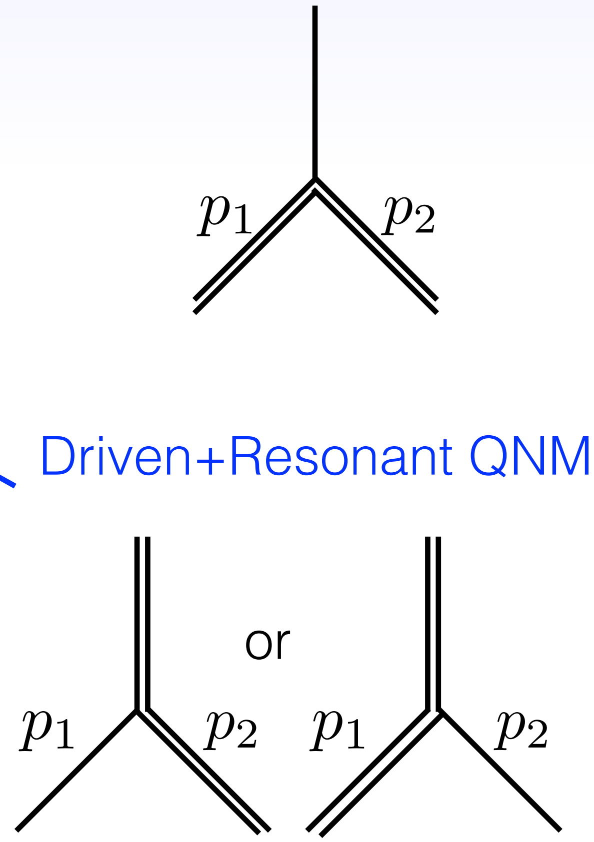
$$Re\left[r_3\left(\frac{p+s}{2},\frac{p-s}{2}\right)\right]_{s=fixed}$$

Fine tuned



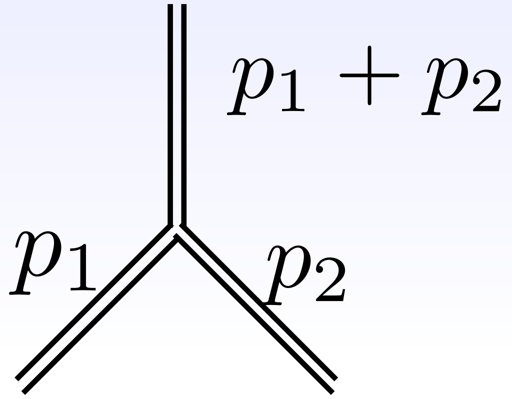
Driven+Driven QNM

Driven+Resonant QNM



- In principle triple poles can also present, depending on the details of the theory. Couldn't find any in our model.

Driven+Driven+Resonant QNM



- No extra poles, beyond what we expected, at this level.

4. Extra results

What about 3-point function involving the stress tensor?

- Use Ward identities in the r/a basis, with $J_1 = J_2 = J_r\,, J_a = 0$

$$\eta^{\mu\nu} \left\langle (T_r)_{\mu\nu} \right\rangle_J = (d-\Delta) J_r \left\langle \mathcal{O}_r \right\rangle_J \,,$$
$$\partial^{\mu} \left\langle (T_r)_{\mu\nu} \right\rangle_J = \partial_\nu J_r \left\langle \mathcal{O}_r \right\rangle_J$$

- Expand $\langle (T_r)_{\mu\nu} \rangle_J$, $\langle \mathcal{O}_r \rangle_J$ in powers of J_r and plug into Ward Ident.

G^{TOO}_{raa} in momentum space

$$\eta^{\mu\nu} (r_3^{TOO})_{\mu\nu}(p_1, p_2) = (d-\Delta)(r_2(p_1) + r_2(p_2)),$$
$$(p_1+p_2)^{\mu} (r_3^{TOO})_{\mu\nu}(p_1, p_2) = r_2(p_1)(p_2)_\nu + r_2(p_2)(p_1)_\nu$$

G^{OO}_{ra} in momentum space

- When the spatial momenta are zero, can solve explicitly

$$(r_3^{TOO})_{tt}(\omega_1, \omega_2) = -\frac{r_2(\omega_1)\omega_2 + r_2(\omega_2)\omega_1}{\omega_1 + \omega_2}$$

- Can extend to higher point function.

What about the fluctuation-dissipation theorem?

In momentum space $\tilde{G}_{rr} = (1+2n)\left(\tilde{G}_{ra} - \tilde{G}_{ar}\right)$

Bose-Einstein distribution $n = (e^{\beta\omega} - 1)^{-1}$

This is manifest in the bulk-boundary propagators!

- Express ψ_a, ψ_r and in terms of J_a, J_r and use coefficients $c_{ij}^{R,A}$

$$\tilde{\Delta}_{ra} = \frac{1}{2}\left(\tilde{\Delta}_{11} + \tilde{\Delta}_{12} + \tilde{\Delta}_{21} + \tilde{\Delta}_{22}\right) = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} G^R(\omega, r),$$

$$\tilde{\Delta}_{rr} = \frac{1}{2}\left(\tilde{\Delta}_{11} - \tilde{\Delta}_{12} + \tilde{\Delta}_{21} - \tilde{\Delta}_{22}\right) = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} (1+2n)\left(G^R(\omega, r) - G^A(\omega, r)\right),$$

$$\tilde{\Delta}_{aa} = \frac{1}{2}\left(\tilde{\Delta}_{11} + \tilde{\Delta}_{12} - \tilde{\Delta}_{21} - \tilde{\Delta}_{22}\right) = 0,$$

$$\tilde{\Delta}_{ar} = \frac{1}{2}\left(\tilde{\Delta}_{11} - \tilde{\Delta}_{12} - \tilde{\Delta}_{21} + \tilde{\Delta}_{22}\right) = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} G^A(\omega, r)$$

5. Summary & Outlook

Summary

- We have computed retarded thermal 3-point functions holographically: $G_{raa}^{T\mathcal{O}\mathcal{O}}, G_{raa}^{\mathcal{O}\mathcal{O}\mathcal{O}}$.
- We proved that real-time holography reduces to ingoing boundary conditions on a single sheet.
- QNMs appear as single and double poles. The universality of black hole ringdown extends to non-linear response.

Outlook

- For 2-point function, only 1 independent component. For 3-point functions there are 3; we computed one of these, the other two? G_{ara}, G_{aar}

- Understand the generalised Fluctuation-Dissipation theorem from the bulk.

- Extend to conserved current operators, e.g. G^{TTT}

- Relationship between Skenderis-van Rees and Crossley-Glorioso-Liu prescriptions. Recent related work [Loganayagam,Rangamani,Virrueta '22]

Thanks for listening!