Relativistic hydrodynamics: a singulant perspective

Based on [arXiv:2110.07621] and [arXiv:2112.12794] with Michal Heller, Michał Spaliński, Viktor Svensson and Ben Withers

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Alexandre Serantes, ICCUB
HoloTube, 01/11/2022

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Introduction

- Relativistic hydrodynamics: EFT description of the infrared behavior of any relativistic medium with a conserved stress-energy tensor.
- Key organizational principle: gradient expansion around local thermal equilibrium.
- Pivotal tool to model real-world phenomena: high-energy nuclear collisions, neutron star mergers...
- Venerable subject with plenty of open problems to explore!

➥ In this talk: **the nature of the gradient expansion.**

➤ Past studies: general fluid flows in linear response regime & nonlinear comoving flows.

Motivation: where is relativistic hydrodynamics applicable?

The gradient expansion, truncated to low orders, can work successfully away from local thermal equilibrium

First principles AdS/CFT studies crucial!

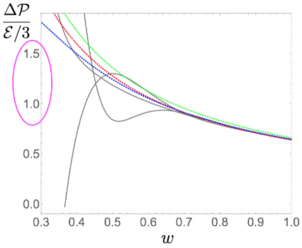
[Chesler & Yaffe, '08, '10, '15] [Heller, Janik & Witaszczyk, '11 '12] [van der Schee, '12] [Casalderrey-Solana, Heller, Mateos & van der Schee, '13] [Jankowski, Plewa & Spalinski, '14] [Chesler, '15, '16] [Attems, Bea, Casalderrey-Solana, Mateos, Triana & Zilhao, '17 '18] ...

hydrodynamization time

$\neq$

local equilibration time

[Casalderrey-Solana, Mateos, Rajagopal & Liu, '11]



From arXiv:1610.02023 by Heller

What is the mechanism setting the applicability regime of RH?

The gradient expansion in the linear response regime: momentum space

A *mode* $\omega = \omega(k)$ is a singularity of the retarded two-point function $G(\omega, \mathbf{k})$ ($k \equiv |\mathbf{k}|$)

$$\rho(t, \mathbf{k}) \supset e^{-i\omega(k)t + i\mathbf{k}\cdot\mathbf{x}} J(\mathbf{k})$$

Two classes:

- ⋆ Hydrodynamic mode, $\omega_H(k) \to 0$ for $k \to 0$ → long-lived & slowly varying pert.
- ⋆ Nonhydrodynamic mode, $\omega_{NH}(0)$ finite → transient pert.

Hydrodynamics predicts the small-k expansion of $\omega_H(k)$.

Is it convergent?

Intense scrutiny in recent years in AdS/CFT [Withers, '18] [Grozdanov, Kovtun, Starinets & Tadic, '19] [Abbasi & Tahery, '20] [Jansen & Pantelidou, '20] [Areán, Davison, Goutéraux & Suzuki, '20] [Baggioli, Gran & Tornso, '21] [Wu, Baggioli & Li, '21] [Asadi, Soltanpanahi & Taghinavaz, '21] [Grozdanov, Starinets & Tadic, '21] [Jeong, Kim & Sun, '21] [Huh, Jeong, Kim & Sun, '21] [Liu & Wu, '21] [Cartwright, Amano, Kaminski, Noronha & Speranza, '21]...

Common observation: **the small-k expansion has a finite r.o.c.**

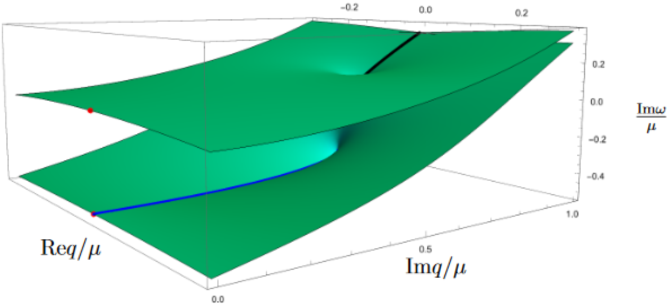
What is the mechanism setting it?

Overarching picture for AdS/CFT, and also MIS-like models and RTA kinetic theory ([Romatchske, '15] [Heller, AS, Spalinski, Svensson & Withers, '20]):

- $\omega_H(k)$ has branch point singularities in the complex k-plane.
- The r.o.c. of the small-k expansion of $\omega_H(k)$ is set by the branch point closest to $k = 0$.
- This branch point appears at the $k \in \mathbb{C}$ for which $\omega_H(k)$ first collides with a nonhydrodynamic singularity of $G(\omega, k)$.

The large-order behavior of RH and nonhydrodynamic modes are deeply intertwined!

- Useful viewpoint: think of $\omega_H(k)$, $\omega_{NH}^{(1)}(k)$, $\omega_{NH}^{(2)}(k)$,... as different sheets of a unique Riemann surface.
- ➥ Analytic continuation allows to reconstruct the transient modes from the hydrodynamic data! [Withers, '18], [Grozdanov & Lemut, '22]



From arXiv:1803.08058 by Withers

The gradient expansion in the linear response regime: position space

What is the counterpart of these results in position space?

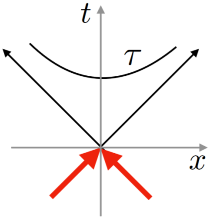
[Heller, AS, Spalinski, Svensson & Withers, '20]

- The finite r.o.c. of the small-k expansion of $\omega_H(k)$ implies the factorial divergence of the position-space gradient expansion for generic fluid flows.
- If the momentum-space support of the flow is capped, there is no factorial divergence, just geometric growth.

➥ See Ben's Holotube talk in 2020 for a detailed discussion of these results!

The gradient expansion in the nonlinear regime

- Beyond the linear response regime, past studies have been restricted to $(0+1)$-dimensional comoving flows: Gubser flow & **Bjorken flow**.



$$\tau = \sqrt{t^2 - x^2}$$

$$\langle T^{\mu}_{\nu} \rangle = \mathrm{diag}\left(-\mathcal{E}(\tau), P_T(\tau), P_T(\tau), P_L(\tau)\right)$$

$$P_L = -\mathcal{E} - \tau\dot{\mathcal{E}} \quad P_T = \mathcal{E} + \frac{1}{2}\tau\dot{\mathcal{E}}$$

- Relativistic hydrodynamics predicts $\mathcal{E}(\tau)$ in a near-equilibrium large-τ expansion:

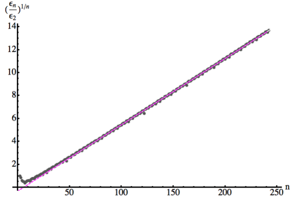
$$\mathcal{E}(\tau) = \tau^{-\frac{4}{3}}\left(\epsilon_2 + \epsilon_3\tau^{-\frac{2}{3}} + \epsilon_4\tau^{-\frac{4}{3}} + \dots\right)$$

Is this a convergent series?

Heller, Janik & Witaszczyk, 2013: computed series to large-order in $\mathcal{N}=4$ SUSY YM in the 't Hooft limit using AdS/CFT.

End result: factorial divergence!

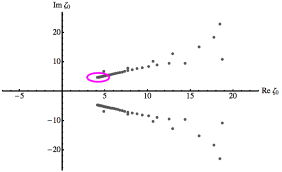
$$\left|\frac{\epsilon_n}{\epsilon_2}\right|^{\frac{1}{n}} \sim n, \quad n \to \infty$$



The large-order behavior of the hydrodynamic gradient expansion is again deeply intertwined with the nonhydrodynamic transient sector!

$$\mathcal{E}(\tau) = \sum_{n=2}^{\infty} \epsilon_n \tau^{-\frac{2n}{3}} \xrightarrow{\text{Borel transform}} \tilde{\mathcal{E}}(\zeta) = \sum_{n=2}^{\infty} \frac{\epsilon_n}{n!}\zeta^n$$

$\tilde{\mathcal{E}}$ has a finite r.o.c. governed by the lowest NH QNM at $k=0$!



(Almost) universal picture for comoving flows in Müller-Israel-Stewart theories, kinetic theory and AdS/CFT

- The gradient expansion is factorially divergent.
- The large-order factorial growth is governed by nonhydrodynamic d.o.f.

[Aniceto, Basar, Baggioli, Behtash, Buchel, Casalderrey-Solana, Cruz-Camacho, Florkowski, Denicol, Dunne, Gushterov, Heller, Kamata, Kurkela, Jankowski, Meiring, Martnez, Noronha, Ryblewski, Shi, Spalinski, Svensson ...]

Natural language: transseries and resurgent analysis

$$\mathcal{E}(\tau) = \mathcal{E}_H(\tau) + \sum_q e^{-\frac{3i}{2}\omega^{(q)}_{QNM}(k=0)\tau^{\frac{2}{3}}} \tau^{\alpha^{(q)}} \mathcal{E}^{(q)}_{NH}(\tau) + \dots$$

➥ Hydrodynamic attractors [Heller, Spalinski, Romatschke, Brewer, Jefferson, Mitra, Mondkar, Mukhopadhayay, Rebhan, Soloviev, Strickland, van der Schee, Wiedemann, Wu, Yan, Yin ...]

We need to bridge the gap between the studies of generic flows in the linear response regime and the studies of nonlinear comoving flows

Demands novel computational techniques & novel conceptual insights!

In this talk, I will report novel progress in this direction:

- Part I will present the first explicit computations of the gradient expansion at the nonlinear level for non-comoving flows [arXiv:2110.07621] (PRL).
- Part II will discuss a new perspective into the large-order behavior of the gradient expansion based on **singulants** [arXiv:2112.12794] (PRX).

The question we will address

- We work in the Landau frame,

$$T^{\mu\nu} = T^{\mu\nu}_{ideal} + \Pi^{\mu\nu}, \quad T^{\mu\nu}_{ideal} = \mathcal{E} U^\mu U^\nu + P(\mathcal{E})(\eta^{\mu\nu} + U^\mu U^\nu), \quad T^\mu_\nu U^\nu = -\mathcal{E} U^\mu.$$

- For a CFT, $T^\mu_\mu = 0$ and

$$P(\mathcal{E}) = \frac{1}{d-1}\mathcal{E}, \quad \Pi^\mu_\mu = 0.$$

- Classical hydrodynamics as an EFT is defined by the constitutive relations

$$\Pi_{\mu\nu} = \sum_{n=1}^{\infty} \Pi^{(n)}_{\mu\nu}(\mathcal{E}, U) = -\eta\,\sigma^{\mu\nu} + \ldots,$$

  which express the dissipative tensor as a gradient expansion in terms of the hydrodynamic fields: energy density $\mathcal{E}$ and fluid velocity U^μ.

- The operational question we will focus on is the large-order behavior of the gradient-expanded constitutive relations when evaluated on a particular fluid flow.

New explicit computations of the gradient expansion

Longitudinal flows

- I will describe a new computational method to obtain the gradient expansion in MIS-like theories.
- The method is valid for generic fluid flows. Case study: **longitudinal flow in BRSSS theory** [Baier, Romatschke, Son, Starinets & Stephanov, '07].
- Longitudinal flow: non-boost invariant dynamics confined to the t-x plane.

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \quad \mathcal{E}, \quad U^\mu = (\cosh u, \sinh u, 0, 0),$$

$$\Pi^{\mu\nu} = \begin{pmatrix} -2\sinh^2(u)\Pi_\star & \sinh(2u)\Pi_\star & 0 & 0 \\ \sinh(2u)\Pi_\star & -2\cosh^2(u)\Pi_\star & 0 & 0 \\ 0 & 0 & \Pi_\star & 0 \\ 0 & 0 & 0 & \Pi_\star \end{pmatrix}.$$

- Three functions, $\mathcal{E}$, u and $\Pi_\star$ that only depend on t & x. For a conformal fluid, we trade $\mathcal{E}$ for $T \propto \mathcal{E}^{\frac{1}{4}}$.
- You can think of longitudinal fluid flows as nonlinear sound waves.

Toy model: causal UV-completion of second-order RH.
Our perspective here: *BRSSS as a mock microscopic theory.*

Main idea: promote $\Pi_{\mu\nu}$ to a set of independent dynamical degrees of freedom,

$$(\tau_\Pi(T)U^\alpha \mathcal{D}_\alpha + 1)\Pi^{\mu\nu} = -\eta(T)\sigma^{\mu\nu} + \dots$$

$\mathcal{D}_\mu$: Weyl-covariant derivative [Loganayagam, '08]
$\sigma_{\mu\nu}$: shear tensor (symmetric, transverse and traceless part of $\nabla_\mu U_\nu$).
τ_Π: relaxation time ($\propto T^{-1}$)
η: shear viscosity ($\propto T^3$).

How do we compute the gradient expansion? ➥ Recursion relations

$$t \to \frac{t}{\epsilon}, \quad x \to \frac{x}{\epsilon}, \quad \Pi_\star(t,x) = \sum_{n=1}^{\infty} \Pi_\star^{(n)}(t,x)\epsilon^n$$

$$\Downarrow$$

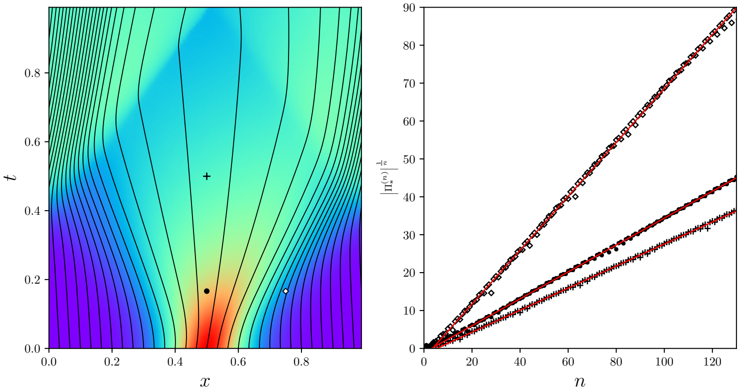
$$\Pi_\star^{(1)} = -\frac{2}{3}\eta\nabla\cdot U, \quad \Pi_\star^{(n+1)} = -\tau_\Pi(U\cdot\nabla)\Pi_\star^{(n)} - \frac{3}{2}(\nabla\cdot U)\tau_\Pi\Pi_\star^{(n)} - \frac{\lambda_1}{\eta^2}\sum_{m=1}^{n}\Pi_\star^{(m)}\Pi_\star^{(n+1-m)}.$$

The gradient expansion

Strategy:

1. Choose $T(0,x)$, $u(0,x)$ and $\Pi_\star(0,x)$.
2. Solve BRSSS equations of motion numerically.
3. Evaluate gradient expansion using recursion relations and numerical solution.

We always find that the gradient expansion is always factorially divergent!



The continuum limit

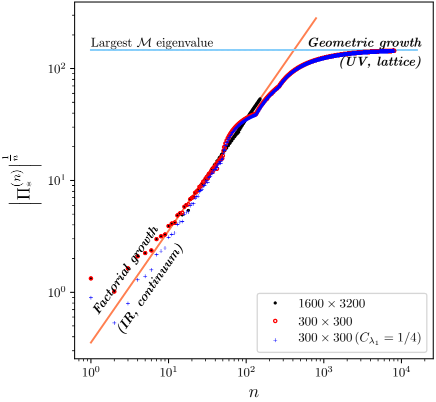
In practice, lattice discretization. $\Pi_\star^{(n)}(t,x) \longrightarrow \tilde{\Pi}_\star^{(n)}{}_{ij} \equiv \Pi_\star^{(n)}(t_i, x_j)$

Recursion relation as matrix operation: $\tilde{\Pi}_\star^{(n+1)} = \mathcal{M}\tilde{\Pi}_\star^{(n)}, \quad \mathcal{M} = \mathcal{M}(T, u)$

Convergence:

- For fixed l.s., $(\tilde{\Pi}_\star^{(n)})_{ij} \sim \Lambda^n$. Λ: largest $\mathcal{M}$ eigenvalue
- $\Lambda \to \infty$ as l.s. $\to 0$: the factorial growth at fixed n is recovered.

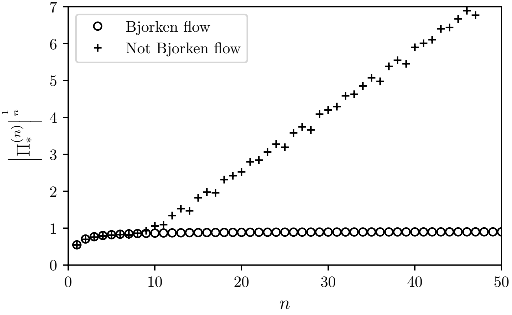
Nonlinear avatar of the finite-momentum space support in linear response!



The case of the DN model

Denicol & Noronha, '19: only known counterexample to gradient expansion being fact. div. in Bjorken flow

When longitudinal boost-invariance is broken,
the gradient expansion becomes factorially divergent again



Explanation: for Bjorken flow, $\Pi_\star^{(1)}$ is a linear combination of eigenfunctions of $\mathcal{M}$!

Take-home points

The nonlinear longitudinal flow results conform to the linear response insights!

- The factorial growth of the gradient expansion *does not need* a factorial growth in the number of transport coefficients with the order.
- The factorial growth of the gradient expansion *does need* that the system supports excitations of arbitrarily short wavelength: killed by k-space cutoff (linear) or spacetime lattice (nonlinear).

Imposing special symmetries (e.g. boost-invariance)
might also halt the factorial growth

Singulants: a new perspective on the asymptotic behavior

The singulant field

In the previous example, the slope of $|\Pi_\star^{(n)}|^{\frac{1}{n}}$ changes accross spacetime.

This slope is governed by an emergent collective field: the **singulant** (χ) [Dingle, '74]

$$\Pi_\star^{(n)}(t,x) \sim A(t,x)\frac{\Gamma(n+\gamma(t,x))}{\chi(t,x)^{n+\gamma(t,x)}}$$
$$|\Pi_\star^{(n)}(t,x)|^{\frac{1}{n}} \sim \frac{n}{e|\chi(t,x)|}.$$

Remaining part of the talk:

1. Singulants obey **simple equations of motion**.
2. Singulants embody a **duality between far-from-equilibrium relativistic hydrodynamics and linear response around global equilibrium**.
3. Singulants provide a **proxy for optimal truncation order & error**.

Singulant equation of motion: the governing principles

1. Recursion relation approach: large order linearization & eikonalization

$$f = \sum_{n=0}^{\infty} f^{(n)}\epsilon^n \qquad\qquad (f^k)^{(n)} \sim (kf^{(0)})^{k-1} f^{(n)}$$

$$\Rightarrow$$

$$f^{(n)} \sim A\frac{\Gamma(n+\gamma)}{\chi^{n+\gamma}} \qquad\qquad (\epsilon^k \partial^{\alpha_1} \dots \partial^{\alpha_k} f)^{(n)} \sim f^{(n)}(-1)^k \partial^{\alpha_1}\chi \dots \partial^{\alpha_k}\chi$$

2. Transseries approach: singulants as non-perturbative actions

$$f = \sum_{n=0}^{\infty} f^{(n)}\epsilon^n + \sum_s e^{-\frac{\chi_s}{\epsilon}} \sum_{n=0} \tilde{f}^{(s,n)}\epsilon^n + \dots.$$

Equivalent to 1., singulant e.o.m as WKB eikonal equation.

Important consequence: multiple singulants! $\qquad f^{(n)} \sim \sum_q A_q \frac{\Gamma(n+\gamma_q)}{\chi_q^{n+\gamma_q}}$

Singulants in phenomenological models: the BRSSS case

For longitudinal flows in BRSSS theory,

$$U^\mu(t,x)\partial_\mu \chi(t,x) = \frac{1}{\tau_\Pi(T(t,x))}$$

$$\chi(\tau,\sigma) = \chi(0,\sigma) + \int_0^\tau \frac{d\tau'}{\tau_\Pi(T(\tau',\sigma))}$$

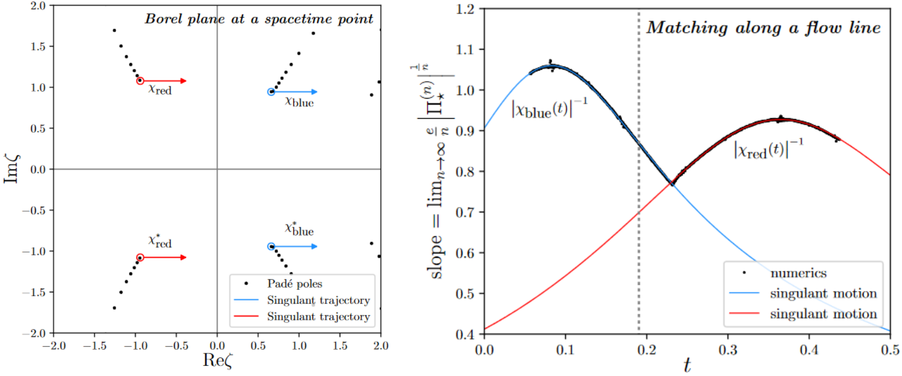
Flow to thermal equilibrium: the real part of χ always increases!

Cross-check: use numerical computation of the gradient expansion.

$$\Pi_\star = \sum_{n=1}^{\infty} \Pi_\star^{(n)} \xrightarrow{\text{B.T.}} \Pi_\star^{(B)} = \sum_{n=1}^{\infty} \frac{\Pi_\star^{(n)}}{n!}\zeta^n \xrightarrow{\text{A.C.}} \tilde{\Pi}_\star^{(B)}(\zeta)$$

Singulant: branch-point singularity of $\tilde{\Pi}_\star^{(B)}$.

In practice: use Padé approximants to perform analytic continuation.


Borel plane at a spacetime point
Im ζ
Re ζ
χred
χblue
χ*red
χ*blue
Padé poles
Singulant trajectory
Singulant trajectory
Matching along a flow line
slope = lim_{n→∞} (e/n) |Π⋆^(n)|^(−1/n)
|χblue(t)|^−1
|χred(t)|^−1
t
numerics
singulant motion
singulant motion

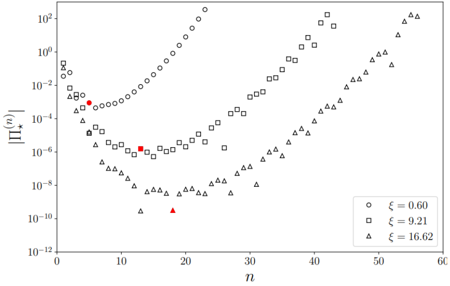
Singulants and the optimal truncation of the gradient expansion

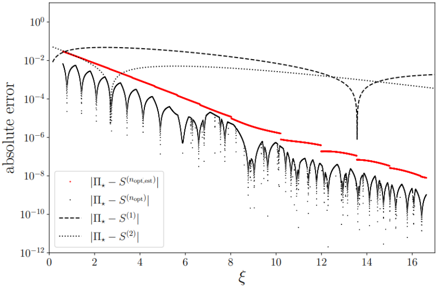
$$|\chi_d| \leq |\chi_2| \leq |\chi_3| \leq \dots \qquad \chi_d\text{: dominant singulant}$$

Estimate: $n_{\text{opt.,est.}} = [|\chi_d|]$

- Dominant singulant moves toward the right in the Borel plane: $n_{opt.,est.}$ increases.
- Leads to a truncation order & error that tracks down the optimal ones correctly!

Singulants govern the optimal truncation of the gradient expansion in BRSSS theory





$$\xi(t) = \int_0^t \frac{dt'}{\tau_\Pi(T(t'),0)}$$

Singulants in phenomenological models & and linear response

The singulant equation of motion is dual to linear response theory problem

$$\text{e.o.m } \Pi_\star + \Pi_\star = \lambda \delta \Pi_\star e^{i q_\mu X^\mu} + T(t,x) \text{ \& } U^\mu(t,x) \to T_0 \text{ \& } U_0^\mu$$
$$\Downarrow$$
Dispersion relation for q^μ, $q^0 = q^0(\mathbf{q})$.

The map

$$T_0 \to T(t,x),\; u_0 \to u(t,x), iq_\mu \to -\partial_\mu \chi$$

transforms this dispersion relation into the singulant equation of motion!

Important: *might or might not be* equivalent to the computation of the sound modes

Singulants and linear response in BRSSS theory

- $\Pi_\star$ fluctuations @ fixed T_0 & u_0 ,

$$i\tau_\Pi(T_0)U_0^\mu q_\mu + 1 = 0 \xrightarrow{map} \tau_\Pi(T(t,x))U^\mu(t,x)\partial_\mu\chi(t,x) - 1 = 0$$

- Sound channel modes,

$$T_0(i\tau_\Pi(T_0)\omega + 1)(3\omega^2 - k^2) - 4i\frac{\eta}{s}\omega k^2 = 0, \quad \omega = -U_0^\mu q_\mu,\ k = -Z_0^\mu q_\mu.$$

- One NH mode with $\omega_{NH}(k=0) = -\frac{i}{\tau_\Pi(T_0)}$: solves original $\Pi_\star$ fluctuation problem!

For any longitudinal flow in BRSSS theory, singulants are governed by the nonhydrodynamic sound mode evaluated at $k = 0$ and at the local temperature.

The movement of χ toward the right in the Borel plane is the far-from-equilibrium counterpart of the decay of the nonhydrodynamic fluctuations around thermal equilibrium.

Towards AdS/CFT: a new phenomenological model

- BRSSS theory: singulant dynamics determined by $\omega^{(NH)}(k=0)$. Reason: e.o.m for $\Pi_{\mu\nu}$ contains only $U^{\mu}\mathcal{D}_{\mu}$.

➥ **Not generic: it will not happen in AdS/CFT!**

- Toy model example of the general case: generalization of the HJSW model [Heller, Janik, Spalinski & Witaszczyk, '14]

$$\left(\left(\frac{\mathcal{D}}{T}\right)^2+2\Omega_I\left(\frac{\mathcal{D}}{T}\right)+|\Omega|^2-\frac{c_{\mathcal{L}}}{2T^2}\left[\Delta^{\alpha}_{\mu}\Delta^{\beta}_{\nu}+\Delta^{\alpha}_{\nu}\Delta^{\beta}_{\mu}-\frac{2}{3}\Delta_{\mu\nu}\Delta^{\alpha\beta}\right](\Delta^{\rho\sigma}\mathcal{D}_{\rho}\mathcal{D}_{\sigma})\right)\Pi_{\mu\nu}= -\eta(T)|\Omega|^2\sigma_{\mu\nu}+\dots$$

➥ Causal and stable for linearized perturbations of thermal equilibrium (all channels)!

➥ Phenomenological utility?

➤ Singulant e.o.m. in a longitudinal flow & dispersion relation of associated linear response problem,

$$U(\chi)^2 - c_{\mathcal{L}} Z(\chi)^2 - 2\Omega_I T U(\chi) + |\Omega|^2 T^2 = 0 \quad \& \quad -\omega^2 + c_{\mathcal{L}} q^2 - 2i\Omega_I T_0 \omega + |\Omega|^2 T_0^2 = 0.$$

$Z = Z^\mu \partial_\mu$: unit-normalized & orthogonal to U

➤ Sound channel dispersion relation,

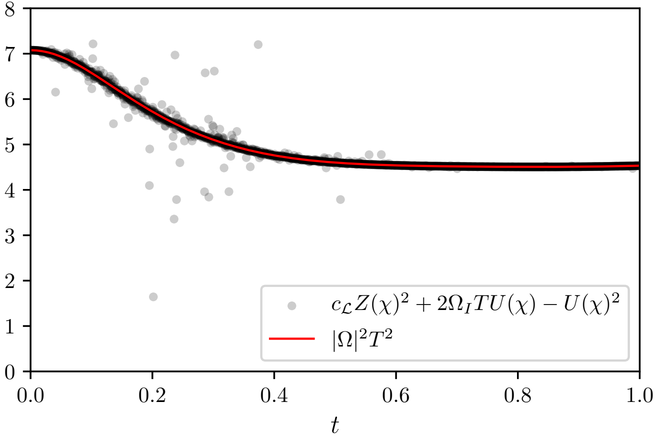
$$\left(-\omega^2 - 2iT_0\Omega_I\omega + c_{\mathcal{L}} q^2 + T_0^2|\Omega|^2\right)\left(\omega^2 - \frac{1}{3}q^2\right) + \frac{4}{3}\frac{\eta}{s}\left(T_0|\Omega|^2 - ic_\sigma\omega\right) i\omega q^2 = 0.$$

➥ Equivalent iff $Z(\chi) = 0$ ($q = 0$ under the map).

In general MIS-like models, the singulant dynamics is not governed by $\omega^{(NH)}(k=0)$ for general longitudinal flows!

Exception: Bjorken flow

We have confirmed our predictions by explicit numerical computations: factorial divergence with right singulant e.o.m



Singulants and γ_s

What is the physical meaning of the linear response problem?

- In any four-dimensional conformal fluid, sound channel modes obey [Grozdanov, Kovtun, Starinets & Tadic, '19]

$$\omega^2 + i\omega q^2 \gamma_s(\omega, q^2) - \frac{q^2}{3} = 0.$$

➤ γ_s: momentum-dependent sound attenuation length. Microscopic theory observable!

$$\delta\hat{\Pi}_\star(\omega, q) = \frac{2}{3}\mathcal{E}_0 \gamma_s(\omega, q^2) iq\delta\hat{u}(\omega, q).$$

In MIS-like models, the singulant e.o.m is set by the poles of γ_s

$$\gamma_s^{\text{BRSSS}} = \frac{\frac{4}{3}\frac{\eta}{s}}{T_0(1 - i\tau_\Pi(T_0)\omega)}, \quad \gamma_s^{\text{gHJSW}} = \frac{\frac{4}{3}\frac{\eta}{s}\left(T_0|\Omega|^2 - ic_\sigma\omega\right)}{-\omega^2 - 2iT_0\Omega_I\omega + c_{\mathcal{L}}q^2 + T_0^2|\Omega|^2}.$$

In the final part of the talk, we will argue that this is also true in AdS/CFT...

AdS/CFT: the geometry dual to a longitudinal flow

What is the bulk dual to a longitudinal flow?

➥ The gradient expansion of $\Pi_\star$ follows directly from the near-boundary behavior of the gradient expansion of the metric: fluid/gravity.

➤ Metric ansatz [Bhattacharyya, Loganayagam, Mandal, Minwalla & Sharma, '08],

$$ds^2 = -2U_\mu(X)dX^\mu(dr + \mathcal{V}_\nu(r,X)dX^\nu) + \mathcal{G}_{\mu\nu}(r,X)dX^\mu dX^\nu, \quad \mathcal{G}_{\mu\nu}U^\mu = 0.$$

➤ Flow-adapted boundary coordinate system, $X = (\tau, \sigma, x_\perp^1, x_\perp^2)$,

$$dh^2 = -e^{2a(\tau,\sigma)}d\tau^2 + e^{2b(\tau,\sigma)}d\sigma^2 + d\vec{x}_\perp^2, \quad U^\mu\partial_\mu = e^{-a(\tau,\sigma)}\partial_\tau \;\&\; Z^\mu\partial_\mu = e^{-b(\tau,\sigma)}\partial_\sigma.$$

$$\mathcal{V}_\mu dx^\mu = V_\tau d\tau + V_\sigma d\sigma, \quad \mathcal{G}_{\mu\nu}dx^\mu dx^\nu = \Sigma^2\left(e^{-2B}d\sigma^2 + e^{B}d\vec{x}_\perp^2\right)$$

➤ IR b.c.: infalling. UV b.c.: dh^2 boundary metric + Landau frame ($V_{\sigma,2} = 0$).

➤ Holo. ren.: $\boldsymbol{\Pi_\star = B_4}$.

AdS/CFT: the gradient expansion of the metric

1. Divide Einstein equations into *dynamical equations* & *constraint equations*.

- Dynamical equations: $\Pi_{\mu\nu} = \Pi_{\mu\nu}[\mathcal{E}, U]$. AdS/CFT analogue of $\Pi_{\mu\nu}$ e.o.m. in MIS-like models.

- Constraint equations: $\nabla^\mu T_{\mu\nu} = 0$.

2. Introduce the auxiliary bookkeeping parameter ϵ into the dynamical equations,

$$\tau \to \frac{\tau}{\epsilon}, \quad \sigma \to \frac{\sigma}{\epsilon}, \quad \vec{x}_\perp \to \frac{\vec{x}_\perp}{\epsilon}, \quad r \to r.$$

3. Introduce the formal power series ansatze

$$V_\tau = \sum_{n=0}^{\infty} V_\tau^{(n)} \epsilon^n, \quad V_\sigma = \sum_{n=0}^{\infty} V_\sigma^{(n)} \epsilon^n, \quad \Sigma = \sum_{n=0}^{\infty} \Sigma^{(n)} \epsilon^n, \quad B = \sum_{n=0}^{\infty} B^{(n)} \epsilon^n,$$

into 2. and expand around $\epsilon = 0$.

1+2+3: recursion relations for the gradient expansion of the metric.

Singulants in AdS/CFT

- We solve the recursion relations for the metric with the singulant ansatz

$$V_\tau^{(n)}(r,\tau,\sigma) \sim \sum_q \bar{V}_{\tau,q}(r,\tau,\sigma) \frac{\Gamma(n+\gamma_{V_\tau,q}(r,\tau,\sigma))}{\chi_q(r,\tau,\sigma)^{n+\gamma_{V_\tau,q}(r,\tau,\sigma)}}, \quad \text{etc.}$$

- Main assumption: same χ for V_τ, V_σ, Σ & B.

➥ Main consequence: χ is independent of r.

End result: eigenvalue problem for $U(\chi)$!

$$\partial_r^2 \bar{V}_\tau + \frac{4\partial_r \bar{V}_\tau}{r} + \frac{2\bar{V}_\tau}{r^2} - \frac{2e^{a-b}Z(\chi)}{3r^2}\partial_r \bar{V}_\sigma - \frac{2e^{a-b}Z(\chi)}{3r^3}\bar{V}_\sigma - \frac{e^a Z(\chi)^2}{3r^2}\bar{B} = 0,$$

$$\partial_r^2 \bar{B} + \left(\frac{1}{r} + \frac{4r}{f^{(0)}} - \frac{2U(\chi)}{f^{(0)}}\right)\partial_r \bar{B} - \left(\frac{3U(\chi)}{rf^{(0)}} + \frac{Z(\chi)^2}{3r^2 f^{(0)}}\right)\bar{B} - \frac{2e^{-b}Z(\chi)}{3r^2 f^{(0)}}\partial_r \bar{V}_\sigma - \frac{2e^{-b}Z(\chi)}{3r^3 f^{(0)}}\bar{V}_\sigma = 0,$$

$$\partial_r^2 \bar{V}_\sigma + \frac{\partial_r \bar{V}_\sigma}{r} - \frac{4\bar{V}_\sigma}{r^2} + 2e^b Z(\chi)\partial_r \bar{B} = 0.$$

$f^{(0)}$: (part of) 0-th order solution, set by local $\mathcal{E}$

Analogous properties to MIS-like models!

- Ultralocal in the boundary directions: at (r, X): depend on collective fields @ X, do not depend on their boundary spacetime derivatives.

- Provided that $\mathcal{E}$, U, and $Z(\chi)$ are known at X, the eigenvalue problem can be solved to find $U(\chi)$.

Is this eigenvalue problem related to the poles of γ_s?

➥ How do we compute γ_s in Holography?

The computation of γ_s in AdS/CFT

γ_s can be computed in all-orders linearized hydrodynamics

[Bu & Lublinsky, '14]

- Exact constitutive relations of shear & sound channel in linear response:

$$\mathcal{E} = \mathcal{E}_0 + \delta\mathcal{E}, \quad U = \partial_t + \delta\vec{u}\cdot\vec{\partial}, \quad \Pi_{\mu\nu}dx^\mu dx^\nu = \delta\Pi_{ij}dx^i dx^j, \quad \frac{\delta\mathcal{E}}{\mathcal{E}_0}, |\delta\mathbf{u}|, \frac{\delta\Pi_{ij}}{\mathcal{E}_0} \ll 1$$

$$\boldsymbol{\delta\hat{\Pi}_{ij} = -\eta(\omega, k^2)\hat{\sigma}_{ij} - \xi(\omega, k^2)\hat{\pi}_{ij},}$$

$$\hat{\sigma}_{ij} = \frac{i}{2}\left(k_i\delta\hat{u}_j + k_j\delta\hat{u}_i - \frac{2}{3}\delta_{ij}k_l\delta\hat{u}^l\right), \quad \hat{\pi}_{ij} = -i\left(k_ik_j - \frac{1}{3}\delta_{ij}k_lk^l\right)k_m\delta\hat{u}^m,$$

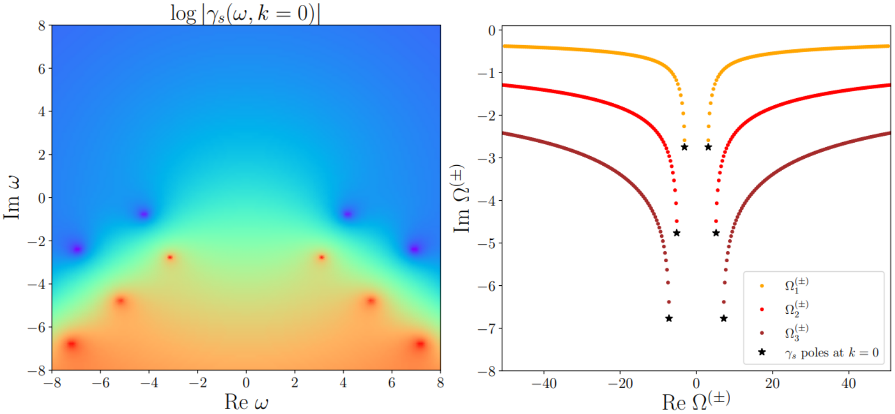
➤ η & ξ: momentum-dependent transport coefficients. Computed by solving a system of four coupled radial ODEs in a black brane background.

➤ $\boldsymbol{2\mathcal{E}_0\gamma_s = \eta - k^2\xi}$.

➥ Bu & Lublinsky construction can be adapted right away!

New results for γ_s @ complex ω & k

- γ_s: meromorphic function of ω @ fixed k. Real k: Christmass tree of poles.
- $\sim$ but $\neq$ QNM (agreement at $k = 0$ only).



Poles of γ_s: a new eigennvalue problem

$$P'' + \frac{P'}{r} - \frac{4P}{r^2} - 2ikQ' = 0,$$

$$Q'' + \frac{f + 4r^2 - 2ir\Omega_p}{rf} Q' + \frac{k^2 - 9i\Omega_p r}{3r^2 f} Q + \frac{2ik}{3r^2 f} P' + \frac{2ik}{3r^3 f} P = 0,$$

$$f = r^2 - \mu^4 r^{-2}$$

Dual to the singulant e.o.m!

Map: $\boldsymbol{P \to \pm e^{-b} \bar{V}_\sigma, \quad Q \to \bar{B}, \quad \mu \to r_h, \quad \Omega_p \to -iU(\chi), \quad k \to \pm iZ(\chi)}$

- **Large-order factorial growth for the gradient expansion in longitudinal flows is self-consistent in AdS/CFT.**
- Natural expectation based on MIS-like models results: factorial growth would show up!
- Still needs to be checked (any volunteers?)

Take-home points

- The asymptotic behavior of the gradient expansion simplifies dramatically at large order: **singulants**.
- Useful notion in MIS-like models and holography (*see the paper for RTA kinetic theory*).
- Singulants generalize nonhydrodynamic QNM far-away from thermal equilibrium: **simple e.o.m dictated by linear response**.
- Singulants are **intimately connected to the emergence of an optimal truncation** of the gradient expansion.

Open questions

Many open questions & avenues for further exploration!

New explicit computations and analytic insights required!

- Is there a systematic way of relating the initial conditions for the singulant fields to the initial data?
  - ➥ Partial progress in the linear response regime (see backup slides!).

  Can we utilize this putative relationship to place constraints on the hydrodynamization time at a given spacetime point?

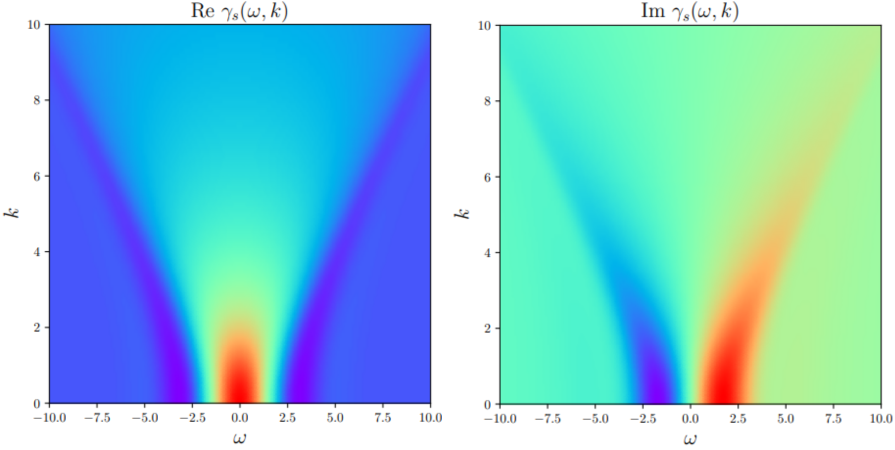
- Singulants beyond longitudinal flows.
- Singulants in other formulations of the hydrodynamic gradient expansion? Map between singulants in different formulations?
- Singulants in non-relativistic hydrodynamics.
- Novel uses of singulants in other perturbative expansions (e.g. large D)?

Many thanks for your time!

Backup slides

Backup slides for singulants

AdS/CFT: γ_s for real ω and k



Matching singulants to initial data in the linear response regime: a BRSSS example

- The fluctuations,

$$T(t,x) = T_0 + \lambda\,\delta T(t,x), \quad u(t,x) = \lambda\,\delta u(t,x), \quad \Pi_\star(t,x) = \lambda\,\delta\Pi_\star(t,x),$$

- The recursion relations,

$$\delta\Pi_\star^{(1)} = \frac{2}{3}\eta(T_0)\partial_x\delta u,\ \delta\Pi_\star^{(n+1)} = -\tau_\Pi(T_0)\partial_t\delta\Pi_\star^{(n)}.$$

- The solution in Fourier space,

$$\delta u(t,x) = \int_{\mathbb{R}} dk\, e^{ikx} \sum_{q=+,-,NH} \delta u_q(k) e^{-i\omega_q(k)t}, \quad \ldots$$

- Individual modes decouple at the level of the recursion relations,

$$\delta\Pi_\star^{(1)}(k) = \frac{2}{3}\eta(T_0) ik\delta u_q(k), \quad \delta\Pi_\star^{(n+1)}(k) = i\tau_\Pi(T_0)\omega_q(k)\delta\Pi_\star^{(n)}(k).$$

- Closed-form solution,

$$\delta\Pi_\star^{(n)}(k) = \frac{2}{3}\eta(T_0) i^n \tau_\Pi(T_0)^{n-1}\omega_q(k)^{n-1} k\delta u_q(k).$$

- Assumption: the analytical continuation of the Borel transform of the gradient expansion,

$$\delta\Pi_{\star}^{(B)}(t,x;\epsilon)=\sum_{n=1}^{\infty}\frac{\delta\Pi_{\star}^{(n)}(t,x)}{n!}\epsilon^{n},$$

has a well-defined Fourier transform, $\delta\Pi_{\star}^{(B)}(t,k;\epsilon)$.

– The contribution of the q-th mode is

$$\delta\Pi_{\star,q}^{(B)}(t,k;\epsilon)=\frac{2\eta(T_0)(e^{i\epsilon\tau_{\Pi}(T_0)\omega_q(k)}-1)}{3\tau_{\Pi}(T_0)\omega_q(k)}k\delta u_q(k).$$

- Main hypothesis: when the Fourier integral

$$\int_{\mathbb{R}}dk\, e^{ikx}\sum_{q=+,-,NH}\delta\Pi_{\star,q}^{(B)}(t,k;\epsilon)e^{-i\omega_q(k)t}$$

ceases to exist for a particular $\epsilon=\epsilon_s$, the original analytically continued Borel transform $\delta\Pi_{\star}^{(B)}(t,x;\epsilon)$ becomes singular. ϵ_s gives the value of the singulant.

- The convergence depends on: i) the large-k behaviour of the initial data, which determines the large-k behaviour of $\delta u_q(k)$ and ii) the large-k behaviour of the mode frequencies.

- Example: Lorentzian initial data

$$\delta u(0,x) = \delta \Pi_{\star}(0,x) = 0, \quad \delta T(0,x) = \frac{\alpha}{\pi(x^2+\alpha^2)}, \quad \delta T(0,k) = \frac{1}{2\pi} e^{-\alpha|k|}.$$

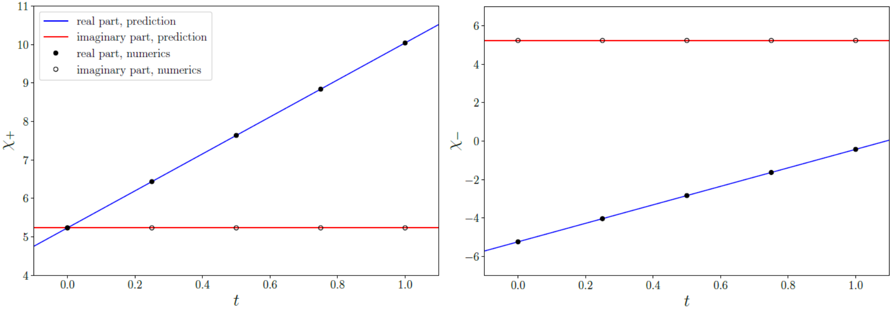
– The Fourier integral ceases to be well-defined whenever the argument of

$$\exp(i(\epsilon \tau_{\Pi}(T_0) - t)\omega_q(k) + ikx - \alpha|k|)$$

vanishes as $|k| \to \infty$ along the real axis.

– Prediction: $\epsilon_s = \chi_+, \chi_-, \chi_+^\star, \chi_-^\star$,

$$\chi_\pm = \frac{T_0 t}{\tau_{\Pi,0}} \pm \frac{T_0 x}{\tau_{\Pi,0} c_{UV}} + i \frac{\alpha T_0}{\tau_{\Pi,0} c_{UV}}.$$



The prediction matches precisely the singulants extracted from an explicit computation of the gradient expansion!