Factorization versus chaos in the IKKT matrix model

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Outline

- Overture: black holes, scrambling, chaos, replicas, factorization, (half-)wormholes... Connect the buzzwords!
- Is the factorization problem more general than holography?
- Factorization in non-perturbative string theory: type IIB (IKKT) matrix model [2203.10697]
- Dynamics, chaos and OTOC in type IIB matrix model
- Relation to proper quantum chaos and random matrices [2202.09443]

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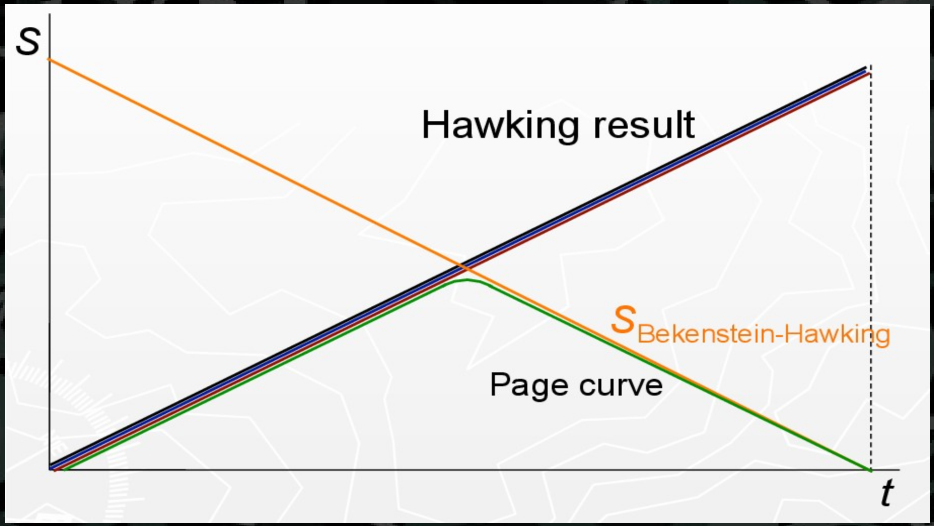
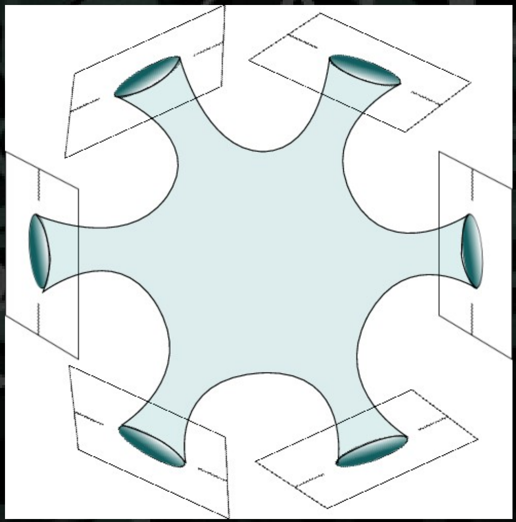
Replica wormholes and all that



Page curve of an evaporating black hole



Sum over saddles, including wormholes

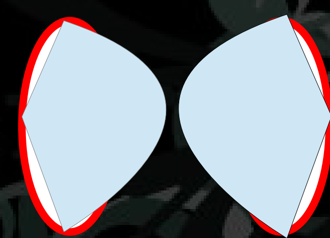
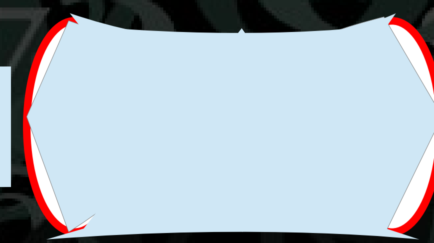
Almheiri, Hartman, Maldacena, Shaghoulian, Tajdini 1911.12333; Penington, Shenker, Stanford and Yang 1911.11977

The factorization puzzle

- Remember AdS/CFT: $Z_{\text{gravity}} = Z_{\text{CFT}}$

  gravity partition function = CFT partiton function
- Wormholes ruin the factorization:

$Z_1 =$ 

$Z_2 =$  $+$ 

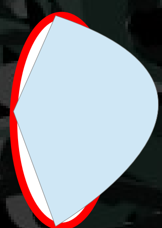
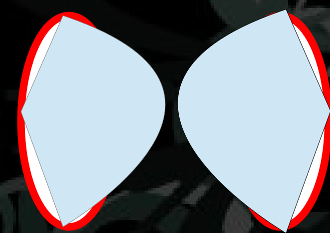
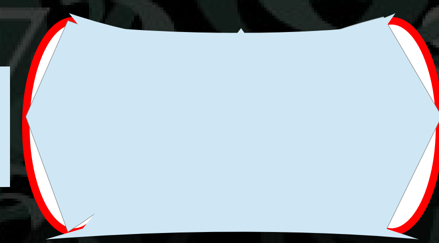
$$Z_2 \neq Z_1^2 \quad ?!$$

- We can live with $Z_{1g}^2 \neq Z_{2g}$ but we do expect $Z_{1\text{CFT}}^2 = Z_{2\text{CFT}}$

Factorization and averaging

- Remember AdS/CFT: $Z_{\text{gravity}} = Z_{\text{CFT}}$

  gravity partition function = CFT partiton function
- Wormholes ruin the factorization:

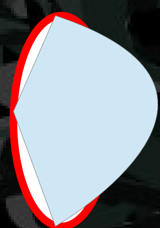
$\langle Z_1 \rangle =$  $\langle Z_2 \rangle =$  $+$ 

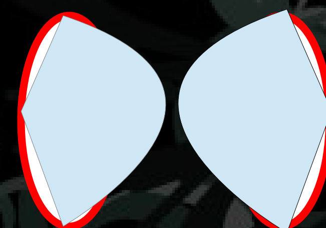
$$\langle Z_2 \rangle \neq \langle Z_1^2 \rangle$$

- We can easily have $\langle Z_{1\text{CFT}}^2 \rangle \neq \langle Z_{2\text{CFT}} \rangle$

Where does averaging come from?

- Averaging over what?
- Is the average fundamental (over quenched disorder) or emergent (coarse-graining or time binning)?

$\langle Z_1 \rangle =$ 

$\langle Z_2 \rangle =$  $+$ 

$$\langle Z_2 \rangle \neq \langle Z_1^2 \rangle$$

- Many ideas: 2008.08570, 2103.16754, 2105.02129, 2105.08270, 2107.13130, 2110.06221, 2111.07863, 2111.11705, 2202.01372, 2203.09537, 2211.09398 ...

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Averaging and type IIB matrix model

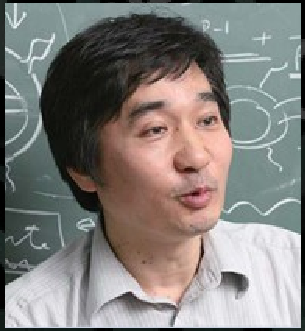
- The matrix formulation of type IIB string theory – Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) model
- Perfect testing ground for our puzzle:

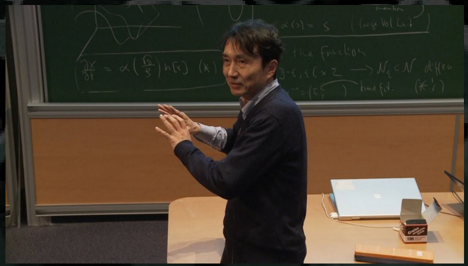
  - rich dynamics, including brane configurations (full nonpertrubative string theory?)

  - 0-dimensional – no derivatives → simpler path integral structure than dynamical models

- Inspired by the 0-dimensional (time-frozen) SYK model (Saad, Shenker, Stanford & Yan [2103.16754])

IKKT model





N. Ishibashi	H. Kawai	Y. Kitazawa	A. Tsuchiya

- Discretization of the Schild action for type IIB string theory in 0 dimensions:

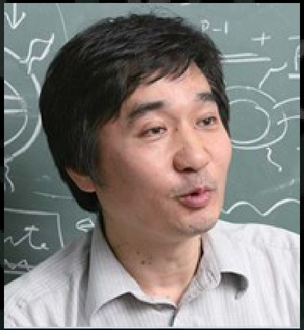
$$S = \frac{1}{4}\left[X^{\mu}, X^{\nu}\right]^2 + \frac{1}{2}\bar{\psi}_{\alpha}\Gamma_{\mu}\left[X^{\mu}, \psi_{\alpha}\right] + \beta$$

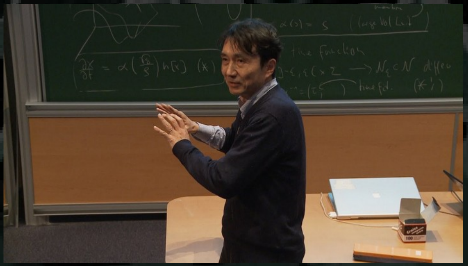
$$\mu = 1\ldots 10, \quad \alpha = 1\ldots 16$$

X^{μ} – bosonic coordinates – NxN Hermitian matrices

ψ_{α} – Majorana-Weyl spinors – NxN Hermitian matrices

IKKT model





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- Discretization of the Schild action for type IIB string theory in 0 dimensions
- Original papers: IKKT [hep-th/9612115]; H. Aoki, IKK & T. Tada [hep-th/9802085]; I. Chepelev, Y. Makeenko & K. Zarembo [he-pth/9701151]
- Review: K. L. Zarembo, Yu. M. Makeenko, An introduction to matrix superstring models, Physics-Uspekhi 41, 1, (1998).

Action and equations of motion

- Discretization of the Schild action for type IIB string theory:

$$S=\frac{1}{4}[A^{\mu},A^{\nu}]^{2}+\frac{1}{2}\bar{\psi}_{\alpha}\Gamma_{\mu}[A^{\mu},\psi_{\alpha}]+\beta$$

$$\mu=1\ldots10,\quad \alpha=1\ldots16$$

- Euclidean signature: well-defined partition function but not always real in presence of fermions:

$$Z_{E}=\sum_{N}\int D[A_{\mu}]\int D[\psi_{\alpha}]\int D[\bar{\psi}_{\alpha}]\exp\left(-S_{E}[A_{\mu},\psi_{\alpha},\bar{\psi}_{\alpha}]\right)$$

- Lorentzian signature: always real but not positive definite because of the time component → sign problem

$$Z_{L}=\sum_{N}\int D[A_{\mu}]\int D[\psi_{\alpha}]\int D[\bar{\psi}_{\alpha}]\exp\left(iS_{L}[A_{\mu},\Psi_{\alpha},\bar{\psi}_{\alpha}]\right)$$

Dp-brane solutions

- Remember: IIB string theory has Dp brane excitations with p odd: $p=-1$ – D-instantons, $p=1$ – strings, etc.
- D-instantons – points in spacetime as elementary degrees of freedom; any configuration is a collection of N instantons
- Single Dp brane solution of the matrix model (IKKT 1997, Aoki, IKK, Tada & Tsuchiya 1999):

$$A_\mu = \left(q_1, k_1, q_2, k_2 \ldots q_{(p+1)/2}, k_{(p+1)/2}, 0, \ldots 0\right), \qquad [q_\mu, k_\nu] = i \frac{L^2}{2\pi N^{2/(p+1)}}$$

- Hermitian random matrices q_i, k_i with compactification radius L_i and eigenvalues bounded as $0 \le \lambda_j^{p_i}, \lambda_j^{p_i} \le L_i$

$j = 1 \ldots N$

D-string solutions

- For our purposes a D-string is good enough:

$$A_\mu = \left(A_{0,} A_1, 0, \ldots 0\right), \qquad [A_0, A_1] = i\frac{L^2}{2\pi N}$$

- A pair of D-strings at distance l with angle θ between them:

$$A_0 = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \qquad A_1 = \begin{pmatrix} p\cos\theta & 0 \\ 0 & p\cos\theta \end{pmatrix}$$

$$A_3 = \begin{pmatrix} l/2 & 0 \\ 0 & -l/2 \end{pmatrix} \qquad A_4 = \begin{pmatrix} -p\sin\theta & 0 \\ 0 & p\sin\theta \end{pmatrix}$$

- BPS and non-interacting for $\theta = 0$

Averaging regimes

- Divide the fields into slow (quenched, semi-classical) and fast degrees of freedom (here given for bosons):

$$A_\mu = A_\mu + a_\mu \qquad Z_E = \int D[a_\mu] \int D[A_\mu] \exp\left(-S_{\text{IKKT}}[A_\mu + a_\mu]\right)$$

elliminate a_μ

$$S_{\text{IKKT}}[A_\mu + a_\mu]$$

elliminate A_μ

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$$S_{\mathrm{IKKT}}[A_\mu + a_\mu]$$

elliminate a_μ

$$\int D[a_\mu] e^{-S_{\mathrm{IKKT}}[A_\mu + a_\mu]} = e^{-W(A_\mu)}$$

elliminate A_μ

$\int D[A_\mu] \delta\left(A_\mu - A_\mu^{(0)}\right)$

$$e^{-S_{\mathrm{IKKT}}[A_\mu + a_\mu]} \rightarrow e^{-S_{A_\mu^{(0)}}(a_\mu)}$$

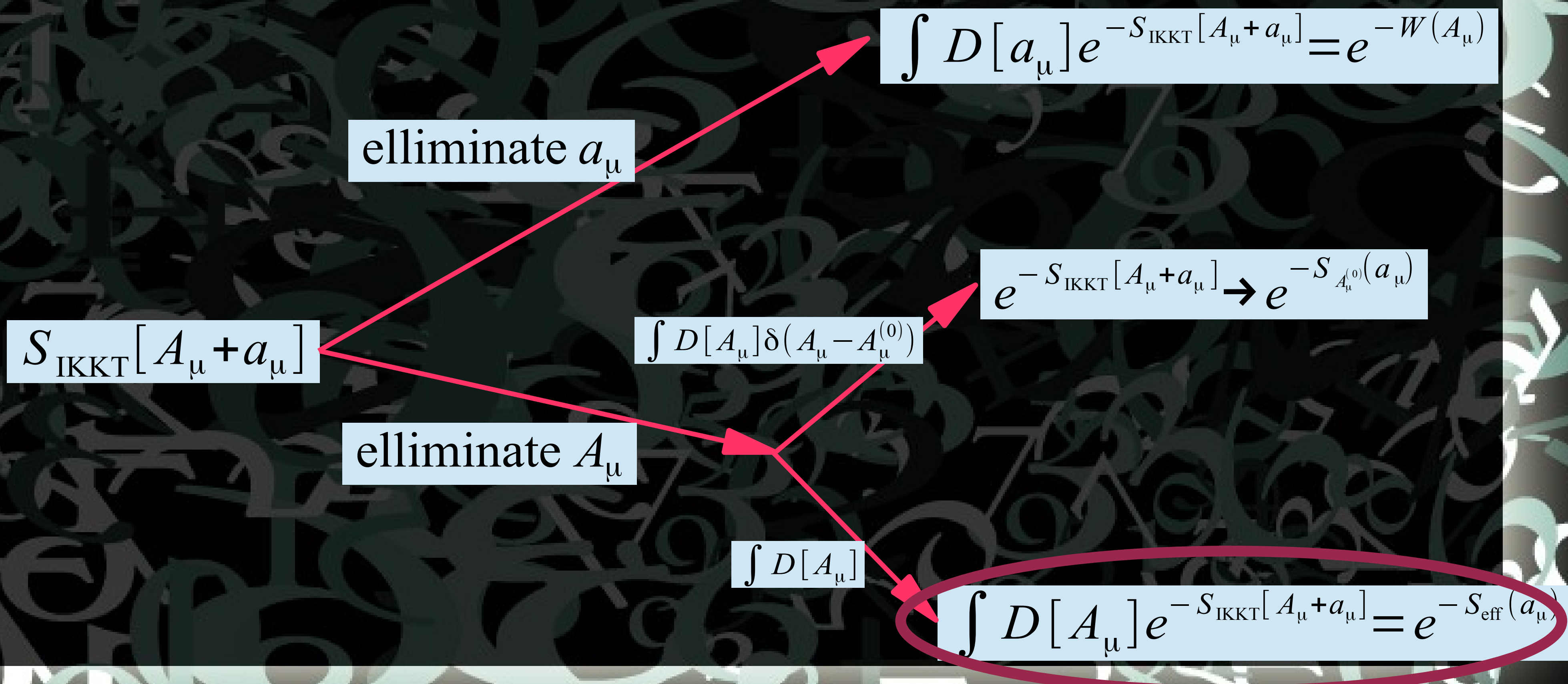
$\int D[A_\mu]$

$$\int D[A_\mu] e^{-S_{\mathrm{IKKT}}[A_\mu + a_\mu]} = e^{-S_{\mathrm{eff}}(a_\mu)}$$

Averaging regimes

- Divide the fields into slow (quenched, semi-classical) and fast degrees of freedom (here given for bosons):

$$\mathcal{A}_\mu = A_\mu + a_\mu \qquad Z_E = \int D[a_\mu] \int D[A_\mu] \exp\left(-S_{\rm IKKT}[A_\mu + a_\mu]\right)$$



Annealed D branes

- Divide the fields into slow (quenched, semi-classical) and fast degrees of freedom (here given for bosons):

$$A_\mu = A_\mu + a_\mu \qquad Z_E = \int D[a_\mu] \int D[A_\mu] \exp\left(-S_{\mathrm{IKKT}}[A_\mu + a_\mu]\right)$$

- Integrate over the slow pseudorandom matrices A_μ → study the configurations of a_μ averaged over "disorder"
- A_μ are random Hermitian matrices with eigenvalue cutoff $\wp(\lambda_i^{1,2}), \quad i = 1 \ldots N$ (Gaussian or hard – no matter)
- Partition function summed first over slow and then over fast variables:

$$Z_E = \int D[a_\mu] \int D[A_\mu] \exp\left(-S[A_\mu + a_\mu]\right) \rightarrow \int D[a_\mu] \exp\left(-S_{\mathrm{eff}}[a_\mu]\right)$$

D branes in presence of disorder – the big issue

- Big issue: Does the replica partition function factorize? This means:

$$\langle Z^n \rangle \ ?\approx? \ \langle Z \rangle^n + \text{small corrections}$$

- Relevant both in the context of AdS/CFT and in general
- The plan: compute Z, Z^2, Z^4 vs. $\langle Z \rangle, \langle Z^2 \rangle, \langle Z^4 \rangle$
- Quenched or annealed it doesn't matter here – can detect the nonfactorization either way (see Engelhardt, Fischetti, Maloney 2007.07444)

Collective field formalism

- The trick: collective fields – used for SYK and similar models (Sachdev et al 2017, Saad-Shenker-Stanford-Yao 2103.16754)

$$\langle Z \rangle = \int Da_\mu \int D\lambda_i \int Dg \exp\left[-a_\mu^\dagger P^2 a_\mu - \frac{4}{L^{2N-2}} \left(\mathrm{Tr}\, g - \mathrm{Tr}\, a_\mu^\dagger a_\mu \right) \right] \delta\left(g - a_\mu^\dagger a_\mu \right) \wp(\lambda_i)$$

$$\langle Z \rangle = \int Da_\mu \int D\lambda_i \int Dg \int Ds \exp\left[-a_\mu^\dagger P^2 a_\mu - \frac{4}{L^{2N-2}} \left(\mathrm{Tr}\, g - \mathrm{Tr}\, a_\mu^\dagger a_\mu \right) - i\, s \left(g - a_\mu^\dagger a_\mu \right) \right] \wp(\lambda_i)$$

$$\langle Z \rangle = \int Dg \int Ds \exp\left[-\frac{1}{2} \log\det s - i\, s\, g - \frac{4}{L^{2N-2}} \mathrm{Tr}\, g \right]$$

- Solution: $s = \frac{2i}{L^{2N-2}} I\,, \quad g = \frac{L^{2N-2}}{4} I$

- Effective action: $S_{\mathrm{eff}}^{(1)} \equiv -\log\langle Z \rangle = \left(N^2 - N \right) \log L + N \log\sqrt{2}$

Four replicas

- Replicas L, R, L', R'
- Two- and four-field combinations for bosons:

$g_{AB'} \equiv a_A^\dagger a_{B'}$ $\quad G_{AAB'B'} \equiv a_A^\dagger a_A a_{B'}^\dagger a_{B'}$ $\quad A, B \in \{L, R\}, \;\; A', B' \in \{L', R'\}$

- Two-field combinations for fermions: $\gamma_{AB} \equiv \frac{1}{N} a_A^\dagger a_B, \gamma_{AB'} \equiv \frac{1}{N} a_A^\dagger a_B'$
- Effective action:

$$S_{\text{eff}}^{(4)} = \frac{1}{2} \log\det s_{AB} + \frac{1}{2} \log\det s_{A'B'} + \frac{1}{2} \log\det S_{AAB'B'} - i S_{AAB'B'} G_{AAB'B'} + \frac{8}{L^{2N-4}} \operatorname{Tr} g_{AA} \operatorname{Tr} g_{BB} - \frac{4}{L^{2N-4}} \operatorname{Tr} G_{AAB'B'}$$

- Hubbard-Stratonovich fields $S_{AAB'B'}, \;\; s_{AB'}, \;\; \sigma_{AB}, \;\; \sigma_{AB'}$
- Wormhole couplings $\sigma_{LR}, \;\; \sigma_{L'R'}$
- Half-wormhole couplings $s_{LR'}, \;\; S_{LLL'L'}, \;\; S_{LLR'R'}$

Four replicas - solutions

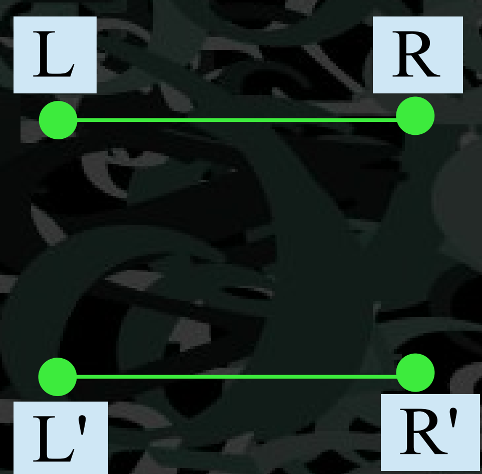
- Trivial solution: $\langle Z^4 \rangle \sim \langle Z \rangle^4$
- Wormhole: $\langle Z^4 \rangle \neq \langle Z \rangle^4$
- Half-wormhole: $\langle Z^4 \rangle \sim \langle Z \rangle^4$
- Wormhole + half wormhole: $\langle Z^4 \rangle \sim \langle Z \rangle^4$
- Full expressions for solutions and partition functions + fermionic contributions can be found in [2203.10697]

Wormhole saddle

$$s = 2iL^{N-2}\,\hat{I} \otimes \hat{E}\,,\; g = \frac{1}{4}L^{N-2}\,\hat{I} \otimes \hat{E}$$

$$S = G = 0$$

$\hat{I}_{N\times N}$ - internal unit matrix; $\hat{E}_{2\times 2}$ - replica space unit matrix

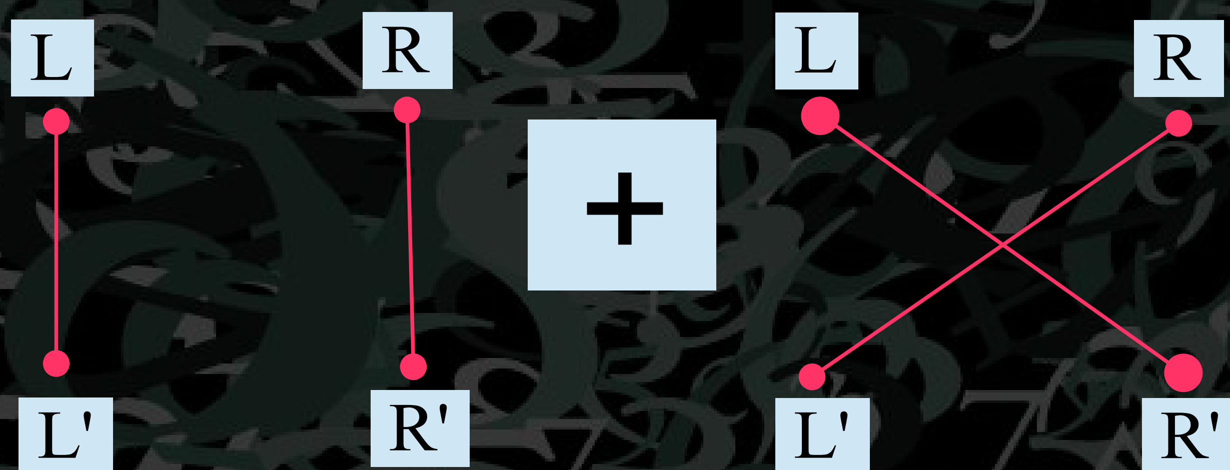


Half-wormhole saddles

$$s=g=0$$

$$S_{LLL'L'}=S_{LLR'R'}=S_{RRL'L'}=S_{RRR'R'}=4iL^{2N-4}\hat{I}$$

$$G_{LLL'L'}=G_{LLR'R'}=G_{RRL'L'}=G_{RRR'R'}=-\frac{1}{8}L^{2N-4}\hat{I}$$



Wormhole+Half-wormhole saddle

$$s=2iL^{N-2}\,\hat{I}\otimes\hat{E}\,,\; g=\frac{1}{4}L^{N-2}\,\hat{I}\otimes\hat{E}$$

$$S_{LLL'L'}=S_{LLR'R'}=S_{RRL'L'}=S_{RRR'R'}=4iL^{2N-4}\,\hat{I}\,,G=-\frac{1}{8}L^{2N-4}\,\hat{I}$$

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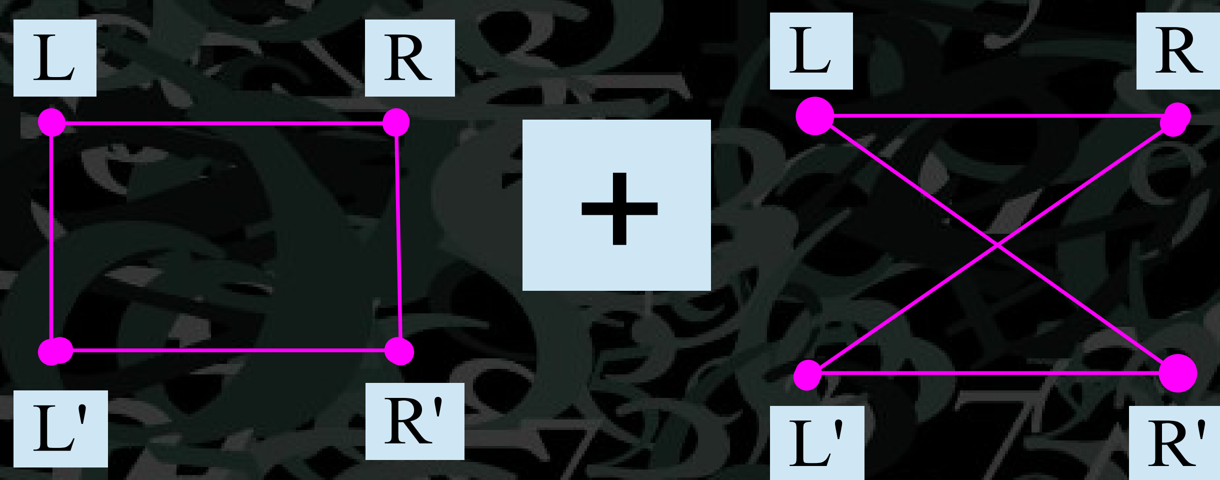
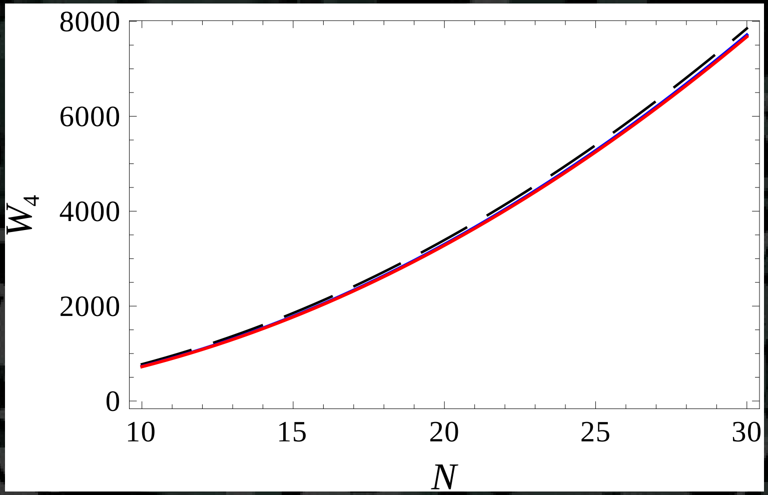
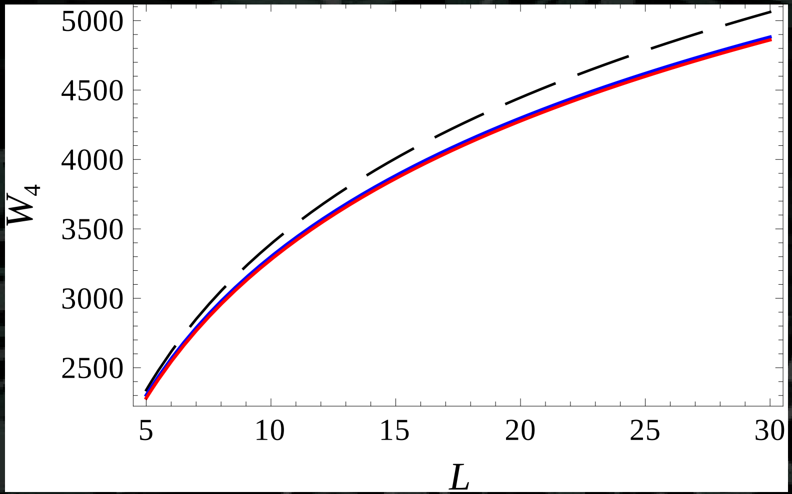


The factorizing solution is prefered

- Trivial saddle: trivially factorizing
- WH saddle: non-factorizing
- HWH saddle: factorizing
- WH+HWH saddle: factorizing





Numerical check: the four-replica action $S_{\mathrm{eff}}^{(4)}$ (blue – analytical, red - numerical) vs. 4 single copies $4\,S_{\mathrm{eff}}^{(1)}$ (black dashed)

Non-BPS effective actions and factorization

- Single-replica:

$$S^1_{\text{eff}} = -\frac{1}{2}\log\det s^2\left(s^2 - \sin^2\theta\, K^2 \zeta^2\right) + \mathrm{Tr}\left[i s g + i \zeta j + 2\omega j + \frac{2\left(1+\cos^2\theta\right)}{L^{2N-2}} g + \frac{l^2}{2}\sin^2\theta\, K^2 g\right]$$

$$j = \sin\theta\left(a_0^\dagger K a_3 + a_0 K a_3^\dagger\right)$$

$$S^1_{\text{eff}} = 2N^2 \log L + N^2 \log 4L\left[\left(1+\cos^2\theta + l^2\sin^2\theta\right)\sin\theta\right]$$

- Four replicas: $S^{(4)}_{\text{eff}} = S^{(4;1)}_{\text{eff}} + S^{(4;2)}_{\text{eff}} + S^{(4;3)}_{\text{eff}}$

$$S^{(4;1)}_{\text{eff}} = -\frac{1}{2}\log\det s^2_{AB}\left(s^2_{AB} - \sin^2\theta\, K^2 \zeta^2_{AB}\right)$$

$$S^{(4;2)}_{\text{eff}} = \mathrm{Tr}\left[i s_{AA} g_{AA} + i \zeta_{AB} j_{AB} + i S_{AAB'B'} G_{AAB'B'} + S^2_{AAB'B'}\left(S^2_{AAB'B'} - \sin^2\theta\, K^2 \zeta_{AAB'B'}\right)\right]$$

$$S^{(4;3)}_{\text{eff}} = \frac{2}{L^{2N-2}}\mathrm{Tr}\, g_{AA}\,\mathrm{Tr}\, g_{B'B'} - \frac{4}{L^{2N-4}}\mathrm{Tr}\, G_{AAB'B'}$$

$$S^{(n)}_{\text{eff}} = 2nN^2 \log L + nN^2 \log 4L\left[\left(1+\cos^2\theta + l^2\sin^2\theta\right)\sin\theta\right]$$

Non-BPS effective actions and factorization

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$$S^1_{\text{eff}} = -\frac{1}{2}\log\det s^2(s^2 - \sin^2\theta K^2 \zeta^2) + \text{Tr}\left[i s g + i\zeta j + 2\omega j + \frac{2(1+\cos^2\theta)}{L^{2N-2}} g + \frac{l^2}{2}\sin^2\theta K^2 g \right]$$

$$j = \sin\theta\left(a_0^\dagger K a_3 + a_0 K a_3^\dagger\right)$$

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$$S^{(4;2)}_{\text{eff}} = \text{Tr}\left[i s_{AA} g_{AA} + i\zeta_{AB} j_{AB} + i S_{AAB'B'} G_{AAB'B'} + S^2_{AAB'B'}\left(S^2_{AAB'B'} - \sin^2\theta K^2 \zeta_{AAB'B'}\right)\right]$$

$$S^{(4;3)}_{\text{eff}} = \frac{2}{L^{2N-2}}\text{Tr}\, g_{AA}\,\text{Tr}\, g_{B'B'} - \frac{4}{L^{2N-4}}\text{Tr}\, G_{AAB'B'}$$

$$S^{(n)}_{\text{eff}} = 2nN^2 \log L + nN^2 \log 4L\left[(1+\cos^2\theta + l^2\sin^2\theta)\sin\theta\right]$$

Dominant contribution to $S^{(n)}_{\text{eff}}$ for any n

Interacting vs. non-interacting

- Non-interacting $\theta = 0$ case is nearly identical to the single-string case
- ***Outcome: in all cases the optimal (minimal-action) solution factorizes.***
- Trivial factorization in interacting systems vs. nontrivial in a BPS-protected system

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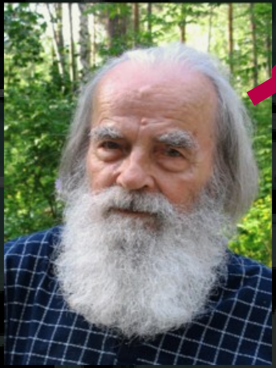
The confusing terminology of quantum chaos

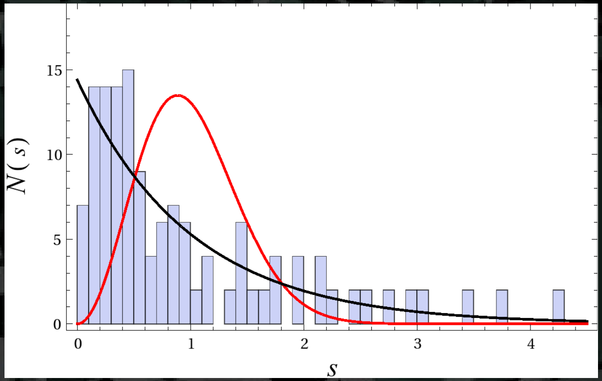
Anyone who uses words "quantum" and "chaos" in the same sentence should be hung on a tree in the park behind the Niels Bohr institute!



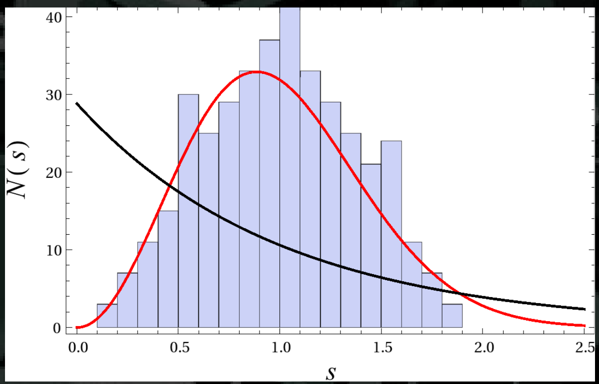
Boris A. Chirikov, Ups. Fiz. Nauk 71, 112, (1973).

Factorizing solutions have chaotic level statistics

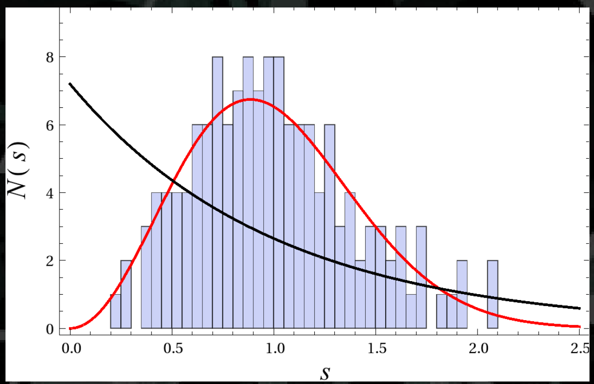
- Trivial saddle: trivially self-averaging, factorizing, regular
- WH saddle: self-averaging, regular
- HWH saddle: factorizing, chaotic
- WH+HWH saddle: self-averaging+factorizing, chaotic



WH - regular



HWH – chaotic



WH+HWH - chaotic

Black – Poisson, Red – Gaussian Unitary Ensemble

Dynamics in the IKKT model

- Lorentzian signature $Z_L = \sum_N \int D[A_\mu] \int D[\psi_\alpha] \int D[\bar{\psi}_\alpha] \exp\left(iS_L[A_\mu, \psi_\alpha, \bar{\psi}_\alpha]\right)$
- Zero-dimensional → no time. How to introduce dynamics?

- Type IIB
- Annealed regime
- The correlations between the eigenvalues of A_0 and A_i lead to dynamics
- $S_L \sim N \int D[A_\mu][A_\mu, A_\nu]^2$
- $a_\mu \sim O(1)$

- Quenched Eguchi-Kawai
- Quenched regime
- The commutator $[A_0, \cdot]$ acts as discretized time derivative: $A_0 \rightarrow L^{-1/4} \operatorname{diag}(\Omega_1 - \Omega_2, \ldots \Omega_n - \Omega_{n-1})$
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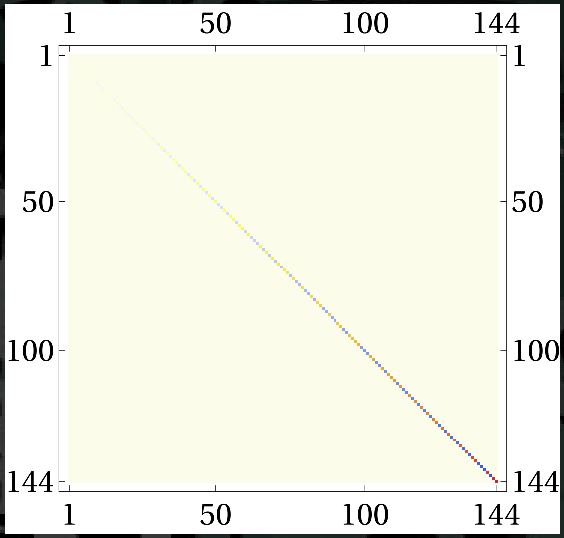
Dynamics in type IIB string regime

- Proposed originally for cosmological applications (Nishimura, Tsuchiya, Kim, Ito, 1108.1540; 1311.5579)
- Consider A_0 as the "time operator" and its eigenvalues α_i as discrete time increments:

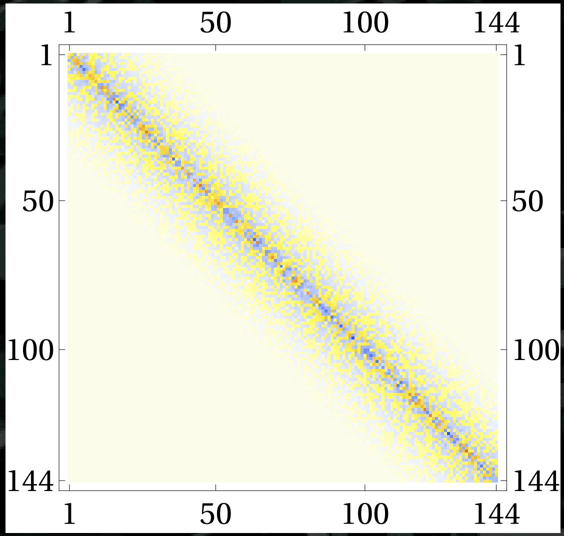
$$0<\alpha_1<\alpha_2<\ldots\alpha_N$$

$$t_I=\frac{1}{n}\sum_{j=0}^{n}\alpha_{I+j}, \qquad I=0\ldots N-n$$

- Time instants ~ $n\times n$ blocks



A_0



A_1

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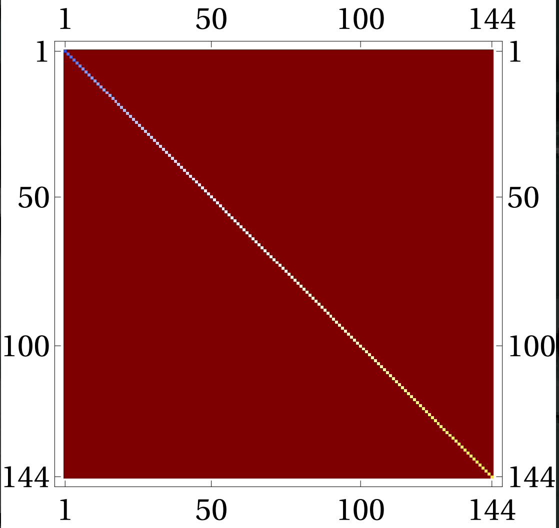
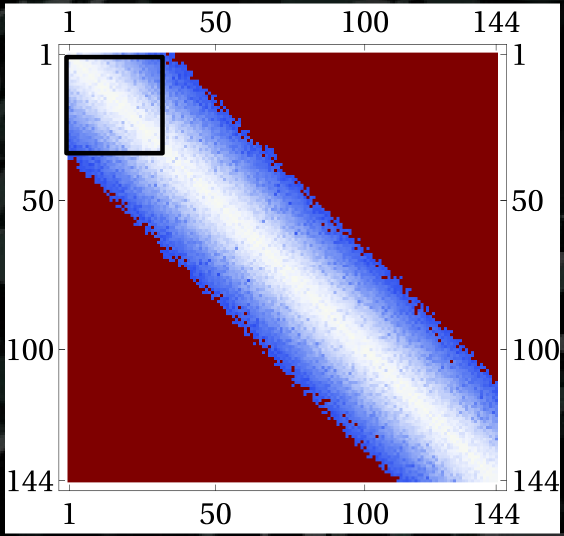


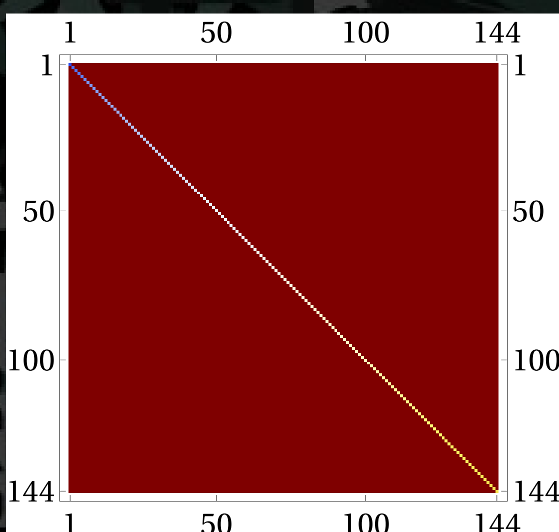
$\log A_0$



$\log A_1$

Dynamics in type IIB string regime

- Can consider dynamics as the series $\widetilde{O}(t_I) \equiv (O_{I+i,I+j})_{n \times n}$
- Numerical result (first found in 1108.1540): off-diagonal elements $O_{I+i,I+j}$ in a typical operator O decay rapidly when $|i-j|<n$ for some n



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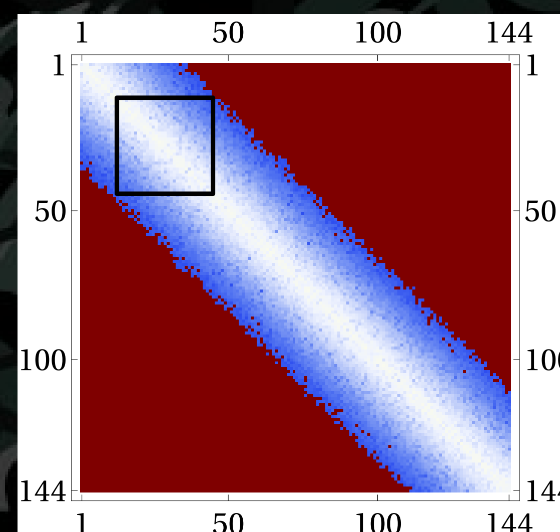

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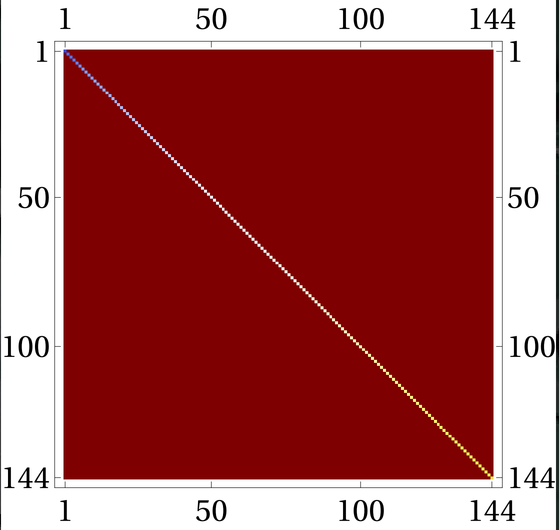

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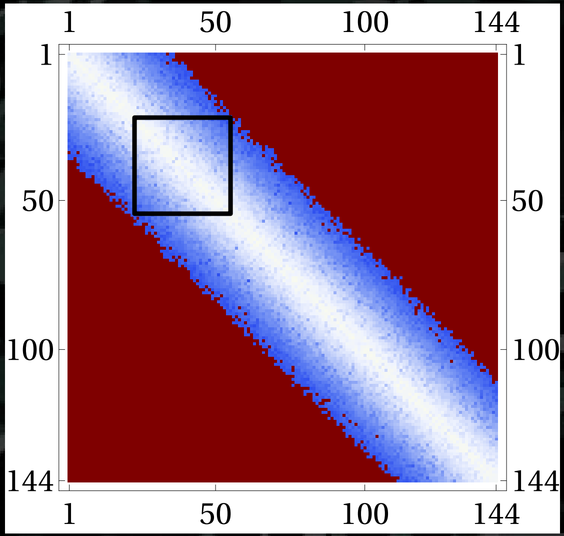

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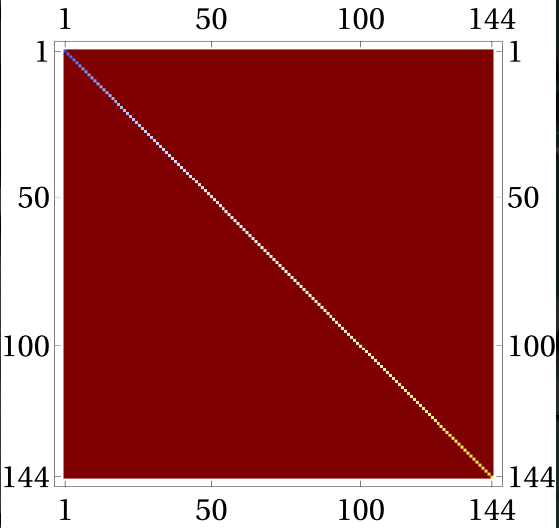
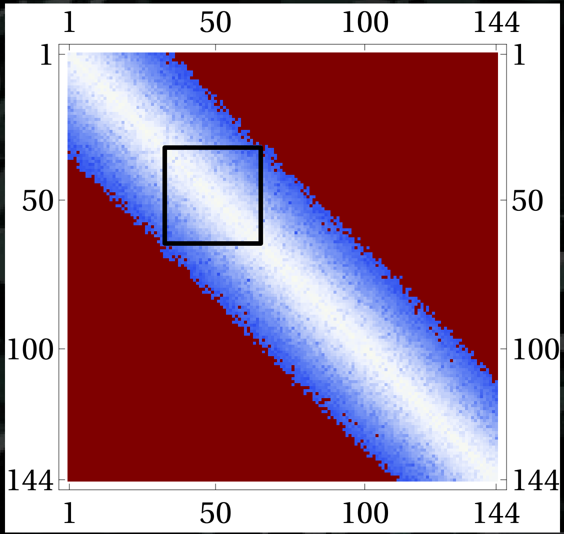


$\log A_0$



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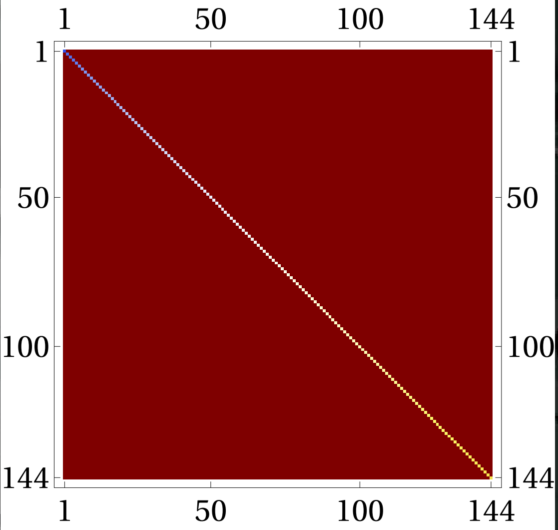
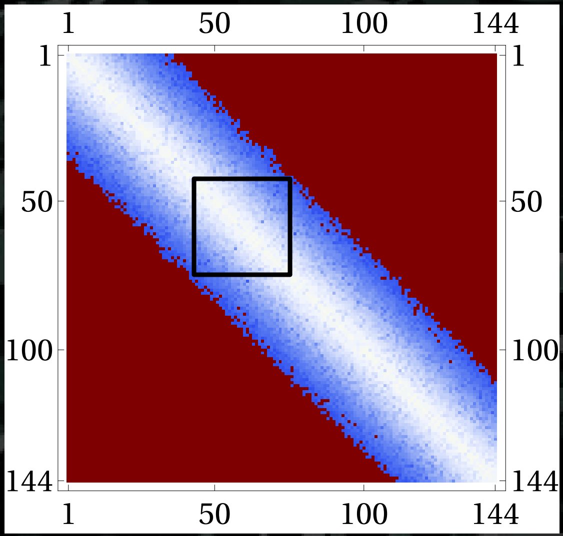


$\log A_0$



$\log A_1$

(O)TOC in type IIB regime

- The usual definition of TOC and OTOC applied to matrices (coordinates $X \equiv A_1, \; Y \equiv A_2$):

$$C(t_I) \equiv \langle |[\widetilde{X}_I, \widetilde{Y}_0]|^2 \rangle = 2(\mathrm{TOC} - \mathrm{OTOC})$$

$$\mathrm{TOC} = \langle \widetilde{X}_I^\dagger \widetilde{X}_i \widetilde{Y}_0^\dagger \widetilde{Y}_0 \rangle = \frac{1}{Z_L} \int D[X_\mu] \widetilde{X}_I^\dagger \widetilde{X}_I \widetilde{Y}_0^\dagger \widetilde{Y}_0 e^{iS_L}$$

$$\mathrm{OTOC} = \langle \widetilde{X}_I^\dagger \widetilde{Y}_0^\dagger \widetilde{X}_I \widetilde{Y}_0 \rangle = \frac{1}{Z_L} \int D[X_\mu] \widetilde{X}_I^\dagger \widetilde{Y}_0^\dagger \widetilde{X}_I \widetilde{Y}_0 e^{iS_L}$$

- Crucial analytical trick: separation into diagonal elements $X_{ii} \equiv q_i, Y_{ii} \equiv p_i$ and off-diagonal elements $x_{ij}, y_{ij} \ll q_i, p_i$

(O)TOC in type IIB regime

- Crucial analytical trick: separation into diagonal elements $X_{ii} \equiv q_i, Y_{ii} \equiv p_i$ and off-diagonal elements $x_{ij}, y_{ij} \ll q_i, p_i$
- Schematically:

$$\mathrm{TOC} = \sum_{i=1}^{n} |q_{I+i}|^2 |p_i|^2 + \sum_{i,j=1}^{n} |q_{I+i}|^2 |y_{ij}|^2 + \sum_{i,j=1}^{n} |p_i|^2 |x_{I+i,I+j}|^2 + \dots$$

$$\mathrm{OTOC} = \sum_{i=1}^{n} |q_{I+i}|^2 |p_i|^2 +$$
$$+ \sum_{i,j=1}^{n} \left(q_{I+i}^* p_{I+j} - \mathrm{c.c.}\right) \left(\tilde{x} \cdot \tilde{y}\right)_{I+i,i} + \sum_{i,j=1}^{n} \left(p_i^* q_i + \mathrm{c.c.}\right) \left(\tilde{x} \cdot \tilde{y}\right)_{I+i,i} + \dots$$

$$C(t_I) = 2 \sum_{i,j=1}^{n} |q_{I+i}|^2 |y_{ij}|^2 + 2 \sum_{i,j=1}^{n} |p_i|^2 |x_{I+i,I+j}|^2 + \text{subleading}$$

$$\tilde{X} = \mathrm{diag}(q_1 \dots q_{N-n}) + \tilde{x}, \quad \tilde{Y} = \mathrm{diag}(p_1 \dots p_{N-n}) + \tilde{y}$$

(O)TOC in type IIB regime

- Crucial analytical trick: separation into diagonal elements $X_{ii} \equiv q_i, Y_{ii} \equiv p_i$ and off-diagonal elements $x_{ij}, y_{ij} \ll q_i, p_i$
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$$\mathrm{TOC} = \sum_{i=1}^{n} |q_{I+i}|^2 |p_i|^2 + \sum_{i,j=1}^{n} |q_{I+i}|^2 |y_{ij}|^2 + \sum_{i,j=1}^{n} |p_i|^2 |x_{I+i,I+j}|^2 + \ldots$$

$$\mathrm{OTOC} = \sum_{i=1}^{n} |q_{I+i}|^2 |p_i|^2 + \\ + \sum_{i,j=1}^{n} \left(q_{I+i}^* p_{I+j} - \mathrm{c.c.}\right) \left(\tilde{x} \cdot \tilde{y}\right)_{I+i,i} + \sum_{i,j=1}^{n} \left(p_i^* q_i + \mathrm{c.c.}\right) \left(\tilde{x} \cdot \tilde{y}\right)_{I+i,i} + \ldots$$

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From numerics and large N arguments $\langle |q_{I+i}|^2 \rangle, \langle |p_{I+i}|^2 \rangle \sim \exp(r t_I)$

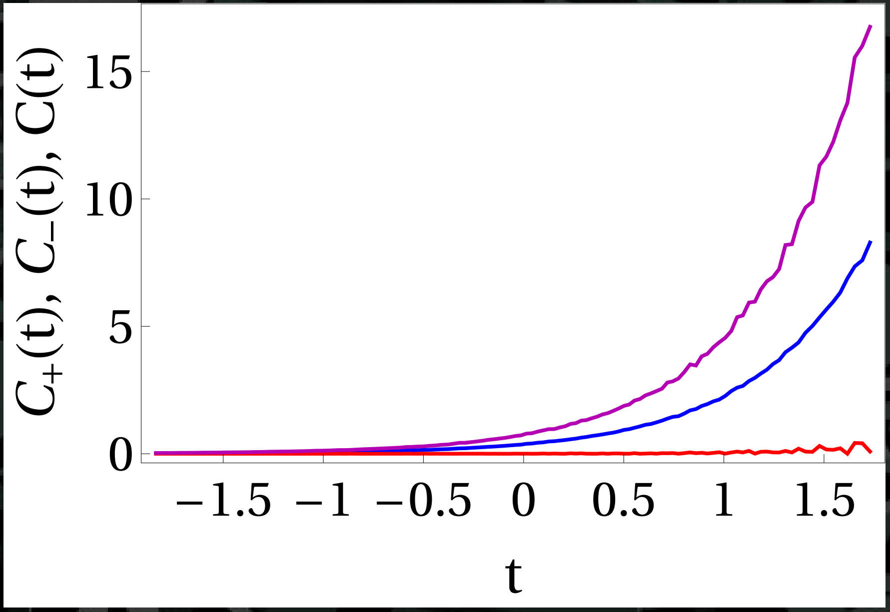
Non-equilibrium dynamics and no maximal chaos

$$C(t_I) = 2\sum_{i,j=1}^{n} |q_{I+i}|^2 |y_{ij}|^2 + 2\sum_{i,j=1}^{n} |p_i|^2 |x_{I+i,I+j}|^2 + \text{subleading}$$

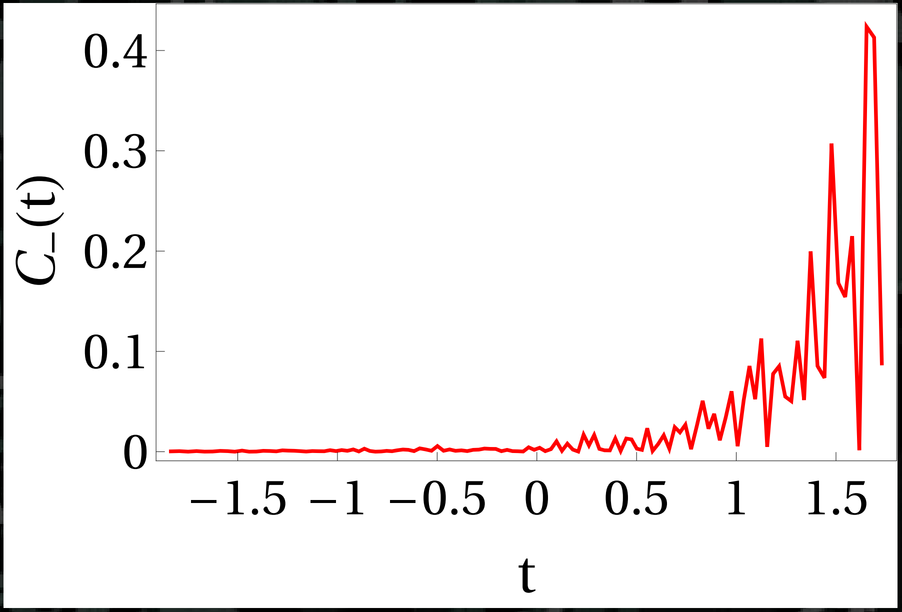
From numerics and large N arguments $\langle |q_{I+i}|^2 \rangle, \langle |p_{I+i}|^2 \rangle \sim \exp(r t_I)$

- Tempting to claim r as the Lyapunov exponent but...
- This is completely wrong! Remember the exponentially growing term ***comes from TOC, not OTOC!***
- Non-stationary TOC: no equilibrium solution, the geometry is non-stationary (in some contexts interpreted cosmologically)

Non-maximal chaos from Monte Carlo numerics



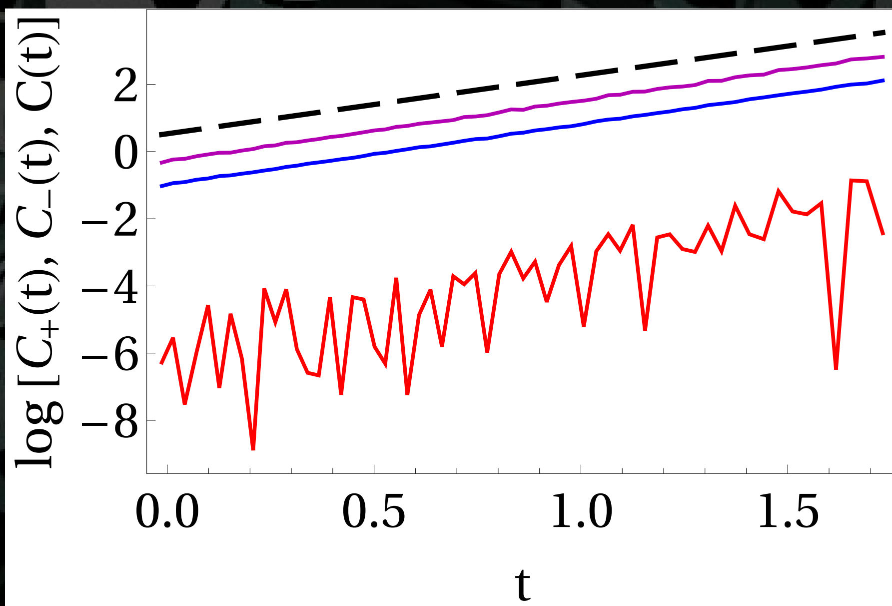
Regular exponential growth of TOC ($C_+(t)$, blue), absence of exponential growth of OTOC ($C_-(t)$, red) and their (doubled) difference ($C(t)$, violet)



Zoom-in onto the slow (sub-exponential) growth of OTOC, the signature of weak chaos

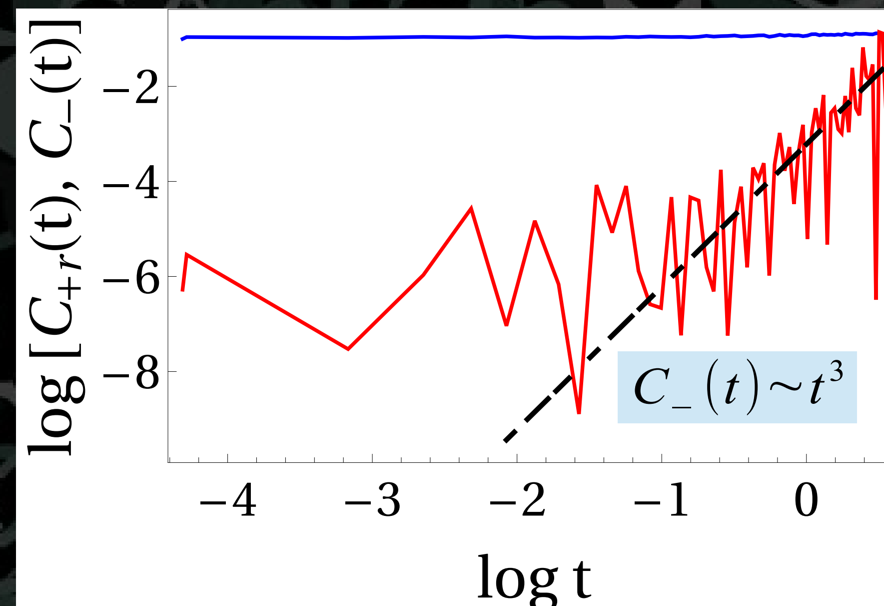
Non-maximal chaos from Monte Carlo numerics

$C_{+}(t), C(t) \sim \exp(rt)$



Log-linear plot confirming the exponential trend (black dashed line – exponential fit)

$C_{+r}(t) \equiv e^{-rt} C_{+r}(t) \sim \text{const.}$



Log-log plot of OTOC and rescaled TOC (by the exponential growth function) – power-law growth of OTOC appears

Annealed or quenched?

- Saddle points and factorization can equivalently be considered either in annealed or in quenched regime but the OTOC differs!
- Reason: OTOC is an expectation value!
- Saddle-point solutions: $\int D[A_\mu] \rightarrow S_{\mathrm{eff}};\ \delta S_{\mathrm{eff}}/\delta a_\mu = 0$
- Correlation functions: $\int D[A_\mu] \rightarrow S_{\mathrm{eff}};\ \int d[a_\mu] \rightarrow \langle O \rangle$
- Annealed: effectively sum over all saddle point solutions for $g_{AB'}$ and $G_{AAB'B'}$
- Quenched case = Eguchi-Kawai discrete Yang-Mills

Is there a morale to the story?

- Factorization is obtained generically in the IKKT matrix model of type IIB string theory, independently of AdS/CFT
- There are non-factorizing (wormhole) solutions but the dominant saddle is always factorizing (half-wormhole)
- Eigenvalue spectrum: chaotic ↔ factorizing saddle
- OTOC: maximal chaos ↔ quenched → dominant saddle → factorizing – realized in Eguchi-Kawai discrete Yang-Mills
- OTOC: weak chaos ↔ annealed → sum over saddles – factorizing and non-factorizing

Appendix I

Dynamics in the IKKT model

- Lorentzian signature $Z_L = \sum_N \int D[A_\mu] \int D[\psi_\alpha] \int D[\bar{\psi}_\alpha] \exp\left(i S_L[A_\mu, \psi_\alpha, \bar{\psi}_\alpha]\right)$
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- Quenched regime
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- $S_L \sim \frac{N}{L} \int D[A_\mu][A_\mu, A_\nu]^2$
- $a_\mu \sim 1/L^{1/4}$

OTOC in quenched EK regime

- Collective field formalism in fixed background allows the analytical treatment of the Schwinger-Dyson formalism (Kitaev 2015 and many others):

$$S = \mathrm{Tr}\left[-\frac{1}{4} a_\mu^\dagger \left(P^2 \delta_{\mu\nu} + 2 F_{\mu\nu}\right) a_\nu - \bar{c} P^2 c - 2\left(P^\mu a_\mu\right)\left(a^\nu a_\nu\right) - \frac{1}{2}\left[a_\mu, a_\nu\right]^2\right] \rightarrow$$

$$\rightarrow \mathrm{Tr}\left[-\frac{1}{4} \log\det\left(P^2 + 2F + \frac{3}{2\sqrt{L}} g + i\Sigma\right) - i\Sigma g - \frac{1}{2\sqrt{L}} g^2 - \frac{1}{4\sqrt{L}}\left(Pg + gP\right)^{-1} + \log\det P^2\right]$$

- Collective field g and the Hubbard-Stratonovich field Σ
- Original fields integrated out
- Saddle-point equations:

$$\frac{\partial S_{\mathrm{eff}}}{\partial g} = i\Sigma + \frac{3}{8\sqrt{L}} \det\left(P^2 + 2F + \frac{3}{2\sqrt{L}} g + i\Sigma\right)^{-1} - \frac{1}{4\sqrt{L}}\left(gPg\right)^{-1} = 0$$

$$\frac{\partial S_{\mathrm{eff}}}{\partial \Sigma} = -ig - \frac{i}{4} \det\left(P^2 + 2F + \frac{3}{2\sqrt{L}} g + i\Sigma\right)^{-1} = 0$$

OTOC in quenched EK regime

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$$\frac{\partial S_{\text{eff}}}{\partial g} = i\Sigma + \frac{3}{8\sqrt{L}} \det\left(P^2 + 2F + \frac{3}{2\sqrt{L}} g + i\Sigma\right)^{-1} - \frac{1}{4\sqrt{L}} (gPg)^{-1} = 0$$

$$\frac{\partial S_{\text{eff}}}{\partial \Sigma} = -ig - \frac{i}{4} \det\left(P^2 + 2F + \frac{3}{2\sqrt{L}} g + i\Sigma\right)^{-1} = 0$$

- Large-N solution along similar lines as SYK model (perturbatively in large N and the cutoff):

$$P_0 \rightarrow -i \frac{N}{L^{1/4}} \partial_t, \; P_0^2 \rightarrow -\frac{N^2}{L^{1/2}} \partial_t^2, \quad F_{0i} = -i \frac{N}{L^{1/4}} \partial_t (P_i \cdot)$$

$$g = \frac{\sqrt{L}}{4N^2} \frac{1}{-\omega^2 + P_i P^i} + O\left(1/L^{1/4}\right)$$

$$\Sigma = \frac{3}{2\sqrt{L}} g - \frac{1}{4\sqrt{L}} (gPg)^{-1} + O\left(1/L^{1/4}\right)$$

Maximal chaos in quenched EK regime

- OTOC from the sum of ladder diagrams:

$$C(t)=\exp\left(2\pi\frac{L^{1/4}}{N}t\right)f(t)$$

- In IKKT/EK model this is a zero-temperature calculation. No clear way to put IKKT at finite temperature. But...
- Duality with BFSS model at temperature T compactified along one direction on radius L:

$$B_\mu \rightarrow B'_\mu \equiv B_\mu / L^{1/4}$$

$$S_{\mathrm{BFSS}}=\int_0^\beta d\tau\left(-\left(D_\tau B_\mu\right)^2+\frac{1}{4}\left[B_\mu, B_\nu\right]^2\right)\rightarrow\frac{\beta L}{4}\left[B'_\mu, B'_\nu\right]^2\rightarrow\frac{N}{4}\left[A_\mu, A_\nu\right]^2$$

$$\beta=N/L, \quad T=L/N$$

$$C(t)=\exp\left(2\pi T t\right)f(t)$$

Maximal chaos in quenched EK regime

- Maximal chaos reconstructed for the discretized strongly coupled Yang-Mills:

$$C(t) = \exp(2\pi T t) f(t)$$

- The N/L (i.e. T) scaling of the result hinges crucially on not integrating over the couplings

Appendix II

Outline

- Overture: black holes, scrambling, chaos, replicas, factorization, (half-)wormholes... Connect the buzzwords!
- Is the factorization problem more general than holography?
- Factorization in non-perturbative string theory: type IIB (IKKT) matrix model [2203.10697]
- Dynamics, chaos and OTOC in type IIB matrix model
- Relation to proper quantum chaos and random matrices [2202.09443]

Few-body systems: chaos vs statistics

Large N quantum field theory is not quantum mechanics!

[arXiv:2202.09443[hep-th]]

Dragan Marković
FF Belgrade

Large N QFT: mean field	QM: no mean field:
Gravity dual	No gravity dual
Self-averaging over the chaotic dynamics or over background fields	Self-averaging only possible over the chaotic dynamics
Factorization expected	Factorization expected

Bose-Hubbard chain

- The motivation: is gravity crucial for (non)factorization?
- Apparently no! As long as the system is big and level statistics is Wigner-Dyson the n-replica partition function factorizes: $\langle Z^n \rangle \sim \langle Z \rangle^n$
- The Hamiltonian:

$$H = -J \sum_k \left(b^+_{k+1} b_k + b^+_k b_{k+1} \right) + \frac{U}{2} \sum_k n_k (n_k - 1)$$

- The appropriate regime to compare to the large-N IKKT model: quasiclassical limit $b_k \rightarrow b_k / \sqrt{N}, \quad [b_k, b^+_{k+1}] \rightarrow 1/N, \quad N \rightarrow \infty$

$$H = -\tilde{J} \sum_k \left(\psi^*_{k+1} \psi_k \right) + \frac{\tilde{U}}{2} \sum_k |\psi_k|^4, \qquad \sum_k |\psi_k|^2 = 1$$

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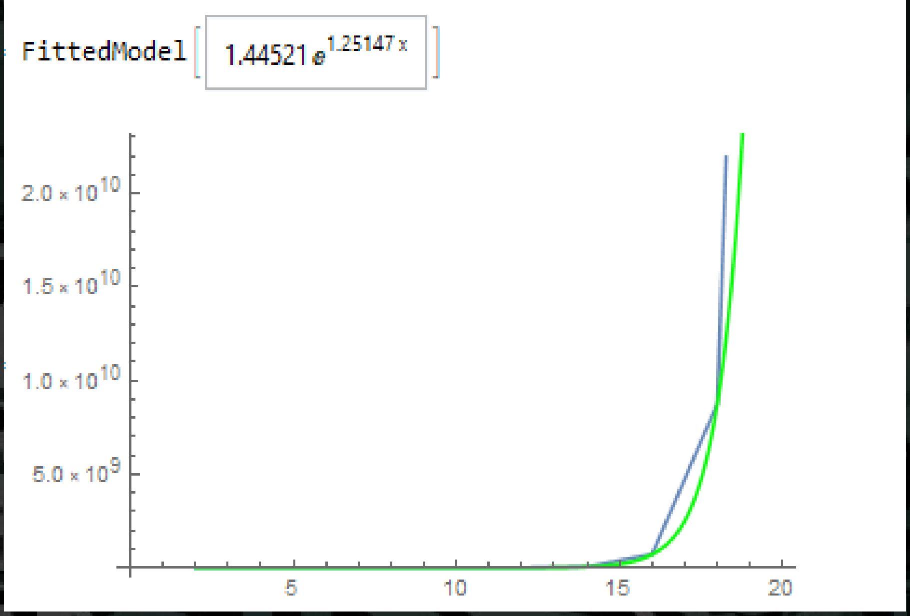
W_1, W_4



N

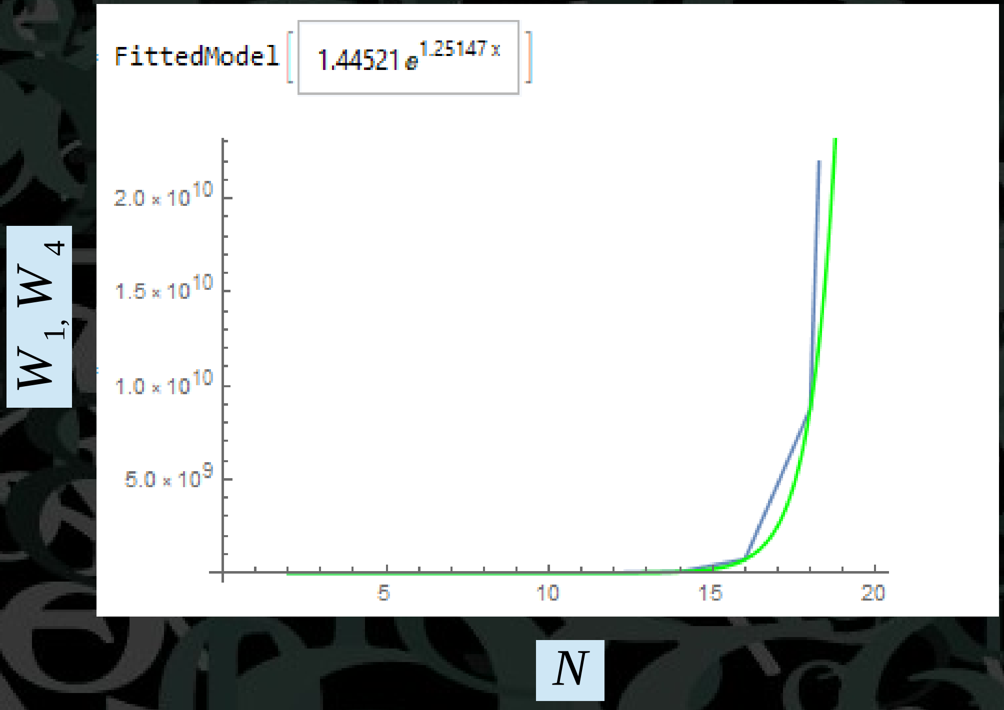
Light green: W_1

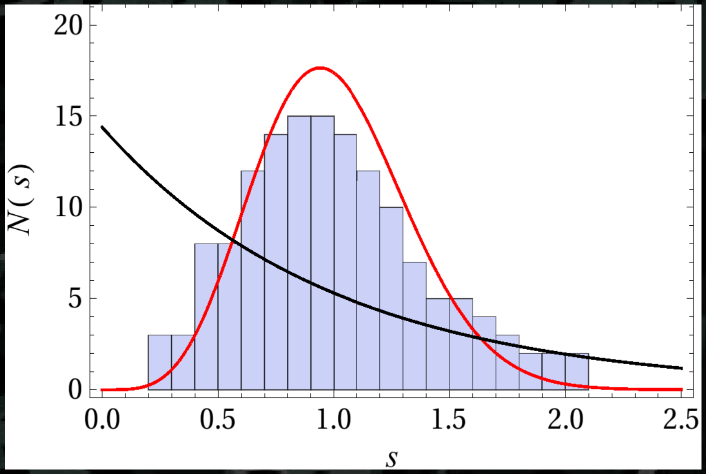
Dark green: W_4

$U = 10$

Strong chaos in the BH model

- Factorizing solution is the chaotic solution
- High-U regime, strongly nonintegrable





Red: Wigner-Dyson

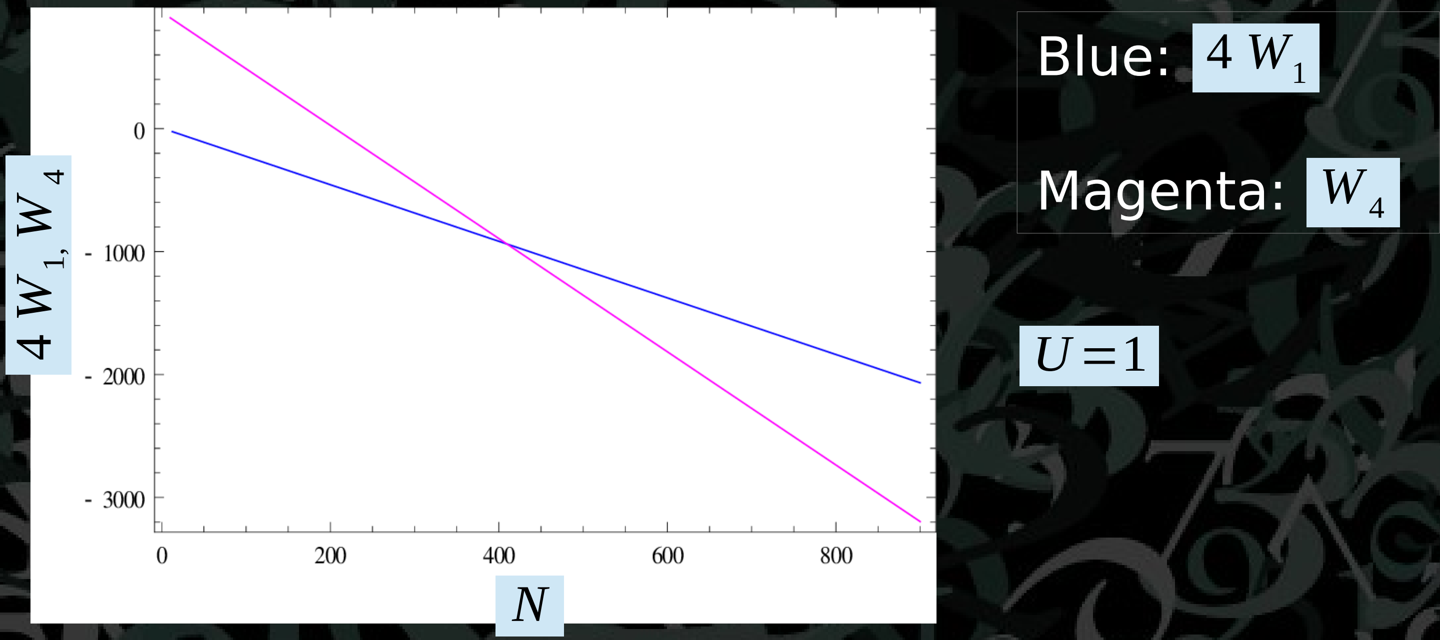
$N(s) \sim s^2 \exp(-\pi s^2)$

Black: Poisson

$N(s) \sim \exp(-s)$

Bose-Hubbard chain

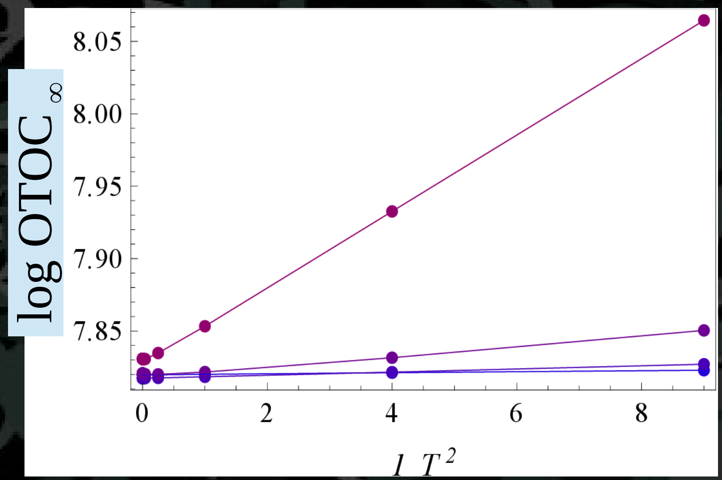
- Low U, near-integrable regime
- No factorization

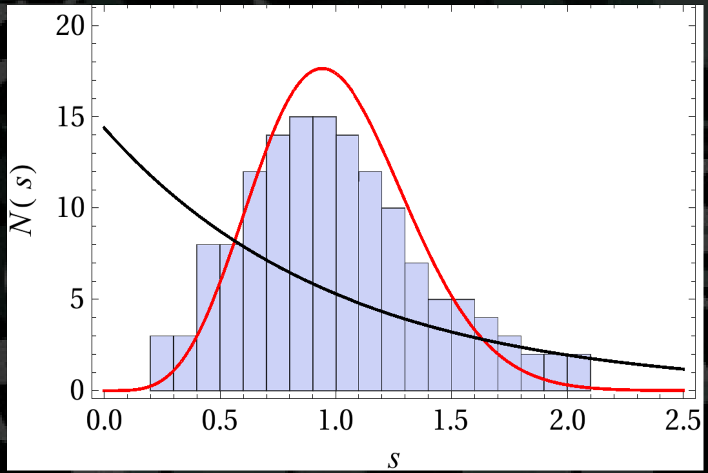


OTOC for strong vs weak chaos

Strong chaos

$$\log \mathrm{OTOC}_\infty = c_0 + c_1/T^2$$





Weak chaos

$$\log \mathrm{OTOC}_\infty = c_0 + c_1/T$$