COVARIANT PRESCRIPTIONS FOR HOLOGRAPHIC ENTANGLEMENT

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[based on work w/ Matt Headrick, 2208.10507]

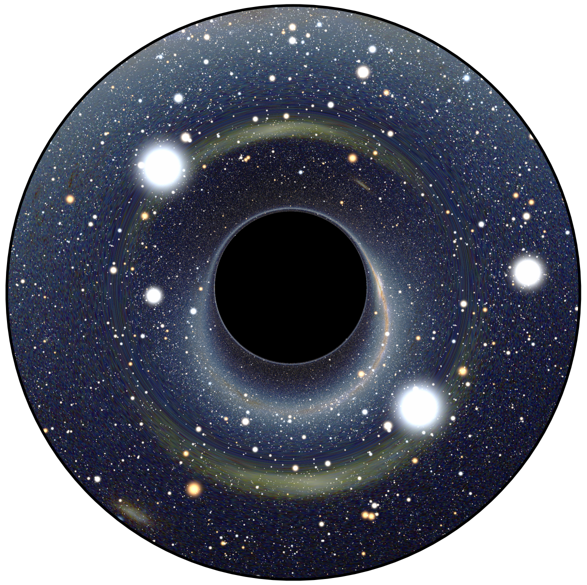
AdS/CFT after 25 years

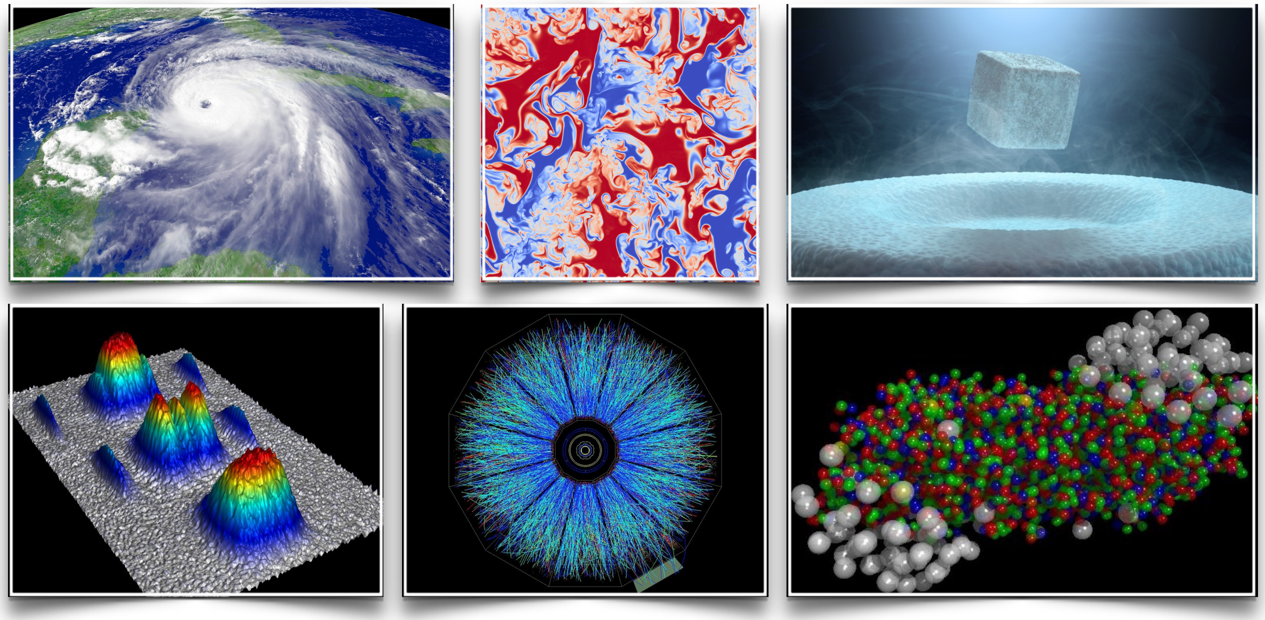
String theory (gravity) $\Longleftrightarrow$ field theory (no gravity)

"in bulk" = higher dimensions *"on boundary" = lower dimensions*

describes gravitating systems, e.g. black holes

describes experimentally accessible systems





Invaluable tool to:

- Study strongly interacting field theory (hard, but describes many systems) by working with higher-dimensional gravity on AdS (easy).
- Study quantum gravity in AdS (hard, but needed to understand spacetime) by using the field theory (easy for certain things)

Pre-requisite:

We need to understand the AdS/CFT dictionary…

- How does bulk spacetime emerge from the CFT?
  - Which CFT quantities give the bulk metric?
  - What determines bulk dynamics (Einstein's eq.)?
  - How does one recover a local bulk operator from CFT quantities?

- What part of bulk can we recover from a restricted CFT info?
  - What bulk region does a CFT state (at a given instant in time) encode?
  - What bulk region does a spatial subregion of CFT state encode?

- (How) does the CFT "see" inside a black hole?
  - (How) does it unitarily describe black hole formation & evaporation process?
  - How does it resolve curvature singularities?

Recent hints / expectations: entanglement plays a crucial role…

Entanglement Entropy (EE)

Suppose we only have access to a subsystem A of the full system = A + B. The amount of entanglement is characterized by Entanglement Entropy S_A :

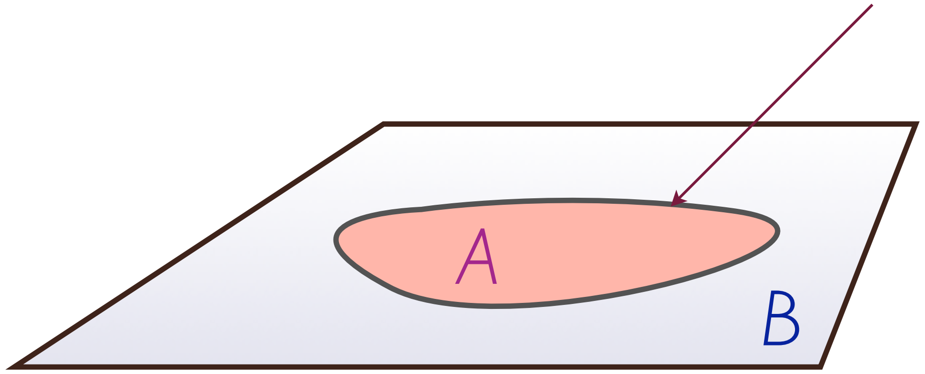
- reduced density matrix $\rho_A = \mathrm{Tr}_B \, |\psi\rangle\langle\psi|$

  (more generally, for a mixed total state, $\rho_A = \mathrm{Tr}_B \rho$)

- EE = von Neumann entropy $S_A = -\mathrm{Tr}\, \rho_A \, \log \rho_A$

- e.g. in local QFT:

  A and B can be spatial regions, separated by a smooth entangling surface



The good news & the bad news

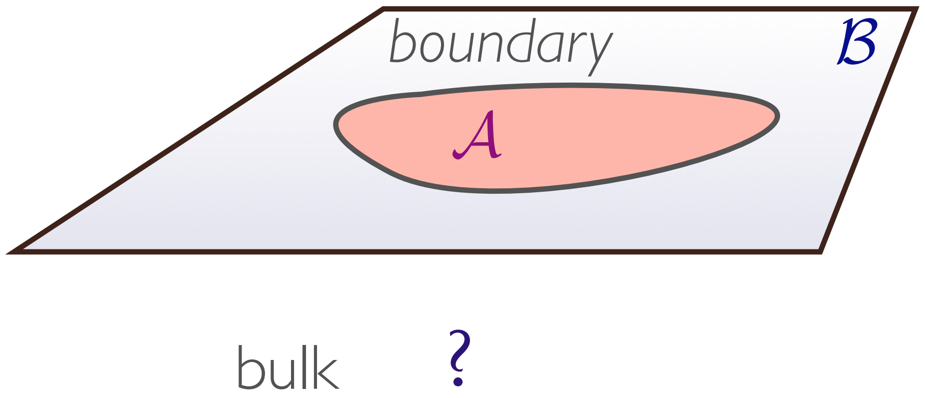
- But EE is hard to deal with…
  - non-local quantity, intricate & sensitive to environment
  - difficult to measure
  - difficult to calculate

… especially in strongly-coupled quantum systems

- AdS/CFT to the rescue?

~ Is there a natural bulk dual of EE?

(= "Holographic EE")



Yes! - described geometrically…

Motivation

- Holography:
  - Tool to elucidate quantum gravity -- dictionary?
- Entanglement:
  - Expected to underlie bulk spacetime emergence -- how?
  - Usefully characterized via entanglement entropy -- underlying structure?
  - ➙ Reformulate holographic entanglement entropy (HEE)
- General covariance:
  - Required for any physical quantity
  - Needed to probe time dependence
  - Complementary toolkit; breaks degeneracy
  - May yield crucial insights

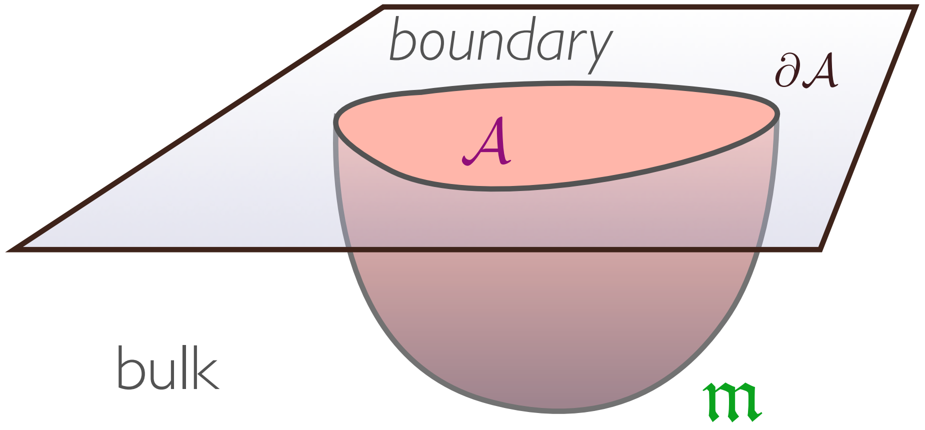
OUTLINE

- ☑ Motivation
- ☐ Background
  - Surfaces
  - Threads
  - Expectations
- ☐ Setup & Toolkit
- ☐ Results
- ☐ Summary

RT and HRT

Proposal [RT=Ryu & Takayanagi, '06] for *static* configurations:

In the bulk, entanglement entropy $S_{\mathcal{A}}$ for a boundary region $\mathcal{A}$ is captured by the area of a minimal co-dimension-2 bulk surface $\mathfrak{m}$ *at constant t* homologous to $\mathcal{A}$

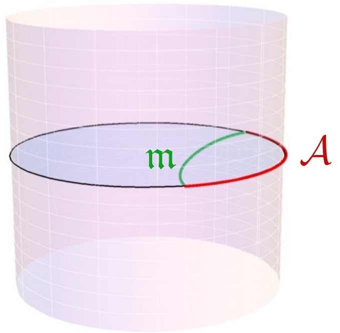


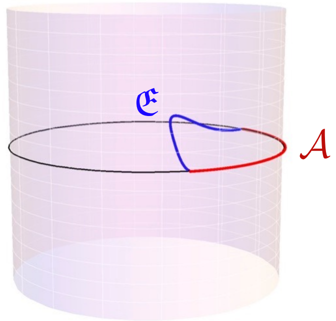
$$S_{\mathcal{A}} = \min_{\partial\mathfrak{m}=\partial\mathcal{A}} \frac{\text{Area}(\mathfrak{m})}{4\,G_N}$$

In *time-dependent* situations, RT prescription needs to be covariantized:

[HRT =VH, Rangamani, Takayanagi '07]

minimal surface $\mathfrak{m}$ at constant time → extremal surface $\mathfrak{E}$ in the full bulk



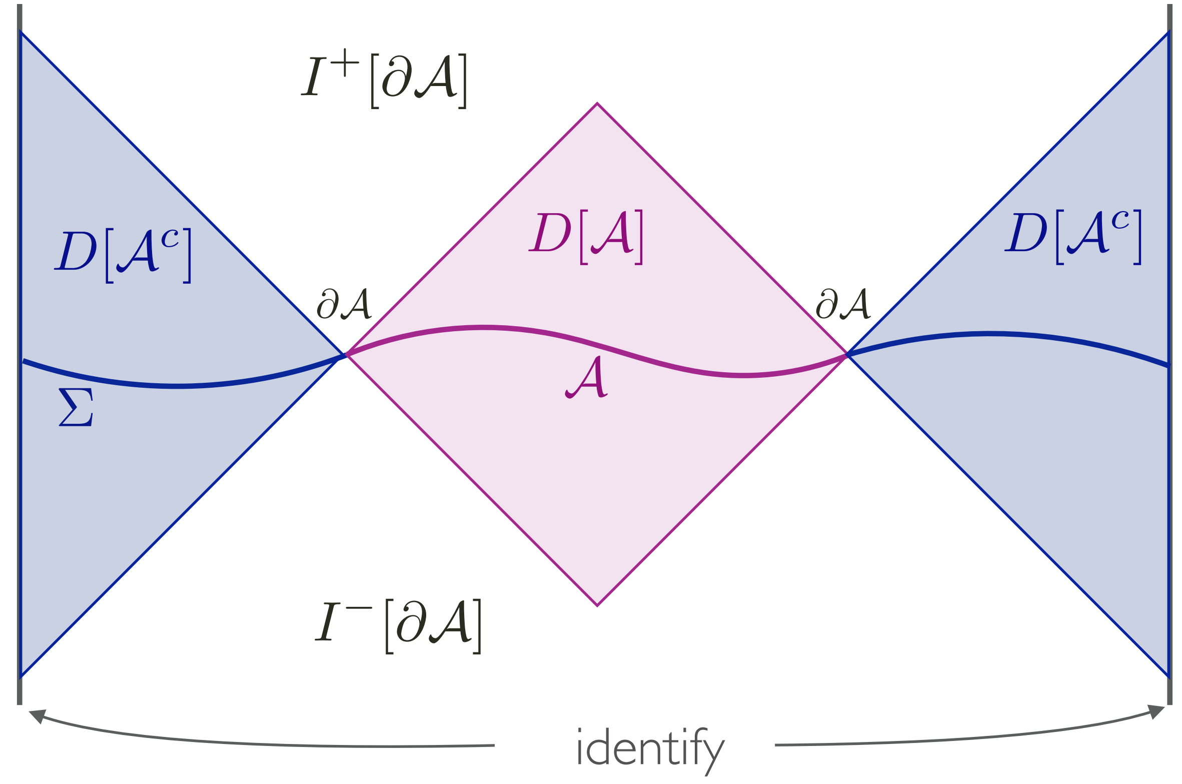


This gives a well-defined quantity in any (arbitrarily time-dependent asymptotically AdS) spacetime.

Entanglement wedge

- Boundary spacetime partition from entangling surface $\partial\mathcal{A}$:

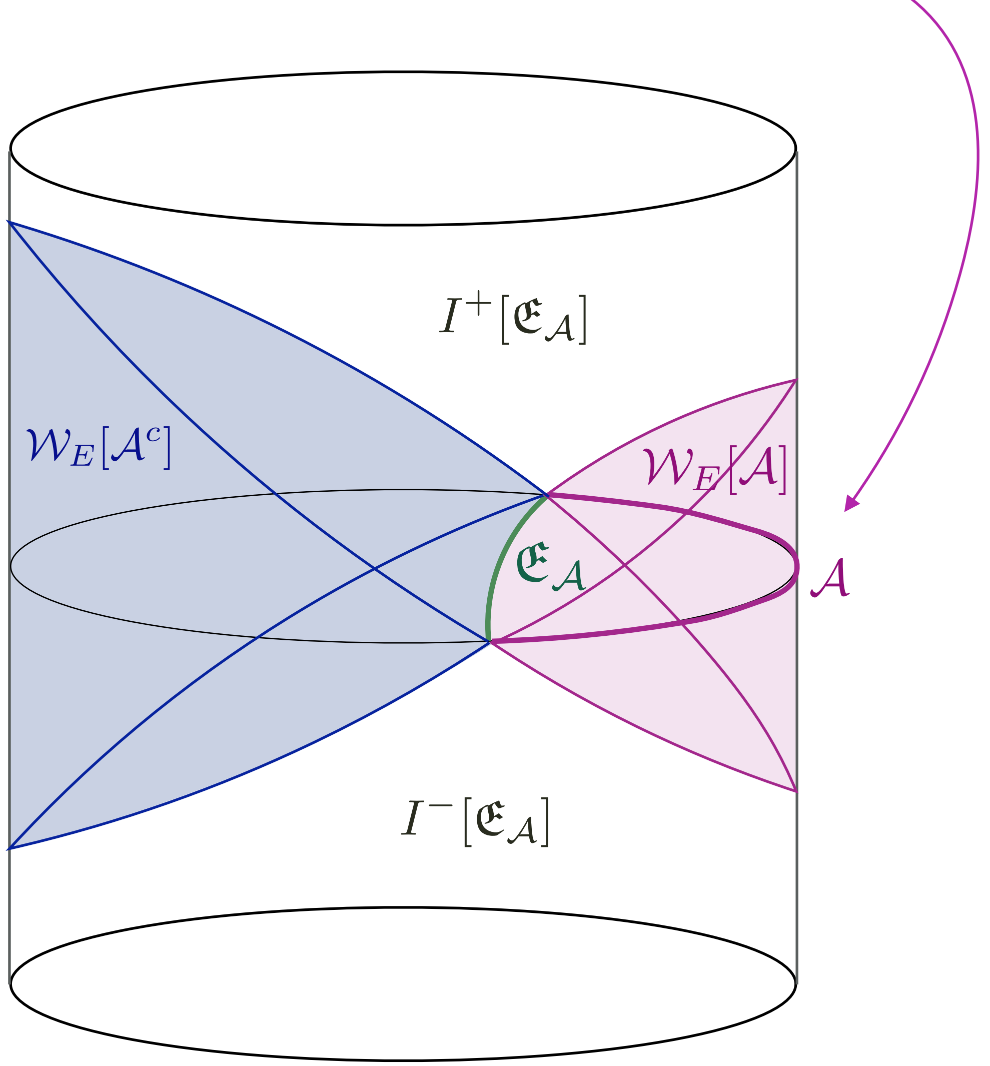
$$\partial\mathcal{M} = D[\mathcal{A}] \cup D[\mathcal{A}^c] \cup I^-[\partial\mathcal{A}] \cup I^+[\partial\mathcal{A}]$$



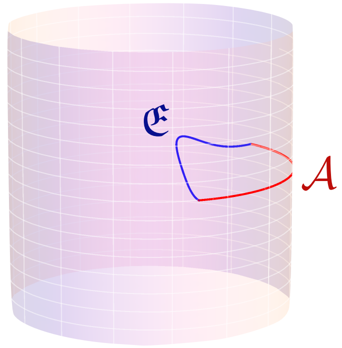
- Induces a corresponding partition into 4 bulk regions from $\mathfrak{E}_\mathcal{A}$:

$$\mathcal{M} = \mathcal{W}_E[\mathcal{A}] \cup \mathcal{W}_E[\mathcal{A}^c] \cup I^-[\mathfrak{E}_\mathcal{A}] \cup I^+[\mathfrak{E}_\mathcal{A}]$$

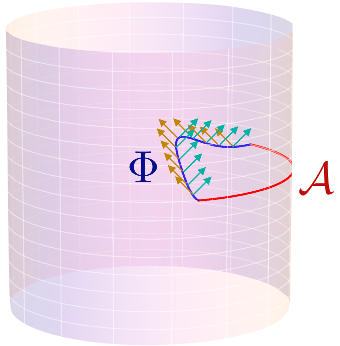
entanglement wedge of $\mathcal{A}$



Extremal surface reformulations



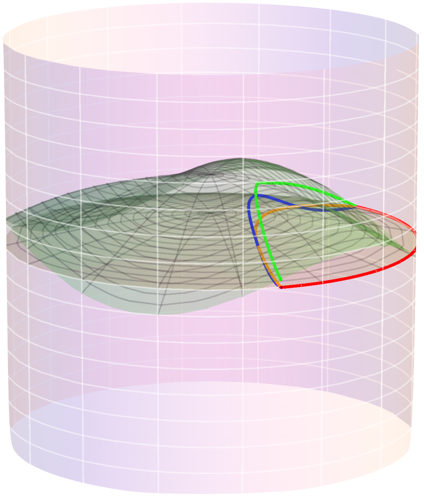
- $\mathfrak{E}$ = Extremal surface
  - (relatively) easy to find
  - minimal set of ingredients required in specification
  - need to include homology constraint as extra requirement



- Φ = Surface with zero null expansions [HRT]
  - (cf. light sheet construction & covariant entropy bound [Bousso, '99]:

    Bulk entropy through light sheet of surface $\sigma \leq$ Area(σ)/4

    Φ = surface admitting a light sheet closest to bdy

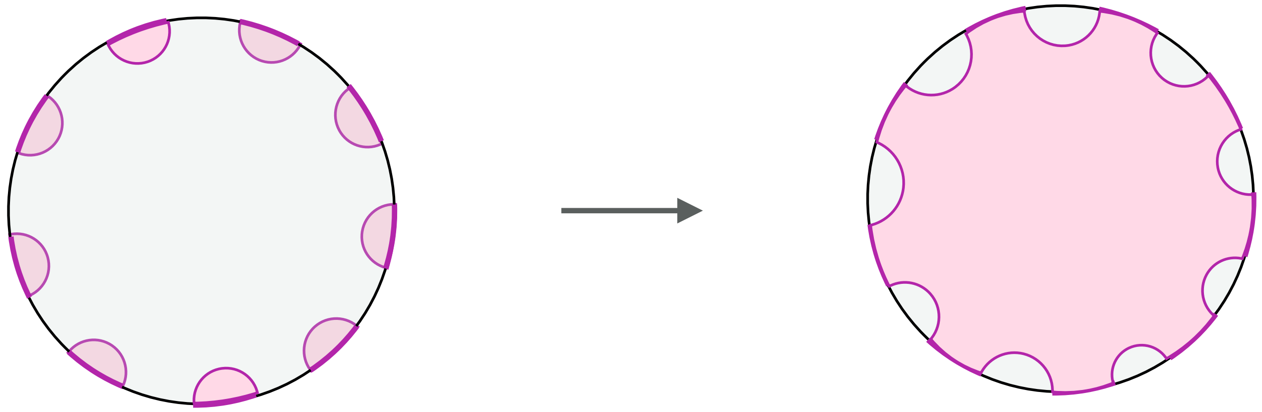


- Maximin surface [Wall, '12]
  - maximize over minimal-area surface on a spacelike slice
  - requires the entire collection of slices & surfaces
  - implements homology constraint automatically
  - useful for proofs (e.g. SSA)
- Minimax surface -- discussed later...

Geometrically, all of these are ultimately the same construct...

Curiosities / drawbacks of surfaces

- Naively (from UV/IR intuition) over-localized
- Arbitrarily large jumps in entanglement wedge (at 'phase transitions')

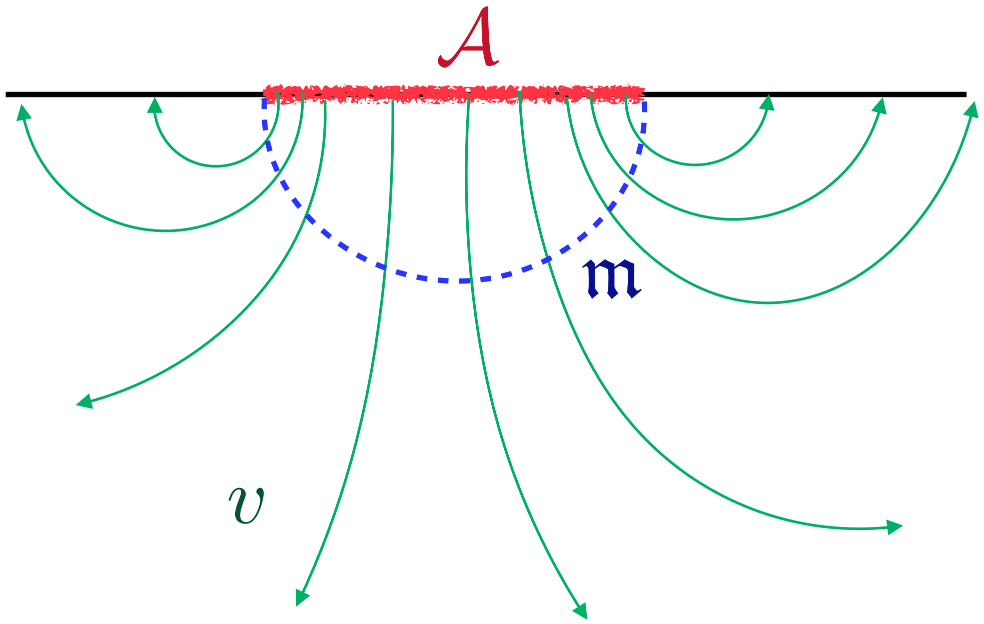


- Does not elucidate the relation to quantum information...
  - Where does the information live?
  - Mutual information $I(A{:}B) = S(A) + S(B) - S(AB)$
    is given by surfaces located in different spacetime regions.
  - Geometric proof of SSA ($S(AB) + S(BC) \geq S(B) + S(ABC)$)
    obscures its meaning as monotonicity under inclusion of correlations...

Riemannian bit-threads

- Reformulate EE in terms of flux of flow v [Freedman & Headrick, '16]
  - let v be a vector field satisfying $\nabla \cdot v = 0$ and $|v| \leq 1$. Then EE is given by

$$S_{\mathcal{A}} = \max_{v} \int_{\mathcal{A}} v$$

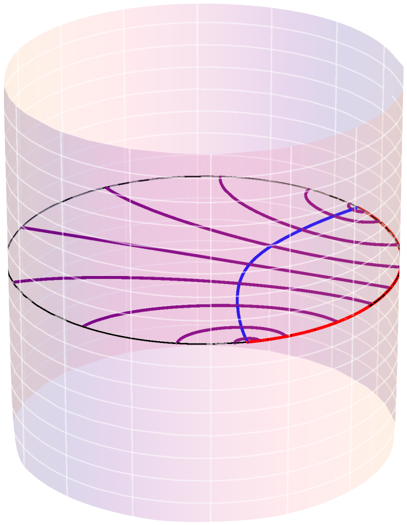


  - By Max Flow - Min Cut theorem, equivalent to RT: (bottleneck for flow = minimal surface)

- Useful reformulation of HEE:
  - flow continuous under varying region (while bottlenecks can jump discontinuously)
  - automatically implements homology constraint and global minimization of RT
  - maximal flow defined even without a regulator (when flux has UV divergence)
  - can be computed more efficiently (via linear programming methods)
  - implements QI meaning of EE and its inequalities more naturally
  - provides more intuition: think of each bit thread as connecting an EPR pair

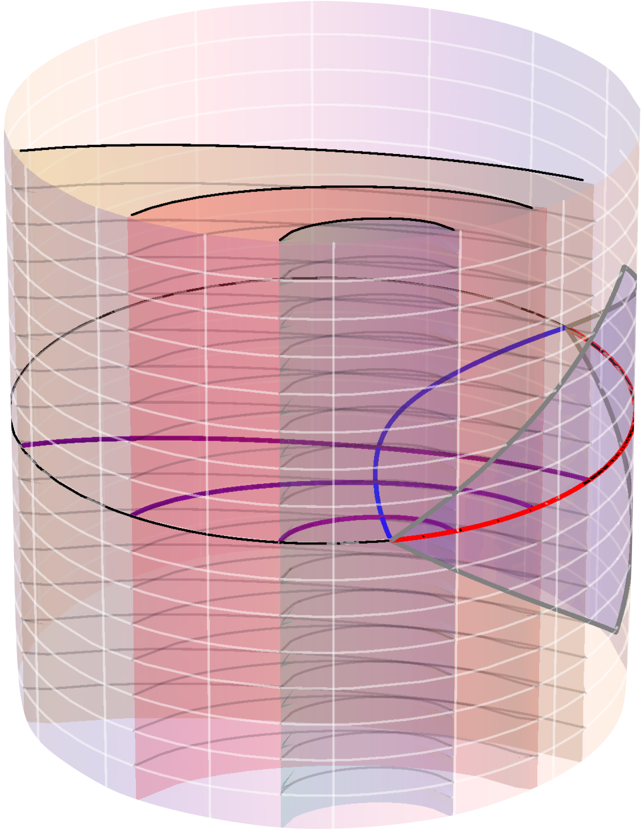
Primary question:

How do Riemannian threads covariantize?

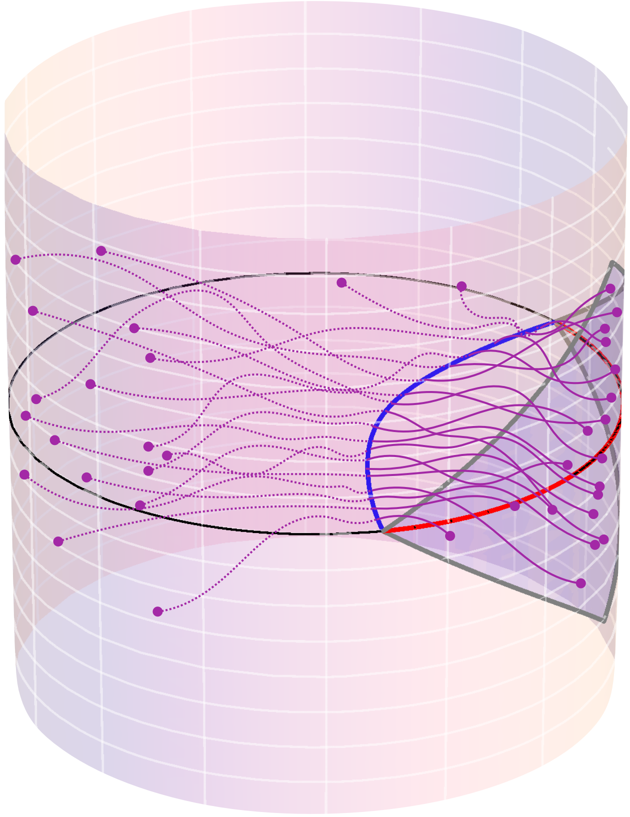


∃ 2 natural possibilities:

extend threads in time
flow sheets

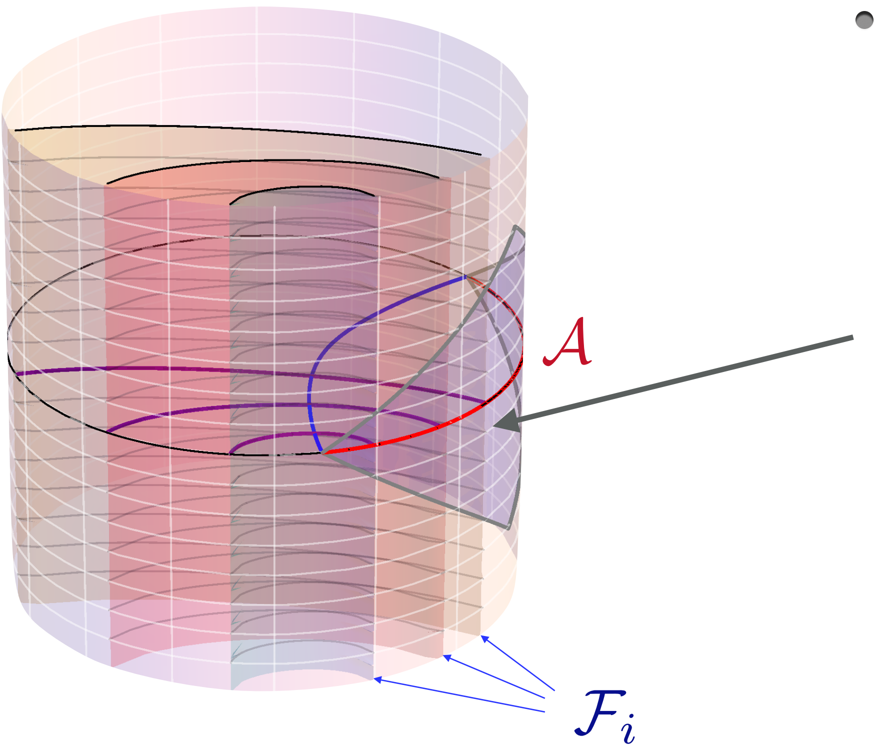


keep 1-d threads
flow lines



Naive expectations

- Extend in time:
  - boundary EPR pair ⤳ pair of worldlines
  - bit thread ⤳ "bit cloth" / "flow sheet" = timelike worldsheet $\mathcal{F}$



- cf. slice codim.-1 RT surf. ⇔ 1-d bit thread

  ⤳ bulk codim.-2 HRT surf. ⇔ 2-d flow sheet

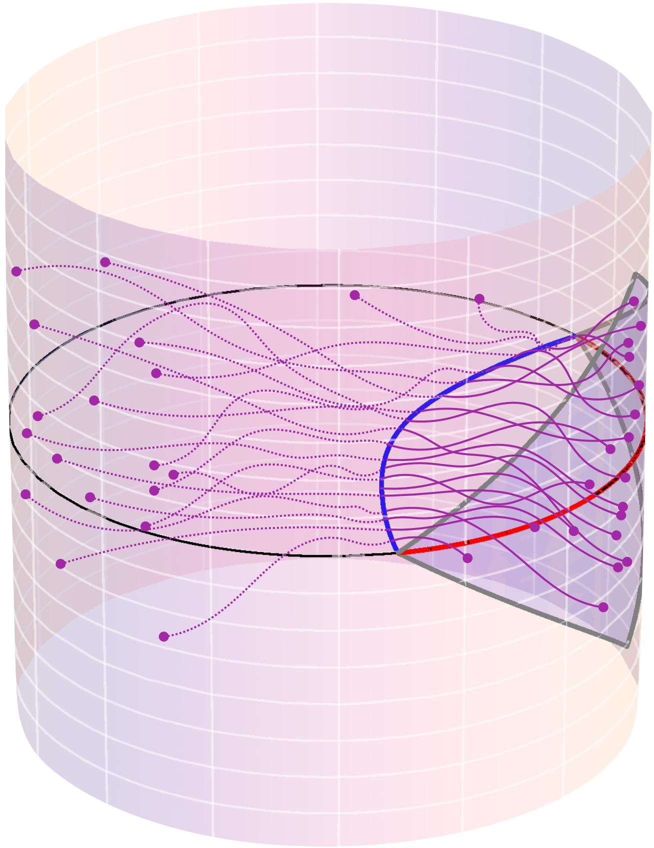
$$S_{\mathcal{A}} \overset{?}{=} \# \text{ sheets through } D[\mathcal{A}]$$

(w/ $\mathcal{F}$ subject to requisite norm bound & can't end in bulk)

- But:
  - norm bound too global
  - generically no canonical way of "evolving" $\partial\mathcal{A}(t)$
  - even for fixed $\{\mathcal{A}(t), \forall t\}$, $\mathcal{F}$ depends on t-foliation...

Naive expectations

- EE pertains to a given instant in time
  - in strongly interacting QFT, EPR pair localizes only for short duration ⤳ bdy events
  - cf. entanglement distillation



Suggests threads from $D[\mathcal{A}]$ to $D[\mathcal{A}^c]$

$$S_{\mathcal{A}} \overset{?}{=} \# \text{ threads from } D[\mathcal{A}]$$

?: what are the restrictions on these threads?

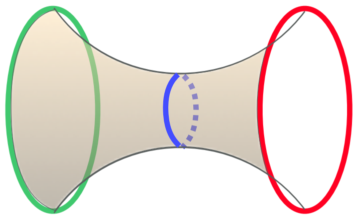
- Are they localized to certain spacetime regions?
- Is their density bounded?
- Can they be timelike?

OUTLINE

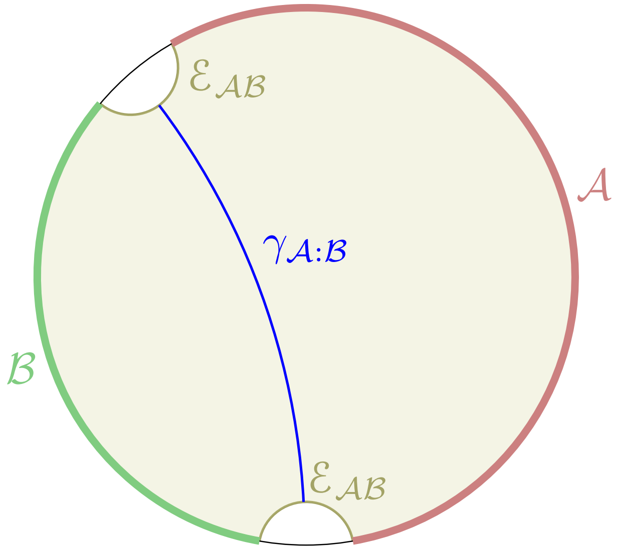
- ☑ Motivation
- ☑ Background
- ☐ Setup & Toolkit
  - HEE regulator
  - Beyond holography
  - Lagrangian duality
- ☐ Results
- ☐ Summary

HEE regulator

- NB. finite HEE for multi-boundary wormholes
  for $\mathcal{A}$ = entire piece of connected bdy Cauchy slice

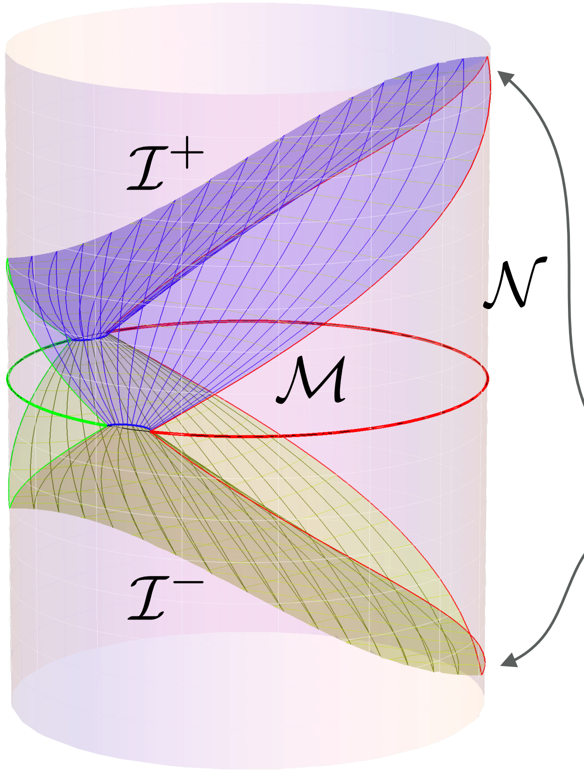


- Regulate HEE via entanglement wedge cross section [Dutta, Faulkner, '19]
  - widen $\partial\mathcal{A}$ to separate $\mathcal{A}$ and $\mathcal{A}^c := \mathcal{B}$



$$S_{\mathcal{A}} := \frac{\mathrm{Area}(\gamma_{\mathcal{A}:\mathcal{B}})}{4G_N}$$

  - relevant part of ST inside EW of $\mathcal{AB}$



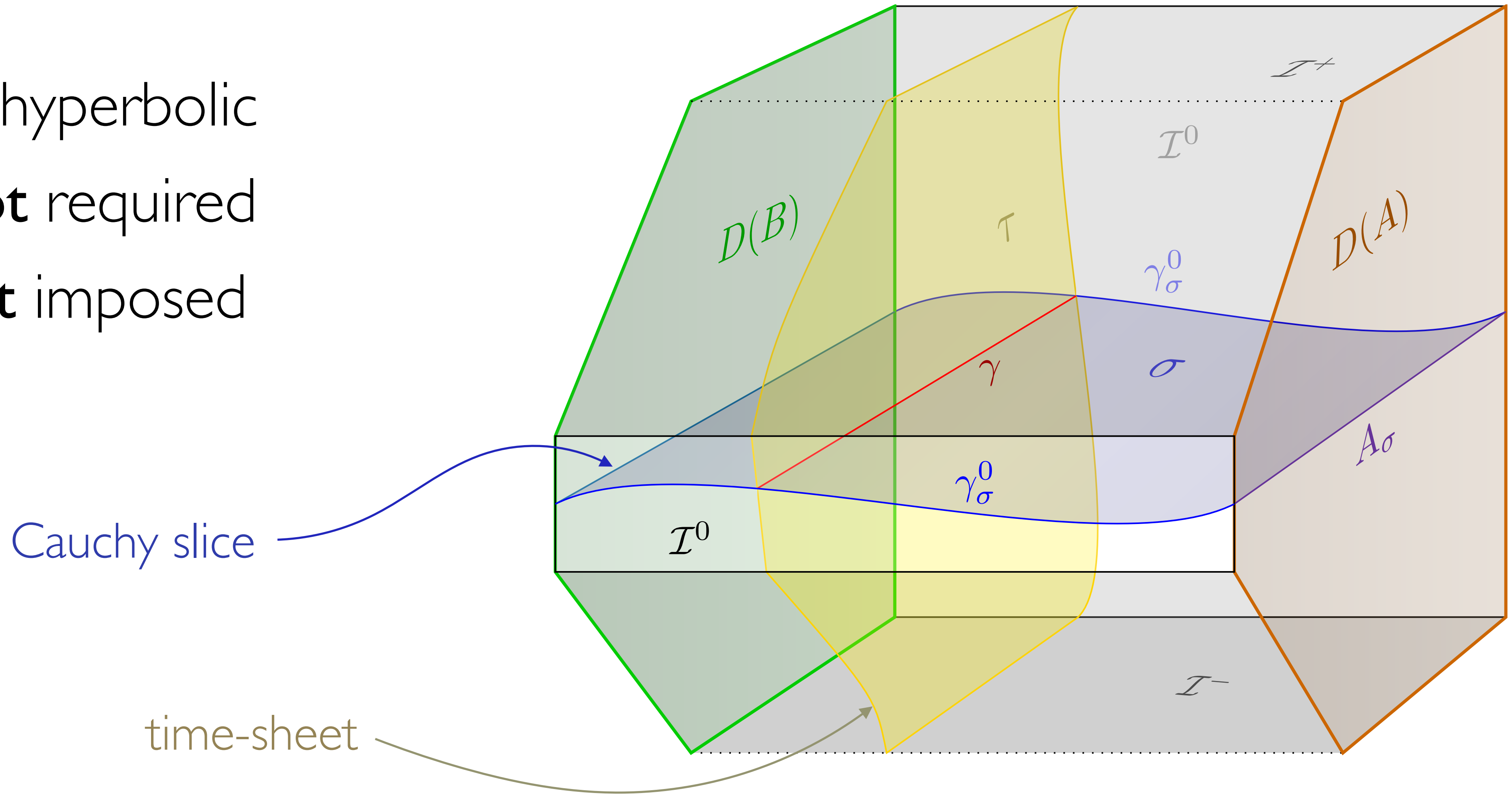
WLOG treat as de-facto boundaries, but full ST can be reinstated later.

Beyond holography

- Consider a Lorentzian d ≥ 3 dimensional spacetime, with:
  - timelike spatial bdy $\mathcal{N} := D[\mathcal{A}] \cup D[\mathcal{B}]$
  - spacelike or null future/past bdy $\mathcal{I}^{\pm}$
  - optionally end of the world brane $\mathcal{I}^0$

$$\mathcal{I} = \mathcal{I}^+ \cup \mathcal{I}^- \cup \mathcal{I}^0$$

  - globally hyperbolic
  - NEC **not** required
  - E.eq. **not** imposed



Convex program & Lagrangian duality

- Convex program:
  - Convex program P: minimize $f_0(y)$ over $y \in \mathcal{D}$ such that $\forall i,\ f_i(y) \leq 0,\ \forall j,\ h_j(y) = 0$

    convex domain

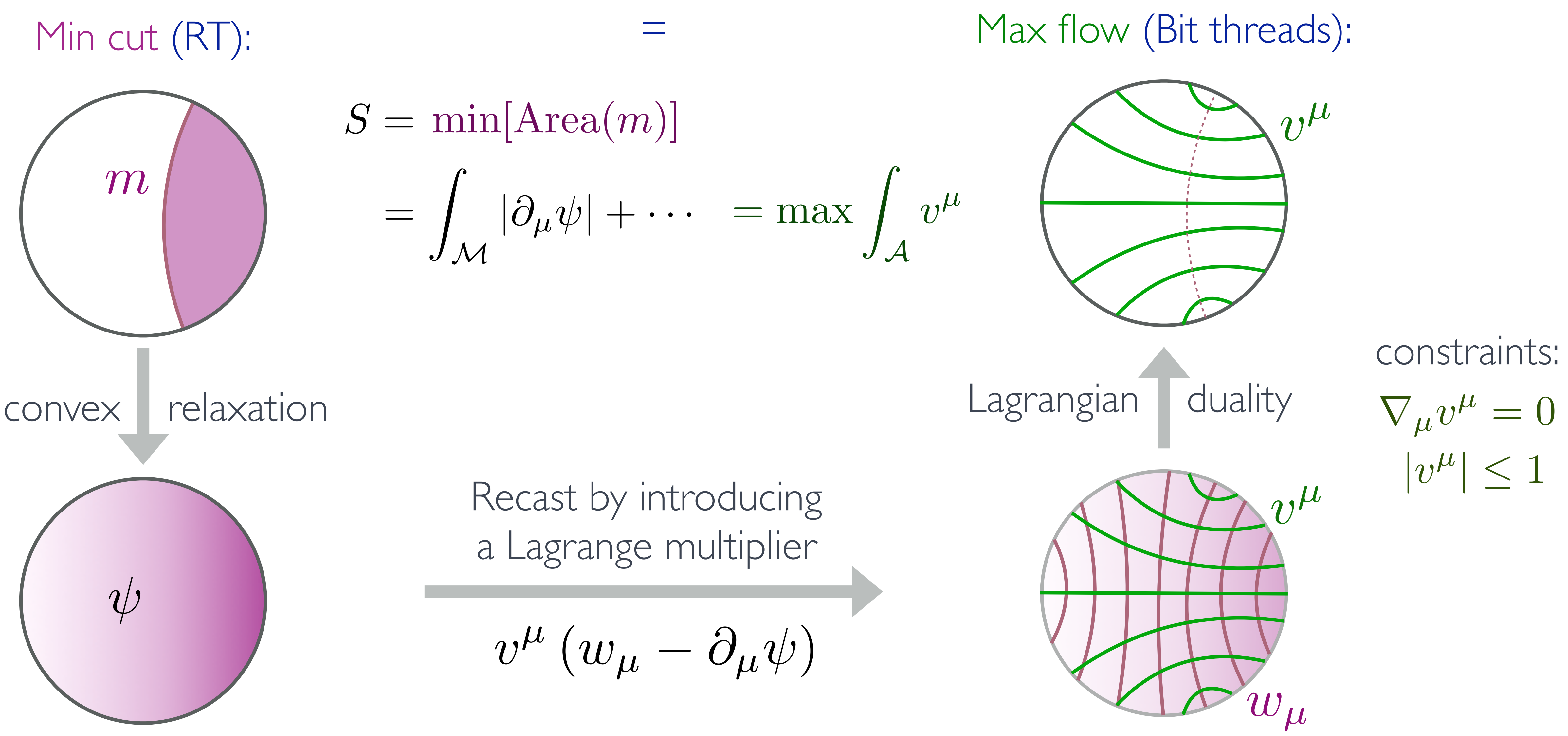
    convex functions

    affine functions
  - More general problems may be converted to the requisite form via convex relaxation
  - use Lagrange multipliers $L(y,\lambda,\nu) \equiv f_0(y) + \sum_i \lambda_i f_i(y) + \sum_j \nu_j h_j(y)$
  - solution via convex optimization: $p^* = \inf_y \sup_{\lambda,\nu} L(y,\lambda,\nu)$
- Lagrangian duality: swap order
  - new extremization problem, in new variables
  - strong duality: primal and dual solutions agree

ex: Max flow - min cut

- Max-flow/min-cut (MFMC) is an example of Lagrangian duality in theory of convex optimization [VH, Headrick, '17]

- Riemannian case:



Min cut (RT): = Max flow (Bit threads):

$$S = \min[\mathrm{Area}(m)]$$

$$= \int_{\mathcal{M}} |\partial_\mu \psi| + \cdots \quad = \max \int_{\mathcal{A}} v^\mu$$

convex relaxation

Recast by introducing a Lagrange multiplier

$$v^\mu \left(w_\mu - \partial_\mu \psi \right)$$

Lagrangian duality

constraints:

$$\nabla_\mu v^\mu = 0$$

$$|v^\mu| \leq 1$$

OUTLINE

- ☑ Motivation
- ☑ Background
- ☑ Setup & Toolkit
- ☐ Results
  - Maximin S_- & minimax S_+
  - Detour: minimax theory
  - Convex relaxed S_{c}
  - Holographic coincidence
  - Optimal flows
- ☐ Summary

Maximin

- Start w/ $S_- := \sup\limits_{\sigma\in\mathcal{S}} \inf\limits_{\gamma\in\Gamma_\sigma} \mathrm{area}(\gamma)$

  codim.-1 slices

  codim.-2 surfaces on σ homol. to A rel. to $\mathcal{I}^0 \cap \sigma$

- Use Riemannian MFMC to reexpress in terms of slice flows ⤳ "maximax":

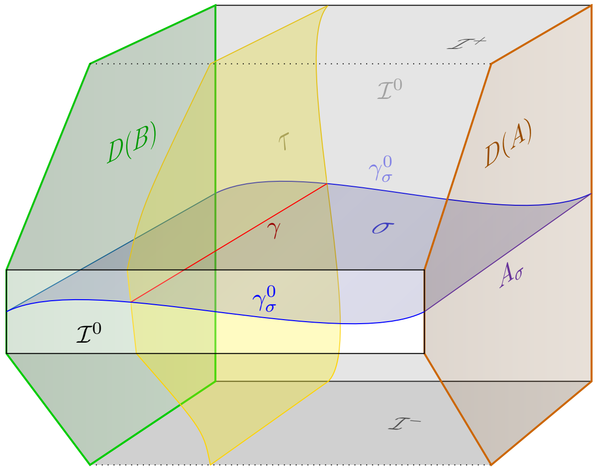
$$S_- = \sup_{\sigma\in\mathcal{S}} \sup_{v\in\mathcal{F}_\sigma} \int_{A_\sigma} *v$$

σ-flows

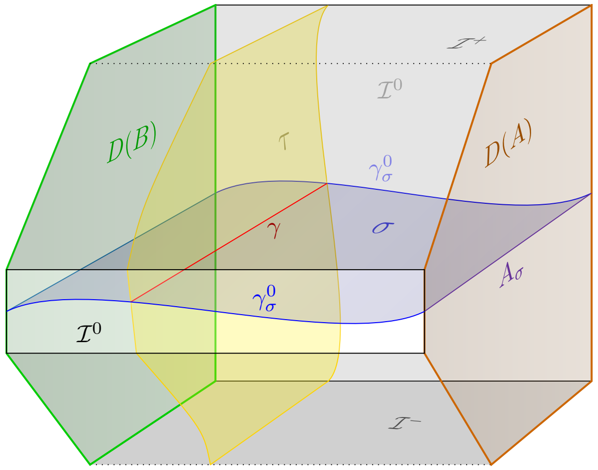
- Put space & time variations on equal footing:

$$S_- = \sup_{\sigma\in\mathcal{S}} \inf_{\tau\in\mathcal{T}} \mathrm{area}(\sigma\cap\tau)$$

codim.-1 time-sheets homol. to D[A] rel. to $\mathcal{I}$



Minimax



- Change order of extremizations from maximin:

$$S_+ := \inf_{\tau\in\mathcal{T}} \sup_{\sigma\in\mathcal{S}} \mathrm{area}(\sigma\cap\tau)$$

- Re-write more analogously to original maximin:

$$S_+ = \inf_{\tau\in\mathcal{T}} \sup_{\substack{\gamma\subset\tau \\ \text{achronal}}} \mathrm{area}(\gamma)$$

- Naively: apply Lorentzian min flow - max cut [VH, Headrick, '17] on time-sheet flow to convert to "minimin"?

  Subtlety: that only requires achronality within time-sheet; for S_+ we have a stronger condition of achronality in bulk... (but OK in holography [Grimaldi, Grado-White, Headrick, VH: W.I.P])

?: What is the relation between S_- and S_+ ?

Minimax theory

- Consider a function $f : X \times Y \to \mathbf{R}$
- Q: what is the relation between maximin & minimax?

$$\sup_{x \in X} \inf_{y \in Y} f(x,y) \qquad \inf_{y \in Y} \sup_{x \in X} f(x,y)$$

  - Not necessarily equal;
    e.g. $X, Y = \{0,1\}$, $f(x,y) = (-1)^{x+y} \implies \quad -1 \qquad\qquad 1$

Minimax theory

- Consider a function $f : X \times Y \to \mathbf{R}$
- Q: what is the relation between maximin & minimax?

$$\sup_{x \in X} \inf_{y \in Y} f(x,y) \leq \inf_{y \in Y} \sup_{x \in X} f(x,y)$$

since $\inf_{y \in Y} f(x_0, y) \leq f(x_0, y_0) \leq \sup_{x \in X} f(x, y_0)$

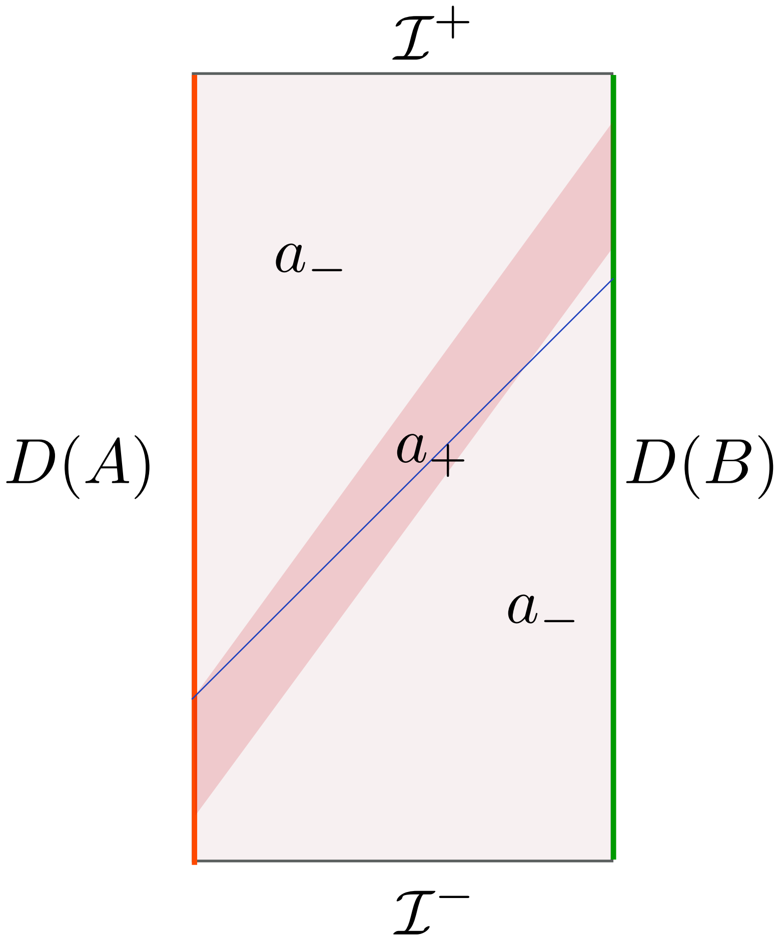
- Q: when is the inequality saturated?
  - when ∃ a global saddle point (x_0, y_0) saturating both inequalities
  - when X and Y are convex subsets of affine spaces, and f is concave-convex

in $x|_y$ in $y|_x$

Toy spacetime

- Easy to find example of spacetime (in our generalized setting) wherein maximin ≠ minimax:

$a_- < a_+$



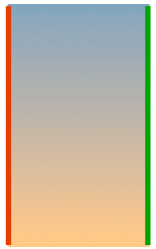
- However, this ST does not satisfy NEC.
- Already if bottom region has $a_b > a_+$, then maximin = minimax = a_b

Convex relaxation

- Instead of $\mathrm{area}(\sigma \cap \tau)$, we want to define concave-convex f

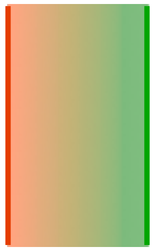
$\downarrow \quad \downarrow$

- convex-relax hypersurfaces to level sets of scalar fields ϕ, ψ



$$\sigma \rightsquigarrow \qquad \mathcal{S}_{\mathrm{c}} := \left\{ \phi : \bar{\mathcal{M}} \to \left[-\frac{1}{2}, \frac{1}{2}\right] \;\middle|\; \phi|_{\mathcal{I}^{\pm}} = \pm\frac{1}{2} \,,\, d\phi \in \mathrm{j}^{+} \right\}$$

future-directed causal 1-form



$$\tau \rightsquigarrow \qquad \mathcal{T}_{\mathrm{c}} := \left\{ \psi : \bar{\mathcal{M}} \to \left[-\frac{1}{2}, \frac{1}{2}\right] \;\middle|\; \psi|_{D(A)} = -\frac{1}{2} \,,\, \psi|_{D(B)} = \frac{1}{2} \right\}$$

- generalize the area of intersection of hypersurfaces:

$$f[\phi, \psi] := \int_{\mathcal{M}} \sqrt{g}\, |d\phi \mathbin{\wedge\!\!\!\cdot} d\psi| \qquad (\phi \in \mathcal{S}_{\mathrm{c}}, \psi \in \mathcal{T}_{\mathrm{c}})$$

wedgedot pairing $\quad |W \mathbin{\wedge\!\!\!\cdot} X| := \max\{|W \wedge X|, |W \cdot X|\}$

$$f[\phi, \psi] = \int_{-1/2}^{1/2} dt \int_{-1/2}^{1/2} ds \ \mathrm{area}'(\sigma_t, \tau_s) \qquad \text{with} \qquad \mathrm{area}'(\sigma, \tau) := \int_{\sigma \cap \tau} \sqrt{h} \times \begin{cases} 1 & (\tau \text{ timelike or null at } \sigma) \\ \coth \chi & (\tau \text{ spacelike at } \sigma) \end{cases}$$

Convex relaxed "entropy"

$$S_{\rm c} := \sup_{\phi \in \mathcal{S}_{\rm c}} \; \inf_{\psi \in \mathcal{T}_{\rm c}} \; f[\phi, \psi] = \inf_{\psi \in \mathcal{T}_{\rm c}} \; \sup_{\phi \in \mathcal{S}_{\rm c}} \; f[\phi, \psi]$$

convex-maximin

convex-minimax

- must lie between non-convex values: $S_- \leq S_{\rm c} \leq S_+$
- More suggestive formulation? Lagrange dualize!

(various possibilities...)

dualize on ψ for fixed ϕ

⤳ V-flow program

(⤳ covariant bit threads)

dualize on ϕ for fixed ψ

⤳ U-flow program

V-flows & U-flows

$$S_{\rm c} := \sup_{\phi\in\mathcal{S}_{\rm c}} \inf_{\psi\in\mathcal{T}_{\rm c}} f[\phi,\psi] = \inf_{\psi\in\mathcal{T}_{\rm c}} \sup_{\phi\in\mathcal{S}_{\rm c}} f[\phi,\psi]$$

V-flow:

$$S_{\rm c} = \sup_{V\in\mathcal{F}} \int_{D(A)} *V$$

$$d{*}V = 0\,, \qquad *V|_{\mathcal{I}} = 0$$

$$\exists\, \phi\in\mathcal{S}_{\rm c} \text{ s.t. } d\phi \pm V \in \mathfrak{j}^{+}$$

∀ bulk timelike curve,

$$\int dt\,|V_{\perp}| \leq 1$$

U-flow:

$$S_{\rm c} = \inf_{U\in\mathcal{G}} \int_{\mathcal{I}^{+}} *U$$

$$d{*}U = 0\,, \qquad *U|_{\mathcal{I}^{0}\cup\mathcal{N}} = 0\,,$$

$$\exists\, \psi\in\mathcal{T}_{\rm c} \text{ s.t. } U \pm d\psi \in \mathfrak{j}^{+}$$

∀ bulk spacelike curve from D(A) to D(B),

$$\int ds\,|U_{\perp}| \geq 1$$

V and U threads

- Define "threads"
  - integral curves of vector field dual to V, or (D-1)-form *V (oriented, can't intersect)
  - unoriented curves obeying a density bound

$$\Delta(q,p) := \int_q dt \int_p ds\, \delta(x(t),y(s))\, |(-\dot{x}) \wedge \dot{y}|$$

- V-thread = curve between $D(A)$ and $D(B)$

Maximize $\mu(\mathcal{P})$ over measure μ on $\mathcal{P}$ subject to: $\forall\, q \in \mathcal{Q}\,, \quad \int_{\mathcal{P}} d\mu(p)\, \Delta(q,p) \leq 1$

- U-thread = causal curve between $\mathcal{I}^-$ and $\mathcal{I}^+$

Minimize $\nu(\mathcal{Q})$ over measure ν on $\mathcal{Q}$ subject to: $\forall\, p \in \mathcal{P}\,, \quad \int_{\mathcal{Q}} d\nu(q)\, \Delta(q,p) \geq 1$

  - density bounds are non-local, and reinforce each other
  - U-threads effectively form a barrier separating D(A) from D(B)
  - V-threads covariantize the original bit threads

Holographic coincidence

- For general ("beyond holography") spacetimes, $S_- \leq S_c \leq S_+$

↓

(= globally hyperbolic w/ timelike parts of bdy but no energy conditions or field equations...)

S_- ↓ maximin

S_c ↓ convex relaxed m.m.

S_+ ↓ minimax

- BUT for holographic spacetimes, $\boxed{S_- = S_c = S_+ = S_{\text{HRT}}}$

↓

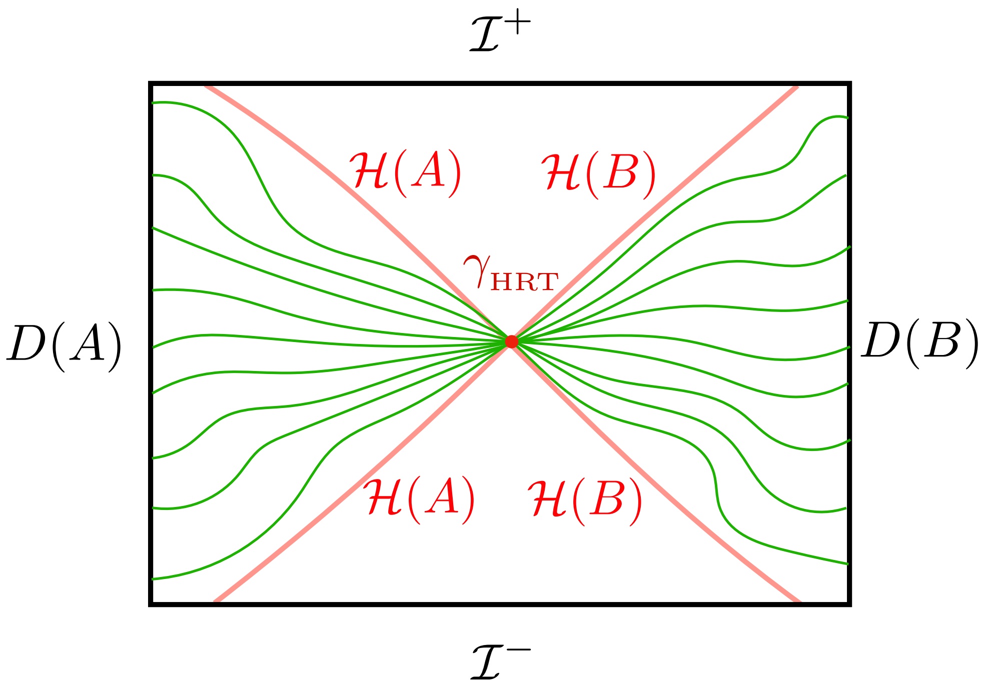
  - NEC $\rightsquigarrow$ congruence from HRT surf. has non-positive & generically decreasing expansion
  - AdS b.c. $\rightsquigarrow$ no null generators emanate from EOWB.

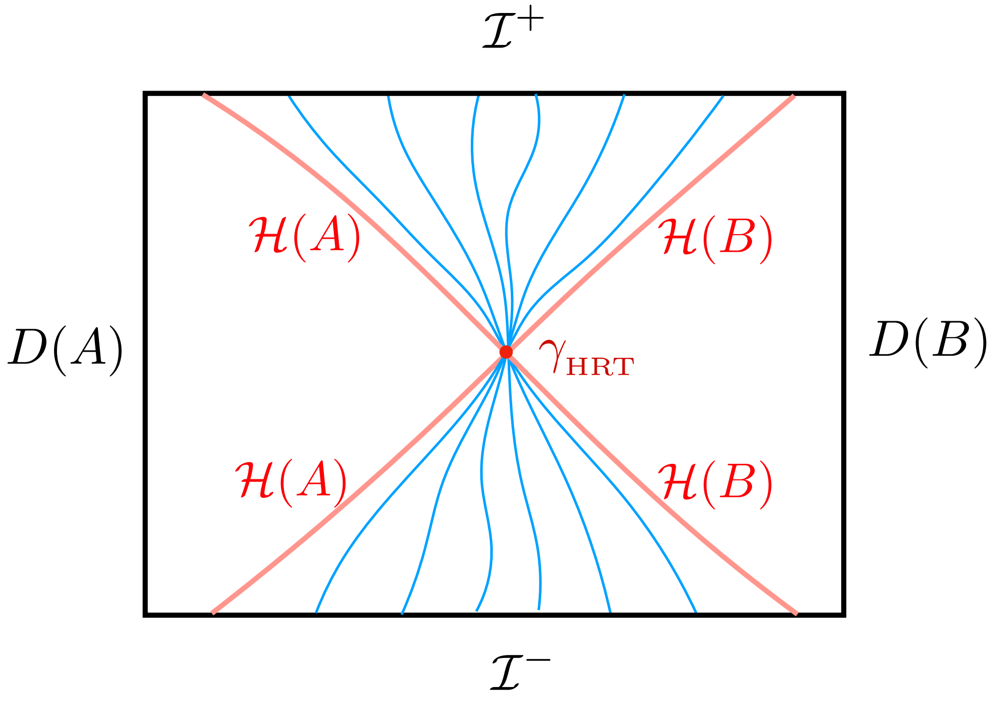
Pf:

- HRT surf. is area-maximizing within entanglement horizon (= time-sheet τ)
- HRT surf. is area-minimizing within maximin slice (= Cauchy slice σ)
- $\rightsquigarrow$ HRT surf. is a global saddle point $\implies$ $S_- = S_+ = S_{\text{HRT}}$ $\square$
- Alt.: $S_+ \leq S_{\text{HRT}}$ [minimax] , $S_- \leq S_+$ [min-max ineq.] , & $S_- = S_{\text{HRT}}$ [Wall] $\square$

Optimal flows

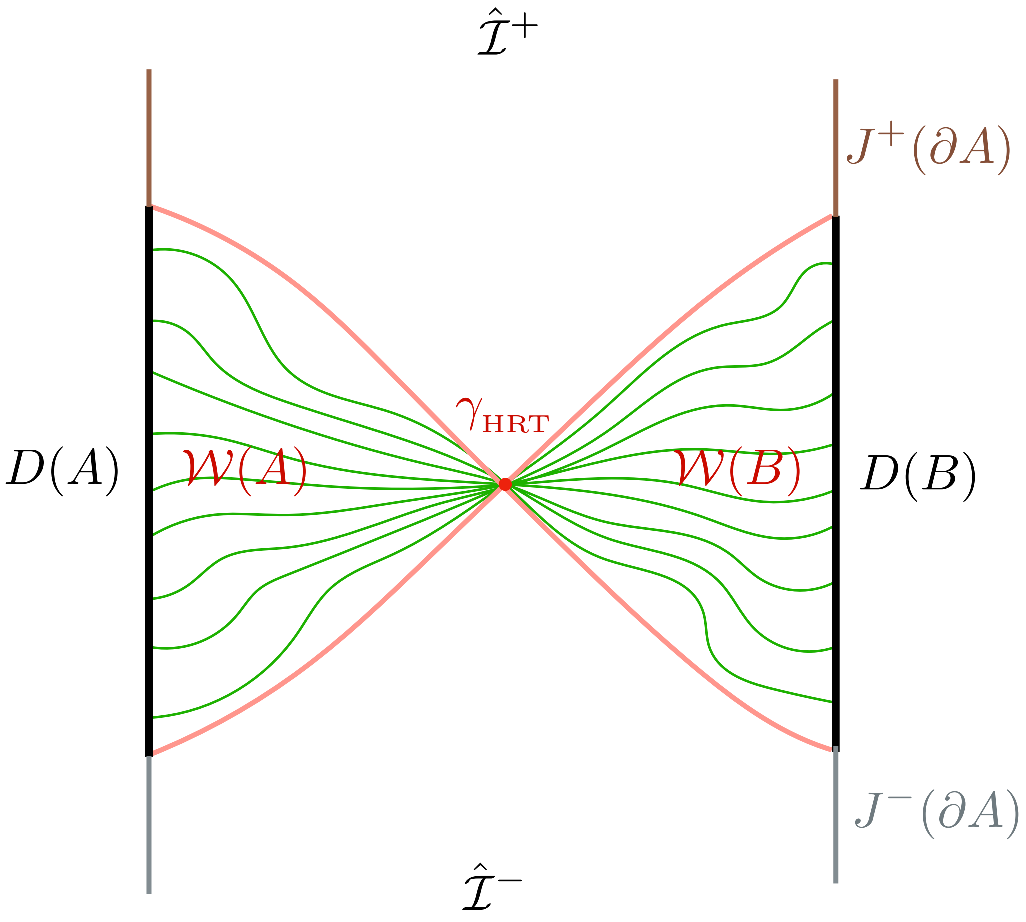
- General V-flows and U-flows are quite floppy and spread out
  - just subject to b.c.s and norm bound (⇝ U-flow lines are timelike everywhere)
- *Optimal* flows in *holographic* ST are more restricted (though still floppy):
  - V-flow lines are confined to entanglement wedges $\mathcal{W}_A$ and $\mathcal{W}_B$
  - U-flow lines are expelled from (interior of) entanglement wedges $\mathcal{W}_A$ and $\mathcal{W}_B$
  - Both flows pass through HRT surface:

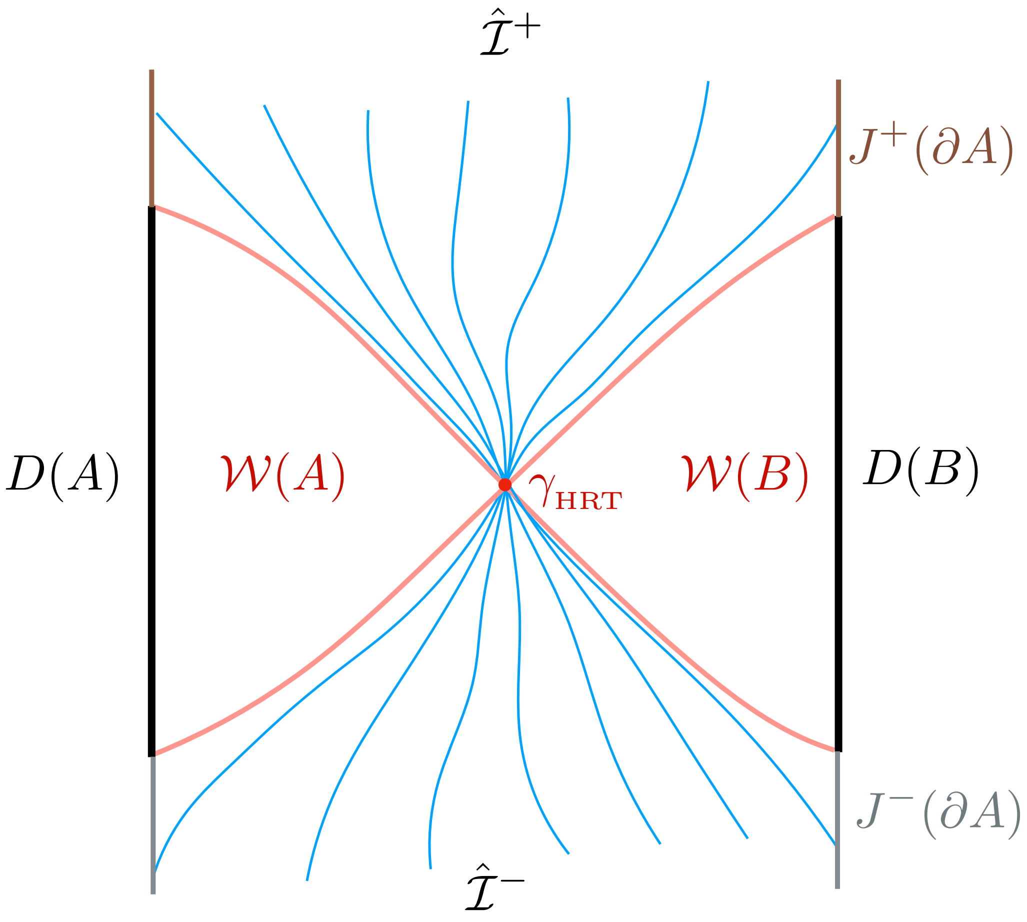




Optimal flows

- In full (unregulated) spacetime:





OUTLINE

- ☑ Motivation
- ☑ Background
- ☑ Setup & Toolkit
- ☑ Results
- ☐ Summary

New covariant HEE prescriptions:

$$\sup_{\sigma\in\mathcal{S}} \ \sup_{v\in\mathcal{F}_\sigma} \int_{A_\sigma} *v$$

$$\sup_{\sigma\in\mathcal{S}} \ \inf_{\tau\in\mathcal{T}} \mathrm{area}(\sigma\cap\tau)$$

S_-

S_+

$$\inf_{\tau\in\mathcal{T}} \ \sup_{\substack{\gamma\subset\tau\\ \text{achronal}}} \mathrm{area}(\gamma)$$

$$\inf_{\tau\in\mathcal{T}} \ \sup_{\sigma\in\mathcal{S}} \mathrm{area}(\sigma\cap\tau)$$

S_c

$$\begin{tikzcd}
\displaystyle\sup_{\phi\in\mathcal{S}_c} \inf_{\psi\in\mathcal{T}_c} \int_{\mathcal{M}} \sqrt{g}\,|d\phi\wedge d\psi| \arrow[rr, leftrightarrow, "\text{minimax}", "\text{theorem}"'] \arrow[d, leftrightarrow, "\begin{array}{c}\psi\leftrightarrow V\\ \text{duality}\end{array}"'] & & \displaystyle\inf_{\psi\in\mathcal{T}_c} \sup_{\phi\in\mathcal{S}_c} \int_{\mathcal{M}} \sqrt{g}\,|d\phi\wedge d\psi| \arrow[d, leftrightarrow, "\begin{array}{c}\phi\leftrightarrow U\\ \text{duality}\end{array}"] \\
\displaystyle\sup_{V\in\mathcal{F}} \int_{D(A)} *V \arrow[rr, leftrightarrow, "(V{,}\phi)\leftrightarrow(U{,}\psi)", "\text{duality}"'] \arrow[d, leftrightarrow, "\begin{array}{c}\text{V-flow}\leftrightarrow\text{V-thread}\\ \text{conversion}\end{array}"'] & & \displaystyle\inf_{U\in\mathcal{G}} \int_{\mathcal{I}^+} *U \arrow[d, leftrightarrow, "\begin{array}{c}\text{U-flow}\leftrightarrow\text{U-thread}\\ \text{conversion}\end{array}"] \\
\displaystyle\sup_{\mu} \mu(\mathcal{P}) \arrow[rr, leftrightarrow, "\mu\leftrightarrow\nu", "\text{duality}"'] & & \displaystyle\inf_{\nu} \nu(\mathcal{Q})
\end{tikzcd}$$

Lessons

- Power of Lagrangian duality & reformulations
  - New hints re. nature of HEE
    (perhaps V-threads ~ entanglement distillation; U-threads ~ entanglement of formation)
  - Important geometrical quantities (HRT surface, entanglement wedge) emerge naturally
  - Dependence on region switches between objective and constraint
  - Dual formulations for proofs (e.g. S_{c} obeys SA) -- looks different, has different advantages, so may be useful in establishing further properties of HEE...
  - Convex programs, so computationally convenient

- Simpler structure for holographic spacetimes
  - HRT surface (= localized on intersection of slice and time-sheet) is a global saddle point, so maximin = minimax even without convex relaxation
  - Convex-relaxed expressions (V-flows, U-flows) retain this via the optimized flows collimated through the HRT surface

Extensions & future directions

- Easy extensions (developed in paper)
    - Embedding into full ST & removing the regulator
    - Multiple regions
- HEE properties from minimax
- Covariant formulations for quantum and stringy corrections to HEE
- Inspire formulations for tensor networks w/ time
- Covariant formulations for generalizations / other contexts
    - weaker norm bounds for multiflows
    - thread / hyperthread constructions for multipartite entanglement measures
    - non-AdS backgrounds

Thank you