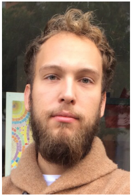
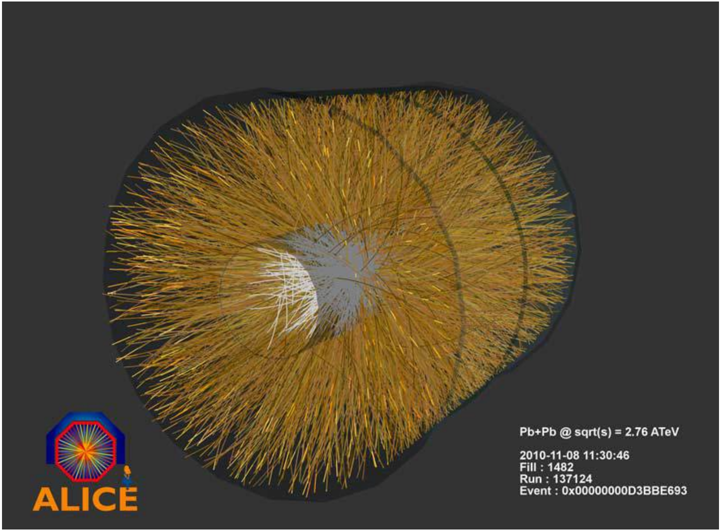
# Hydrodynamics of the $O(4)$ critical point

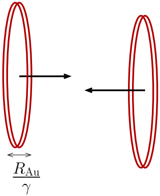
Derek Teaney
Stony Brook University

- Eduardo Grossi, Alex Soloviev, DT, Fanglida Yan: PRD, arXiv:2005.02885
- Eduardo Grossi, Alex Soloviev, DT, Fanglida Yan: PRD, arXiv:2101.10847
- Adrien Florio, Eduardo Grossi, Alex Soloviev, DT, PRD, arXiv:2111.03640

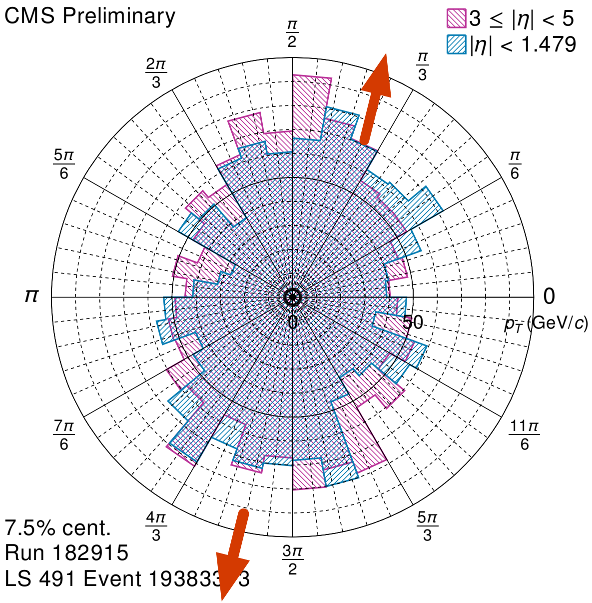


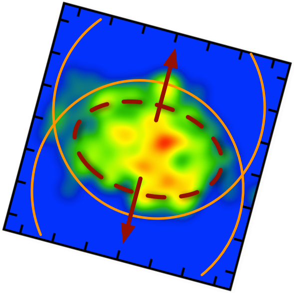
# Colliding Nuclei and Creating Plasma of Quarks and Gluons



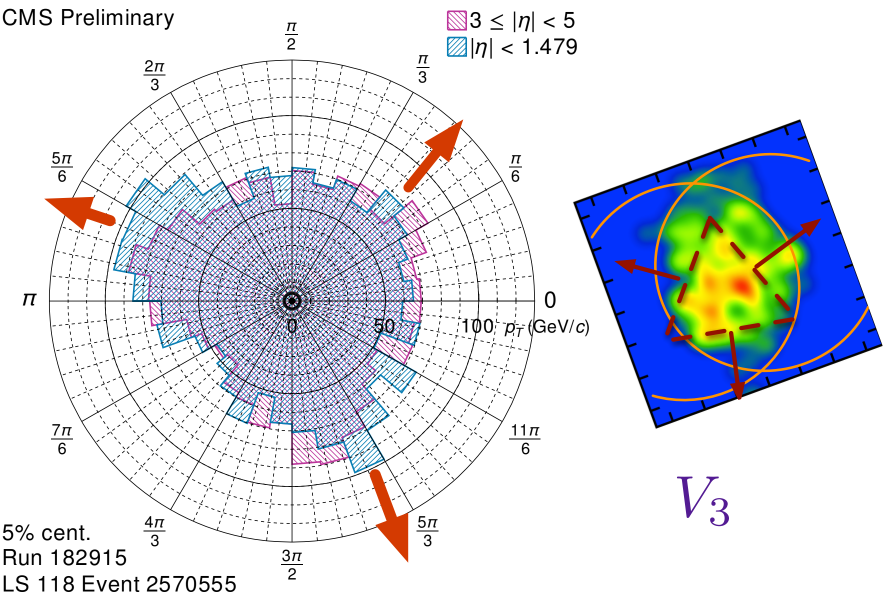



# Measuring the hydrodynamics of the plasma



$V_2$



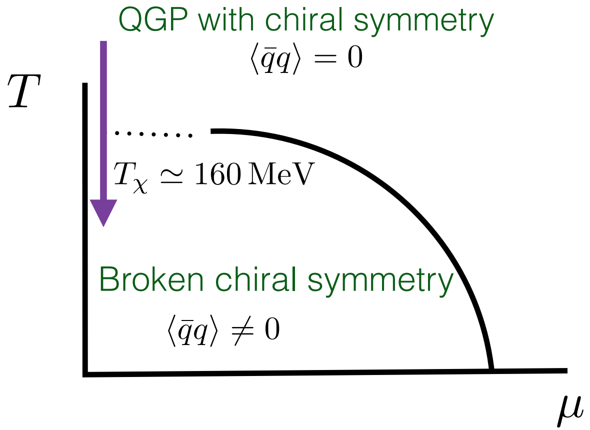
CMS Preliminary
$3 \leq |\eta| < 5$
$|\eta| < 1.479$
$\frac{\pi}{2}$
$\frac{2\pi}{3}$
$\frac{\pi}{3}$
$\frac{5\pi}{6}$
$\frac{\pi}{6}$
$\pi$
0
0
50
100
$p_T$ (GeV/$c$)
$\frac{7\pi}{6}$
$\frac{11\pi}{6}$
$\frac{4\pi}{3}$
$\frac{5\pi}{3}$
$\frac{3\pi}{2}$
5% cent.
Run 182915
LS 118 Event 2570555
$V_3$

# Amazing Hydro: the "Standard" Hydro Model

1. $V_1 \ldots V_6$
2. Momentum dependence $V_n(p)$
3. Probabilities $P(|V_n|^2)$
4. Covariances between harmonics: $\left\langle V_2 V_3 V_5^* \right\rangle$
5. Full covariance matrix: $\left\langle V_2(p_1) V_2^*(p_2) \right\rangle$

# Chiral symmetry breaking and heavy ion collisions

Pisarski, Wilczek




For two massless flavors the chiral symmetry group

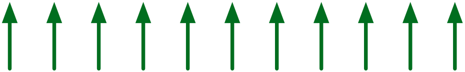
$$SU_L(2) \times SU_R(2) \simeq O(4)$$

is broken, and the transition is 2nd order.

The mass smooths the transition to a crossover

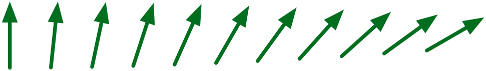
Chiral symmetry plays no role in the “Standard Hydro Model” ...

## Our cold world: T< Tcritical



$$\langle \bar{q}_R q_L \rangle = \bar{\sigma}\, \mathbb{I}_{2\times 2}$$

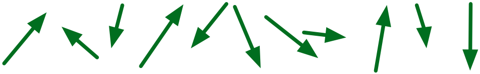
State is ordered. $\langle \bar{q}_R q_L \rangle$ is like the magnetization. $q = u, d$



$$\bar{q}_R q_L = \bar{\sigma}\, e^{i\vec{\tau}\cdot\vec{\varphi}(x)}$$

The slow modulation of the $SU_A(2)$ phase of $\bar{q}_R q_L$ is the pion

## The hot world: T> Tcritical



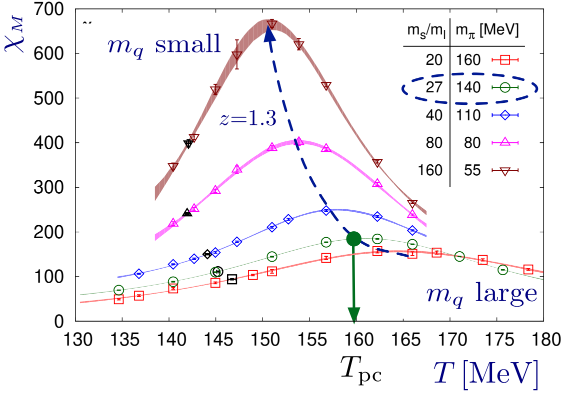
State is disordered: pion propagation is frustrated

This talk will describe pion propagation during the $O(4)$ phase transition

Real world lattice QCD and the $O(4)$ critical point: Hot QCD, PRL 2019

Fluctuations of $\sigma \propto \bar{u}u + \bar{d}d$

$$\chi_M = \langle \sigma^2 \rangle - \langle \sigma \rangle^2$$




O(4) Scaling predictions

$$\chi_M = m_q^{1/\delta - 1} f_\chi(z)$$

$$z \equiv z_0 \frac{(T - T_c)}{T_c} m_q^{-1/\beta\delta}$$

The QCD lattice knows about the $O(4)$ critical point! Hydro should too!

# QCD, the Chiral limit, Broken Symmetry, and Hydro:

Son hep-ph/9912267; Son and Stephanov hep-ph/020422

1. The approximately conserved quantities $\vec{J}^{\mu}_{A} = \bar{\psi}\gamma^5\gamma^{\mu}\vec{\tau}\psi$

$$\underbrace{T^{\mu\nu}}_{\text{stress}} \qquad \underbrace{J^{\mu}_{B}}_{\text{Baryon number}} \qquad \underbrace{\vec{J}^{\mu}_{V}}_{\text{isovector}} \quad \textit{and} \quad \underbrace{\vec{J}^{\mu}_{A}}_{\text{iso-axial vector}}$$

2. There is the phase of the chiral condensate and pion field: $\vec{\varphi} = \vec{\pi}/F$

$$\Sigma = \sigma \cdot U = \sigma \cdot \text{Phase of } \bar{q}_R q_L \equiv \sigma\, e^{i\vec{\tau}\cdot\vec{\varphi}}$$

3. The pion $\vec{\varphi}$ is like $T$, $u^{\mu}$, $\vec{\mu}_I$, *and* $\vec{\mu}_A$ in the constitutive relations
4. Include a mass term so the Goldstone fields decay at large distances

Need to write down a theory of superfluid hydro for $\varphi$ (Son '99)

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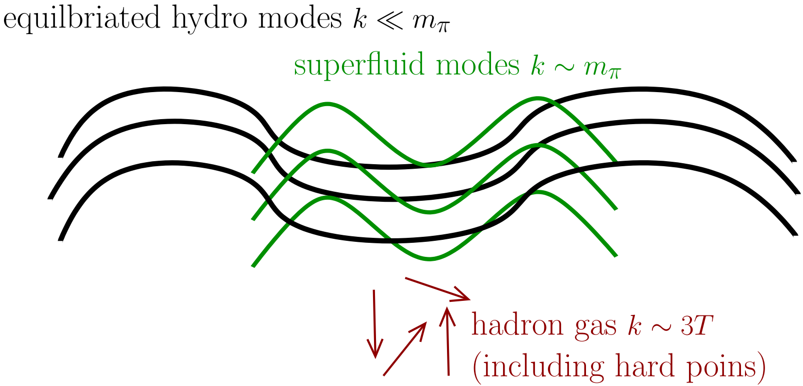
Near the critical point the $\sigma(t, x)$ has dynamics too!

# Picture for $T \lesssim T_c$

Son,Stephanov hep-ph020422

- Work in the regime

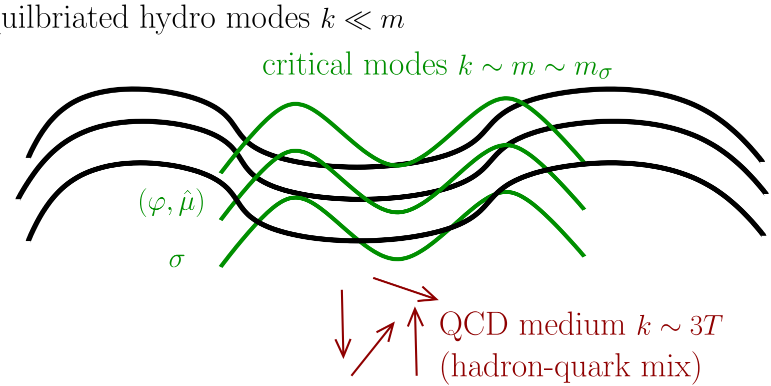
$$k \ll m_\pi \ll \pi T \sim \pi \Lambda_{QCD}$$



# Picture for $T \sim T_c$

- Work in the regime:

$$k \ll m_\pi \sim m_\sigma \ll \pi T_C \sim \pi \Lambda_{QCD}$$



# Hydro below $T_c$

## The pressure from soft pion modes:

- Use 3D dimensionally reduced chiral perturbation theory:

$$Z_{QCD} = \underbrace{e^{\beta p_0(T,\mu_A)V}}_{\text{from hard modes } p\sim T} \times \underbrace{\int^{\Lambda} [D\varphi] \exp\left(-\beta \int d^3\boldsymbol{x}\mathcal{L}_{\rm eff}\right)}_{\text{from soft modes } p\sim m_\pi}$$

where $U = e^{i\vec{\varphi}\cdot\vec{\tau}}$

$$\mathcal{L}_{\rm eff} \simeq \frac{f^2(T)}{4}\,{\rm Tr}\,\nabla U\cdot\nabla U^{\dagger} + \frac{f^2 m^2(T)}{2}{\rm Re}\,{\rm Tr}\,U$$

- For small angular fluctuations

$$\mathcal{L}_{\rm eff} \simeq \frac{f^2}{2}(\nabla\varphi)^2 + \frac{f^2m^2}{2}\varphi^2$$

The parameters $f^2(T)$ and $f^2m^2(T)$ decrease near the $O(4)$ crit. point:

$$f^2m^2 \propto m_q\left\langle\bar{\psi}\psi\right\rangle \propto m_q t^{\beta} \qquad\qquad t\equiv (T-T_c)/T_c$$

# Stress and Current for Superfluids:

Son; Jensen et al 1203.3556, Bhattacharya et al

The pressure in the presence of the phase is $p_\varphi$

$$p_\varphi(T, \nabla\varphi, \varphi^2) = p_0(T) + \frac{1}{2}\chi_A \mu_A^2 - \frac{f^2}{2}\left((\nabla\varphi)^2 + m^2\varphi^2\right)$$

- Derive the ideal stress and current from pressure $W = \int d^4x \sqrt{g} p_\varphi$

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta W}{\delta g_{\mu\nu}} = (e_\varphi + p_\varphi)u^\mu u^\nu + \eta^{\mu\nu} p_\varphi + \underbrace{f^2 \partial^\mu\varphi\partial^\nu\varphi}_{\text{super fluid stress}}$$

$$\vec{J}^\mu_A = \frac{1}{\sqrt{-g}}\frac{\partial W}{\partial \vec{A}_\mu} = \vec{n}_A u^\mu + \underbrace{f^2\partial^\mu\vec{\varphi}}_{\text{super fluid current}}$$

- Requiring that the ground state is stable, $[\bar{q}_R q_L, H - \mu_A N_A] = 0$:

$$\underbrace{-u^\mu \partial_\mu \vec{\varphi} = \vec{\mu}_A}_{\text{Josephson constraint}}$$

# Dissipative corrections to EOM

Son, Stephanov hep-ph/020422 + a bit by us

$$\underbrace{\partial_t J^0_A + \nabla\cdot\boldsymbol{J}_A = f^2 m^2 \varphi}_{\text{PCAC}} \qquad \text{and} \qquad \underbrace{-\partial_t\varphi = \mu_A}_{\text{Josephn's constraint}}$$

- Then expand the current in gradients

$$\boldsymbol{J}_A = \underbrace{f^2\nabla\varphi}_{\text{ideal current}} - \underbrace{\lambda_0\,\nabla\mu_A}_{\text{axial conductivity}} + \xi_J$$

and the josephson constraint

$$-\partial_t\varphi = \underbrace{\mu_A}_{\text{ideal}} + \underbrace{\kappa_2\,\nabla^2\varphi + \kappa_1\,m^2\varphi}_{\text{visc correction}} + \xi_m$$

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# Dissipative corrections to EOM

Holo: Baggioli 1905.09488, Anmon et al 2111.1035, Donos et al 2107.03680; Hydro: Delacretz et al 2111.13459, Armas et al 2112.14373

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- To reach the equilibrium fluctuations we must add noise:

$$\left\langle \xi^i_J \xi^j_J \right\rangle = 2T\lambda_0 \, \delta^{ij} \delta^4(x - x') \qquad \langle \xi_m \xi_m \rangle = 2T\lambda_m \, \delta^4(x - x')$$

# Long wavelength pion (superfluid) modes: Son, Stephanov hep-ph/020422 + a bit by us


- Linearizing the equation of motion $\varphi = Ce^{-i\omega t + i\boldsymbol{q}\cdot\boldsymbol{x}}$ one finds

$$\varphi(t, \boldsymbol{q}) = Ce^{-(\Gamma/2)t} e^{-i\omega_q t} \quad \Leftarrow \text{This is second sound}$$

- The quasi-particle energy is:

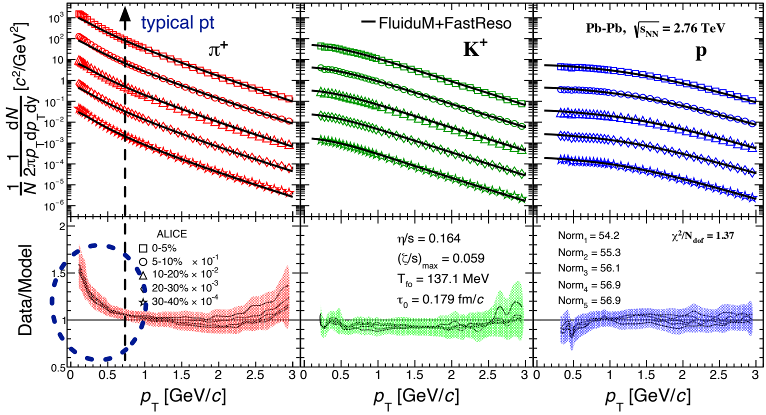
$$\omega_q^2 \equiv v_0^2(q^2 + m^2) \qquad\qquad v_0^2(T) \equiv \frac{f^2}{\chi_A} \quad \Leftarrow \text{pion velocity}$$

The pion velocity has a simple interpretation, $\chi_A = f^2 + \Delta\chi_A$

$$v_0^2 \equiv \frac{f^2}{f^2 + \Delta\chi_A} = \frac{\text{super}}{\text{super} + \text{normal}} \to 0 \text{ near } T_c$$

# Evidence for the chiral crossover in the heavy ion data?

A recent ordinary hydro fit from Devetak et al 1909.10485




Because the pions are the Goldstones expect an enhancement at low $p_T$

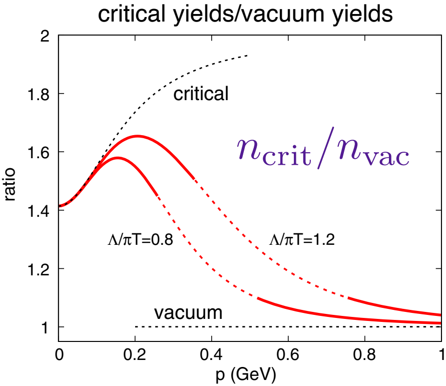
$$n(\omega_q) = \frac{1}{e^{vq/T} - 1} \simeq \frac{T}{vq} \Rightarrow \infty, \qquad \text{Since at } T_c, \text{ the velocity } v \Rightarrow 0 \ !$$

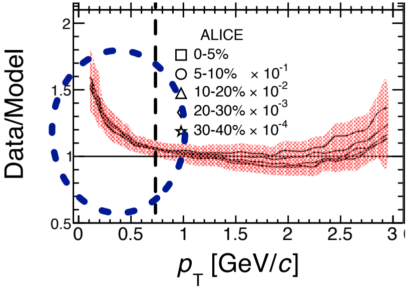
With a modified dispersion curve (relative to vacuum) the yields increase

$$n(\omega(p)) = \frac{1}{e^{\omega(p)/T} - 1} \qquad \omega^2(p) = v^2(T)(p^2 + m^2(T))$$

$v$ goes to zero at $T_c$

We estimated the drop in $v^2(T)$ and $v^2 m^2(T)$ from lattice data ...




Encouraging estimate which motivates additional work on critical dynamics

# Hydro at $T_c$

# Hydrodynamics of the $O(4)$ transition:

Rajagopal and Wilczek '92, Son '99, Son and Stephanov '01, and finally us, arxiv:2101.10847.

1. The order parameter

$$\phi_a = (\sigma, \vec{\pi}) \qquad \Sigma = \sigma - i\vec{\tau}\cdot\vec{\pi}$$

2. The approximately conserved charges quantities:

$$\vec{n}_V = \underbrace{\bar{\psi}\gamma^0\vec{\tau}\psi}_{\text{isovect chrg}} \qquad \textit{and} \quad \vec{n}_A = \underbrace{\bar{\psi}\gamma^0\gamma^5\vec{\tau}\psi}_{\text{isoaxial-vect chrg}}$$

which are combined into an anti-symmetric $O(4)$ tensor $n_{ab}$

$$n_{ab} = (\vec{n}_A, \vec{n}_V)$$

The charge $n_{ab}$ generates $O(4)$ rotations, $\phi \rightarrow \phi_c + \frac{i}{\hbar}\theta_{ab}[n_{ab}, \phi_c]$, implying a Poisson bracket between the hydrodynamic fields:

$$\{n_{ab}(\boldsymbol{x}), \phi_c(\boldsymbol{y})\} = \epsilon_{abcd}\, \phi_d(\boldsymbol{x})\, \delta(\boldsymbol{x} - \boldsymbol{y})$$

## The Landau-Ginzburg Hamiltonian for the $O(4)$ transition:

The Hamiltonian is tuned to the crit. point with $m_0^2(T) < 0$ and $H \propto m_q$:

$$\mathcal{H} = \int d^3x \, \frac{1}{2}\nabla\phi^2 + \frac{1}{2}m_0^2(T)\phi^2 + \frac{\lambda}{4}\phi^4 - H\sigma + \frac{n^2}{4\chi_0}$$

and gives the equilibrium distribution with the correct critical EOS:

$$Z = \int D\phi \, Dn \, e^{-\mathcal{H}[\phi,n]/T_c}$$

The hydro equations of motion take the form

$$\frac{\partial\phi}{\partial t} + \{\phi, \mathcal{H}\} = 0 + \text{visc. corrections} + \text{noise}$$

$$\frac{\partial n_{ab}}{\partial t} + \{n_{ab}, \mathcal{H}\} = 0 + \text{visc. corrections} + \text{noise}$$

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## The hydro equations of motion take the form

$$\frac{\partial\phi}{\partial t} + \{\phi, \mathcal{H}\} = -\Gamma\frac{\delta\mathcal{H}}{\delta\phi_a} + \xi_a$$

$$\frac{\partial n_{ab}}{\partial t} + \{n_{ab}, \mathcal{H}\} = \underbrace{\sigma_0\nabla^2\frac{\delta\mathcal{H}}{\delta n_{ab}}}_{\text{dissipation}} + \underbrace{\nabla\cdot\xi_{ab}}_{\text{noise}}$$

## The equations and the simulations:

We have a charge diffusion equation coupled to order parameter:

$$\partial_t n_{ab} + \underbrace{\nabla \cdot (\nabla \phi_{[a} \phi_{b]})}_{\text{poisson bracket}} + H_{[a} \phi_{b]} = \underbrace{D_0 \nabla^2 n_{ab}}_{\text{diffusion}} + \underbrace{\nabla \cdot \xi_{ab}}_{\text{noise}}$$

and a rotation of the order parameter induced by the charge+relaxation:

$$\partial_t \phi_a + \underbrace{\frac{n_{ab}}{\chi_0} \phi_b}_{\text{poisson bracket}} = \underbrace{-\Gamma_0 \left( \nabla^2 \phi_a + \frac{\partial V}{\partial \phi_a} \right)}_{\text{dissipation}} + \underbrace{\xi_a}_{\text{noise}}$$

## Numerical scheme based operator splitting:

1. Evolve the Hamiltonian evolution with a position Verlet type stepper
2. Treat the dissipative Langevin steps as Metropolis-Hastings updates

## Overview of simulations:

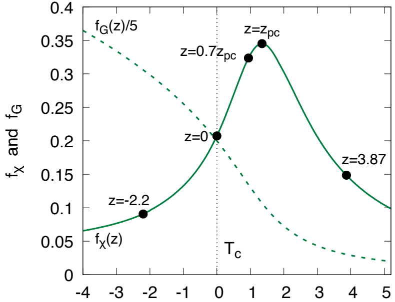
Engels, Fromme, Seniuch; Engels, Karsch; Hasenbuch; Kanaya&Kaya

1. Tune the mass parameter to the critical point $m_0^2 = m_c^2$
2. Measure vev and other static quantities, which take the form:

$$\bar{\sigma} = h^{1/\delta} f_G(z) \qquad z = t_r h^{-1/\beta\delta}$$

   with scaling parameters, $h = H/H_0$ and $t_r = (m_0^2 - m_c^2)/\mathfrak{m}^2$

3. The dynamical measurements scan the phase transition



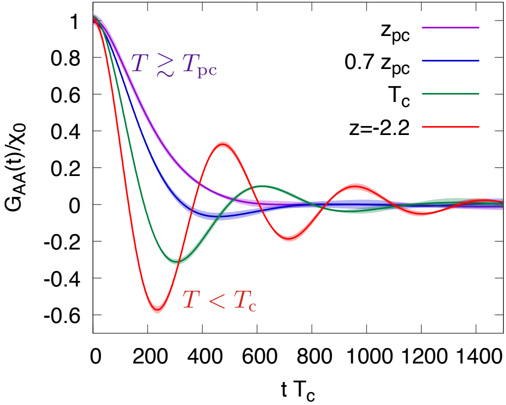

$$z \propto (T - T_c)$$

$$\chi_{\parallel} = \frac{\partial \bar{\sigma}}{\partial H} = \frac{h^{1/\delta - 1}}{H_0} f_\chi(z)\,.$$

## Features of the phase transition in the axial charge correlations:

$$G_{AA}(t) = \int \mathrm{d}^3x \; \langle \vec{n}_A(t, \boldsymbol{x}) \cdot \vec{n}_A(0, \mathbf{0}) \rangle$$

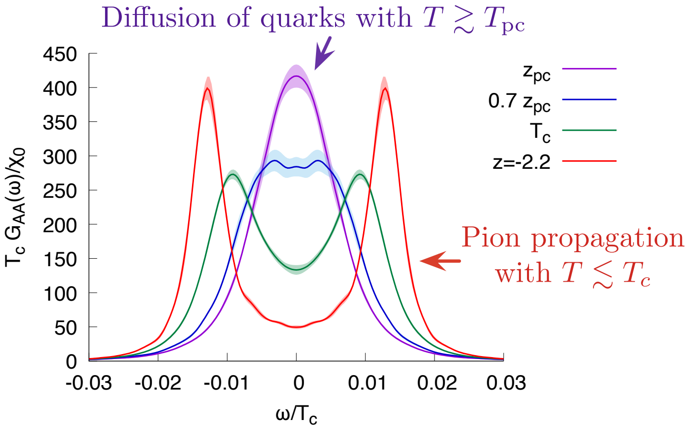
See a change in the dynamics across $T_{\rm pc}$:




Let's take a fourier transform and analyze the transition

Features of the phase transition in the axial charge correlations:

$$G_{AA}(\omega) = \int \mathrm{d}t\, \mathrm{d}^3x\, e^{i\omega t}\, \langle \vec{n}_A(t, \boldsymbol{x}) \cdot \vec{n}_A(0, \boldsymbol{0}) \rangle$$



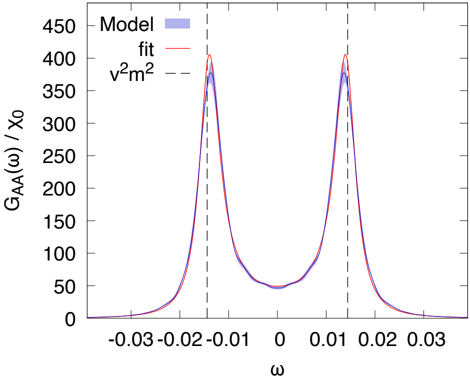

Can see the transition from diffusion of quarks to propagation of pions!

# Quantitative analysis of the pion EFT below $T_c$, at $z = -2.2$:

The predicted pole position $m_p^2$ of pion waves is given by static quantities:

$$m_p^2 = v^2 m^2 = \frac{H\bar{\sigma}}{\chi_0}$$

This is the finite temperature Gell-Mann Oakes Rener relation:



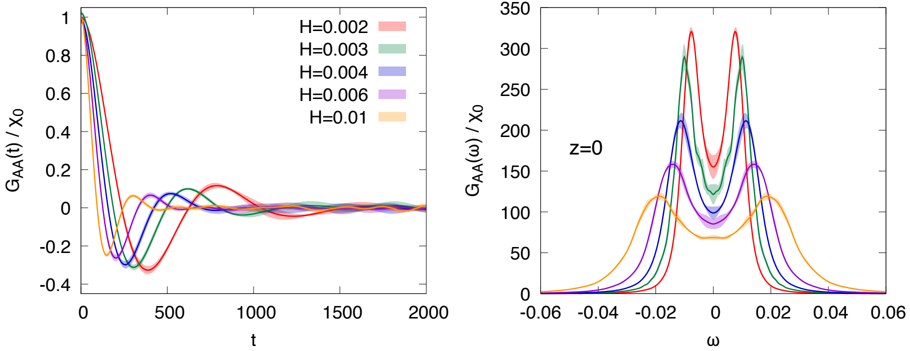

These are static inputs to fit:

$$v^2 = \frac{f^2}{\chi_A}$$

$$m^2 = \text{screening mass}$$

# Scaling of simulations at $T_c$:

At $T = T_c$, we varied the magnetic field, finding the response functions:




See a scaling behavior of the real time correlations

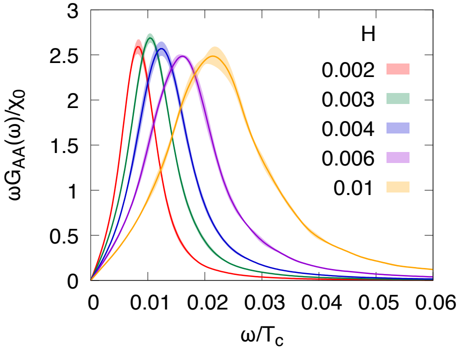
# Dynamical critical exponent of the $O(4)$ transition:

Rajagopal, Wilczek

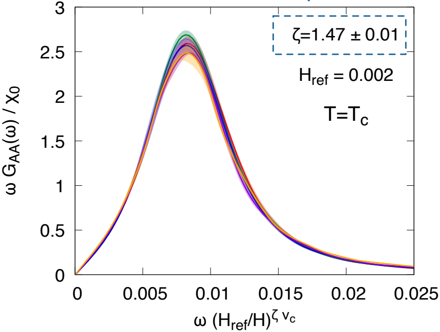
The relaxation time and correlations *scale* with the correlation length $\xi$:

$$\omega\, G_{AA}(\omega,\xi) = \underbrace{f(\omega\,\tau_R)}_{\text{universal fcn}} \quad \text{with} \quad \underbrace{\tau_R \propto \xi^{\zeta}}_{\text{relaxation time}}$$

The correlation length scales as $\xi \propto H^{-\nu_c}$ and the time as $\tau_R \propto H^{-\zeta\nu_c}$:




compare to 3/2



# Summary and Outlook:

1. We are simulating the real-time dynamics of the chiral critical point
   - The numerical method may be useful for stochastic hydro generally
2. We reproduced the expected dynamical scaling laws:

$$\tau_R \propto \xi^{\zeta} \qquad \zeta = \frac{d}{2} \simeq 1.47 \pm 0.01$$

3. The pion waves are well calibrated
4. The next step is to study the expanding case:
   - This will predict soft pions and their correlations with expansion

The hadronization of the pion is the (only) hadronization process that can be studied rigorously, *and only with hydrodynamics!*