The Hilbert space of Chern Simons matter theories

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References

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- Uses large N thermal partition function computed in

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Introduction

- Chern Simons theories coupled to dynamical matter fields are of interest for several reasons.
- First, in parity non invariant theories, the one derivative Chern Simons Lagrangian generically dominates the two derivative Yang Mills kinetic term and so governs gauge dynamics at low energies.
- Second, the Chern Simons coupling, $\frac{1}{k}$, does not flow under the renormalization group, so fine tuning matter masses to zero often results in conformal dynamics.
- Third, Chern Simons matter theories host anyonic excitations with 'non half integer' spins whose S matrices display unusual crossing properties
- Fourth, some of these theories have conjectured AdS/CFT dual descriptions in large N limits.
- Fifth, some of these theories they enjoy invariance under (conjectured) strong weak coupling Bose Fermi duality even without sypersymmetry.

Sixth, and most importantly for this talk, several exact results are available for two interesting limits of these theories.

- (1) When the mass of the matter fields is taken to infinity, our theories reduce to pure Chern Simons theory which has an intricate, beautiful and very thoroughly understood exact solution.
- (2) When all matter fields are in the fundamental, and N and k are taken to infinity with

$$\lambda = \frac{N}{k + \mathrm{sgn}(k)N} \equiv \frac{N}{\kappa}$$

held fixed, the theory is once again exactly solvable. Several interesting dynamical quantities have been exactly computed in this limit.

Introduction

- Solvable limits of theories are special. Hilbert Spaces of theories in such limits often admit mathematically elegant descriptions - sometimes given by imposing natural constraints on free theries.
- In this talk I will attempt to present such a description of the Hilbert space of the Chern Simons fundamental matter theories in the solvable large N limits described in the previous transparency. It is possible that the lessons learnt will also have some value away from the large N limit.
- The all orders expression for the thermal free energy of $S^2 \times S^1$ path integral has already been obtained [1]-[11] by analytically summing all planar diagrams. We will use the most complete expressions - presented in [11] S.M, A. Mishra, N. Prabhakar, 2020 as data for the analysis of this talk.

The Theory

- In this talk I focus attention on a particular Matter Chern Simons theory - the so called Regular Boson theory
- This theory is defined by the Langranian

$$U(N_B)_{k_B} + \int D_\mu \bar{\phi} D^\mu \phi + m_B^2 \bar{\phi}\phi + \frac{4\pi b_4}{\kappa_B}(\bar{\phi}\phi)^2 + \frac{(2\pi)^2}{\kappa_B^2}\left(x_6^B + 1\right)(\bar{\phi}\phi)^3, \quad (1)$$

- In our paper we also study the critical (Wilson Fisher) Boson theory as well as the regular fermion and critical fermion theories. In the interest of time, however, I will focus on a single theory in this talk.

Theory: Exact Quantum effective potential

- Even at large N, the interaction of the matter with the gauge field in this theory makes the diagramatic solution of this theory is a challenging - but doable- task.
- One effect of the gauge field interactions is to renormalize the effective potential for the gauge invariant field $\bar{\phi}\phi$. In the last ISM I had described the computation of the exact quantum effective action for $\bar{\phi}\phi$. Though the computation was complicated, the final result was very simple. In the so called un Higgsed phase (on which we focus in this talk) the entire effect of summing all gauge loops is to renormalize the classical potential

$$+m_B^2\bar{\phi}\phi + \frac{4\pi b_4}{\kappa_B}(\bar{\phi}\phi)^2 + \frac{(2\pi)^2}{\kappa_B^2}\left(x_6^B + 1\right)(\bar{\phi}\phi)^3$$

to the exact quantum value

$$+m_B^2\bar{\phi}\phi + \frac{4\pi b_4}{\kappa_B}(\bar{\phi}\phi)^2 + \frac{(2\pi)^2}{\kappa_B^2}\left(x_6^B + \frac{4}{3}\right)(\bar{\phi}\phi)^3$$

Thermal partition function: First Pass 1

- Like the computation of the exact quantum potential, the interaction of matter with the gauge field makes the computation of the thermal partition function of this theory computationally nontrivial.
- As a first pass one might try to compute the thermal partition function of the ungauged theory - but replacing the classical potential with the quantum effective potential.
- I.e. one might try to compute the thermal partition function of the theory

$$S_{RB}^{eff} = \int D_\mu \bar{\phi} D^\mu \phi + m_B^2 \bar{\phi}\phi + \frac{4\pi b_4}{\kappa_B} (\bar{\phi}\phi)^2 + \frac{(2\pi)^2}{\kappa_B^2} \left(x_6^B + \frac{4}{3} \right) (\bar{\phi}\phi)^3, \quad (2)$$

- This computation is trivial to perform as I now explain

Thermal partition function: First pass 2

- Notice that the ungauged scalar large N matter theory

$$S^{eff}_{RB}[\phi] = \int d^3x \, \left((\partial_\mu \bar{\phi})(\partial^\mu \phi) + m_B^2 \bar{\phi}\phi \right) + \frac{4\pi b_4}{\kappa_B} (\bar{\phi}\phi)^2 + \frac{(2\pi)^2}{\kappa_B^2} (x_6^B + \frac{4}{3})(\bar{\phi}\phi)^3 \right) \quad (3)$$

  can be recast, with the aid Lagrange Multipliers c_B^2 and σ, as

$$S = \int d^3x \left[(\partial_\mu \bar{\phi} \partial^\mu \phi) + c_B^2 (\bar{\phi}\phi) + \frac{N_B}{2\pi} (m_B^2 - c_B^2)\sigma + \frac{N_B \tilde{b}_4 \lambda_B}{\pi} \sigma^2 + \frac{N_B \lambda_B^2}{2\pi} (x_6^B + \frac{4}{3})\sigma^3 \right] . \quad (4)$$

  The equivalence of these two expressions can be seen as follows. The c_B^2 equation of motion sets $\sigma = \frac{2\pi \bar{\phi}\phi}{N}$. Substituting in this value then reduces (4) to (3)

- It follows that the thermal partition function of this theory is given by the extremization over c_B^2 and σ of

$$Z^{eff} = \left[e^{-N_B \mathcal{V}_2 \beta \left(\frac{1}{2\pi} (m_B^2 - c_B^2)\sigma + \frac{\tilde{b}_4 \lambda_B}{\pi} \sigma^2 + \frac{\lambda_B^2}{2\pi} (x_6^B + \frac{4}{3})\sigma^3 \right)} \, \mathrm{tr}_{\mathcal{H}^{Fock}_{c_B}} \left(e^{-\beta H_{c_B}} \right) \right] \quad (5)$$

  where $\mathcal{H}^{Fock}_{c_B}$ is the free Fock space of a scalar of mass c_B

Thermal partition function: Second Pass

- The computation of the previous section works on any spatial manifold. Let us now specialize to an S^2 of volume V_2.
- On such a space it is certainly too crude to ignore the $U(N)$ gauge field completely. At the very least we should account for the fact that (naively speaking) the $U(N)$ Gauss law ensures that only $U(N)$ singlets are allowed states on the sphere.
- The formula of the previous slide is easily modified to account for this constraint. We find the extremization over c_B^2 and σ of

$$\left[e^{-N_B \mathcal{V}_2 \beta \left(\frac{1}{2\pi}(m_B^2 - c_B^2)\sigma + \frac{\tilde{b}_4 \lambda_B}{\pi}\sigma^2 + \frac{\lambda_B^2}{2\pi}(x_6^B + \frac{4}{3})\sigma^3 \right)} \int dU \, \mathrm{Tr}_{\mathcal{H}_{c_B}^{Fock}} \left(U e^{-\beta H_{c_B}} \right) \right]_{\{c_B, \sigma\}} \quad (6)$$

where U is a unitary matrix and dU is the Haar measure.

- Let us now compare our crude guesses with the result of a real computation. In the limit $N \to \infty$, $k \to \infty$, $\mathcal{V}_2 \to \infty$, with all ratios of these quantities, as well as the temperature T, chemical potential μ, and all masses and couplings held fixed, we show that the known answer for the S^2 times S^1 partition function $\mathcal{Z}_{S^2 \times S^1}$, in the un Higgsed phase, simplifies to

$$Z_{RB} = \left[e^{-N_B \mathcal{V}_2 \beta \left(\frac{1}{2\pi}(m_B^2 - c_B^2)\sigma + \frac{b_4 \lambda_B}{\pi}\sigma^2 + \frac{\lambda_B^2}{2\pi}(x_6^B + \frac{4}{3})\sigma^3 \right)} I_B(c_B) \right]_{c_B, \sigma, \mathcal{S}} \quad (7)$$

.

- Here

$$I_B = \int [dU]_{\rm CS} \, {\rm Tr}_{H_{NS}} \left[\left(U e^{-\beta(H - \mu Q)} \right) \left(e^{-\frac{N_B T^2 \mathcal{V}_2}{2\pi} \Theta(|\mu| - c_B) \frac{(|\hat{\mu}| - \hat{c}_B)^2 (|\hat{\mu}| + 2\hat{c}_B)}{6|\lambda_B|}} \right) \right] \quad (8)$$

Modified Haar Measure

- The expression for I_B on the previous slide was presented as an integral over unitary matrices. The measure for this integral, $[dU]_{CS}$, is defined as follows.
- $[dU]_{CS}$ is the usual Haar measure subject to the constraint

$$\rho(\alpha) \leq \frac{1}{2\pi|\lambda|} , \quad \rho(\alpha) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \delta(\alpha - \alpha_j) . \tag{9}$$

($e^{i\alpha_j}$ are the eigenvalues of U and $\rho(\alpha)$ is the eigenvalue distribution function).

- Comparing (7) and (6) we that our crude guess (6) was almost correct. The only correction it needed was the replacement

$$\int dU \, \mathrm{tr}_{\mathcal{H}^{Fock}_{c_B}} \left(U e^{-\beta H_{c_B}} \right) \to I_B(c_B)$$

Interpretation of I_B

- I will now explain that I_B is the large N limit of the free Bosonic Fock space, restricted to the space of WZW (or quantum group) singlets.
- In this talk I will focus attention on the so called Type I $U(N)_k$ theory. In this theory the level for the $SU(N)$ factor is k while the level for the $U(1)$ factor is κ. Everything I say here has simple generalizations to $SU(N)_k$ theory as well as the Type II $U(N)_k$ theory, the theory whose $SU(N)$ and $U(1)$ levels are both k.

- Consider pure Chern Simons theory on $S^2 \times S^1$ with m Wilson lines, each at a point on the S^2 but wrapping the S^1 once. The Wilson lines in question transform in the representations $R_1, R_2 \ldots R_m$ where each representation is restricted to be integrable. Over 30 years ago, Witten famously demonstrated that the result of this path integral is an integer which counts the number of WZW conformal blocks on S^2 with primary insertions in the representations $R_1, R_2 \ldots R_m$. This number can be evaluated using the Verlinde formula, but it may also be evaluated directly in 3d in two different ways.

Path integrals to count conformal blocks

- Again, almost 30 years ago, Blau and Thompson used a clever gauge fixing to explicitly evaluate the CS path integral on $S^2 \times S^1$. We have generalized their explicit $SU(2)_k$ results are easily generlized to Type I $U(N)_k$. We list our results on the next slide.
- Alternately it is possible to use the fact that the $\mathcal{N} = 2$ supersymmetric pure Chern Simons theory is identical to ordinary pure Chern Simons theory, to compute the same quantity using supersymmetric localization. Both methods give the same result, and also agree with the Verlinde formula - but rewritten in a simple and physically useful form.
- Caution: All formulae overleaf apply only to integrable insertions. Unsatisfying. Would be good to understand better why.

Counting Type I $U(N)$ conformal blocks on S^2

- We find

$$N_{sing} = \frac{1}{\kappa^N} \sum_{\{w_i\}} \prod_{i<j} |w_i - w_j|^2 \prod_{p=1}^{n} \chi_{R_p}(w_i) \qquad (10)$$
$$w_m^\kappa = (-1)^{N+1} \quad \forall m$$

- The phase $(-1)^{N+1}$ in the formula arises from a careful regulation of one loop determinants, and was absent in the $SU(N)$ path integral evaluated by Blau and Thompson.
- (10) is a very particular discretization of the Weyl integral formula of classical group theory. In the large N limit the spacing between two eigenvalues, $\frac{1}{2\pi\kappa} \to 0$ the discretization spacing $\to 0$ so (10) reduces to the classical Weyl formula except for one constraint;

$$\rho(\theta) \leq \frac{2\pi N}{\kappa} = \frac{2\pi}{\lambda}.$$

$[dU]_{CS}$ from conformal blocks

- In equations, in the t'Hooft large N limit

$$\frac{1}{\kappa^N}\sum_{\{w_i\}}\prod_{\mathrm{i<j}}|w_i-w_j|^2\prod_{p=1}^{n}\chi_{R_p}(w_i)\to\int[dU]_{CS}\prod_{p=1}^{n}\chi_{R_p}(w_i)$$

- Now recall (8)

$$I_B=\int[dU]_{\mathrm{CS}}\,\mathrm{Tr}_{H_{NS}}\left[\left(Ue^{-\beta(H-\mu Q)}\right)\left(e^{-\frac{N_B T^2\mathcal{V}_2}{2\pi}\Theta(|\mu|-c_B)\ \frac{(|\hat{\mu}|-\hat{c}_B)^2(|\hat{\mu}|+2\hat{c}_B)}{6|\lambda_B|}}\right)\right] \tag{11}$$

- We see that the quantity I_B appears to agree precisely with our interpretation - the restriction the partition function of the Bosonic Fock Space restricted to the WZW singlet sector - provided we ignore the term

$$\left(e^{-\frac{N_B T^2\mathcal{V}_2}{2\pi}\Theta(|\mu|-c_B)\ \frac{(|\hat{\mu}|-\hat{c}_B)^2(|\hat{\mu}|+2\hat{c}_B)}{6|\lambda_B|}}\right).$$

$\theta(\mu - c_B)$... from conformal blocks

- Consider a Type I $U(N)$ Chern Simons theory coupled to fundamental bosons on S^2. Let a parameterize the positive energy solutions of the Klein Gordon equation, with mass c_B, on S^2. Clearly the U twisted partition function over the free bosonic Fock Space is given by a product of partition functions, one for every free particle state
- Explicitly

$$\mathrm{Tr}\left(Ue^{-\beta(H-\mu Q)}\right) = \prod_a \left[\left(\prod_{i_a=1}^{N_B} \frac{1}{1-e^{-\beta(E_a-\mu)}w_{i_a}}\right)\left(\prod_{i_a=1}^{N_B} \frac{1}{1-e^{-\beta(E_a+\mu)}w_{i_a}^*}\right)\right] \tag{12}$$

- Note that

$$\frac{1}{1-e^{-\beta(E_a-\mu)}w_{i_a}} = \sum_{n=0}^{\infty} e^{-n\beta(E_a-\mu)}\chi_n^S(U) \tag{13}$$

Truncation for Bosons

- Now the partition function of the bosonic Fock space restricted to WZW singlets is *not* given simply by

$$\frac{1}{\kappa^N}\sum_{\{w_i\}}\prod_{\mathrm{i<j}}|w_i - w_j|^2 \mathrm{Tr}\left(Ue^{-\beta(H-\mu Q)}\right) \tag{14}$$

- Terms with $n > k_B$ in (13) are non integrable insertions. Conformal blocks involving such insertions should vanish. (14) does not correctly account for this fact (see comment in red before (10)), which must thus be inserted by hand.
- The correct truncation of the free boson Fock space to WZW singlets is given by

$$I_B^l = \frac{1}{\kappa^{N_B}}\sum_{\{z_i\}}\prod_{\text{pairs}}|w_i - w_j|^2 \prod_a \left[\left(\prod_{i_a=1}^{N_B}\frac{1}{1-e^{-\beta(E_a-\mu)}w_{i_a}}\right)\Bigg|_{k_B} \left(\prod_{i_a=1}^{N_B}\frac{1}{1-e^{-\beta(E_a+\mu)}w_{i_a}^*}\right)\Bigg|_{k_B}\right]$$

Implication of the Truncation

- It is not difficult to prove that

$$Q(y) = \prod_{i=1}^{N_B} \frac{1}{1 - w_i y}\bigg|_{k_B} = \left(1 + (-1)^N y^\kappa\right) \prod_{i=1}^{N_B} \frac{1}{1 - z_i y} \tag{16}$$
$$= \exp\left(- \mathrm{tr} \ln\left(1 - yU\right) + \ln\left(1 - y^\kappa\right)\right)$$

- In the large N limit it follows that

$$\ln Q(y) = -\mathrm{tr} \ln\left(1 - yU\right) + \kappa \Theta(y - 1) \ln w \, . \tag{17}$$

- In the physical problem of interest $y = e^{-\beta(E-\mu)}$. The second term in (17) is thus nonzero only for states with $E < \mu$. Such states exist only if $c_B < \mu$. Adding up the contribution of all such states (accounting for the density of states) reproduces the extra term in (8) with all factors.

The Bosonic Exclusion Principle

- We have learnt something important here. In large N matter Chern Simons theories, no single particle bosonic state can be occupied more than k_B times. We call this the 'Bosonic Exclusion Principle'. It is the direct level rank dual of a more obvious result for fermionic theories, namely that no single particle fermionic state can be occupied more than N_F.
- Recall that ordinary free boson theories are ill defined at values of the chemical potential greater than the mass, as all states with energies between the mass and the chemical potential are infinitely occupied in such theories. The bosonic exclusion principle cures this singularity in matter Chern Simons theories, rendering Bosonic theories with chemical potential larger than the mass well defined.

I_B at large volume

- Let us recap. The partition function of large N matter Chern Simons theories is given by an expression of the same form as their ungauged counterparts, with the replacement of the Fock Space partition function to its WZW singlet projected counterpart.
- It is clear that the WZW singlet constraint has a huge effect on the partition function of the theory at small values of the sphere volume. But one might naively expect its impact to disappear in the large volume limit. This is indeed what happens for the Gauss Law constraint. Interestingly enough this expectation is incorrect.

‘Saddle Point’ at large Volume

- At finite N and k I_B is given by the formula (15) (I_F is given by a similar formula). The key simplification of the large volume limit is that the summation over choices of eigenvalues in that formula is dominated by a single eigenvalue configuration (this is a sort of saddle point approximation for the summation in that formula).
- The eigenvalue configuration that dominates the sum is

$$w_i = e^{i(\alpha_i)},$$

$$\{\alpha_i\} = \frac{2\pi}{\kappa}\left\{ -\frac{N-1}{2}, -\frac{N-3}{2}, -\frac{N-5}{2}, \ldots \frac{N-5}{2}, \frac{N-3}{2}, \frac{N-1}{2} \right\} \tag{18}$$

This configuration is the correct ‘saddle point’ for all cases; the $SU(N)$, Type II and Type I theories and for fermions and bosons. There is a one to one map between integrable representations and the collection of discretized eigenvalues that are summed over in the Verlinde formula. (18) is the eigenvalue configuration that maps to the Identity representation

Factorization at large volume

- At general values of the volume, the formula (15) expresses the partition function as a sum over products. As we have explained, at large volume the sum local to a single term, leaving us with a simple product, one term for each single particle state.

$$I_B \propto \prod_a Z_B^a \bar{Z}_B^a \,, \quad Z_B^a = \left(\prod_{i=1}^{N_B} \frac{1}{1 - e^{-\beta(E_a-\mu)} w_i} \right) \Bigg|_{k_B} \,, \qquad (19)$$
$$\bar{Z}_B^a = \left(\prod_{i=1}^{N_B} \frac{1}{1 - e^{-\beta(E_a+\mu)} w_i} \right) \Bigg|_{k_B} \,,$$

(the eigenvalues that apear in (19) are those listed on the previous slide)

-

$$Z_B^a = \sum_{r=0}^{k_B} \chi_n^S(U)\, e^{-r\beta(E_a-\mu)} = \sum_{r=0}^{k_B} d_n^S\, e^{-r\beta(E_a-\mu)} \qquad (20)$$

q numbers and quantum dimensions

- d_n^S is the so called quantum dimension of the n box symmetric representation. As mentioned above it equals the character of this representation evaluated on our special 'saddle point' unitary matrix This connection works for every representation, not justthe completely symmetric representation. Explicitly

$$d_n^S = \binom{n}{m}_q = \frac{[n]_q!}{[m]_q![n-m]_q!}$$
$$[m]_q! = [1]_q[2]_q \cdots [m]_q \qquad (21)$$
$$[r]_q = \frac{q^{r/2} - q^{-r/2}}{q^{1/2} - q^{-1/2}}$$

with $q = e^{\frac{2\pi i}{\kappa}}$

- Similar expressions hold for fermions. Using identities involving q factorials, the bosonic and fermionic expressions can be shown to be level rank dual.

Product but not Free

- Ignoring details, for the moment, an immediately striking aspect of the large volume limit is that the partition factorizes (its a product of partition functions, one for each single particle state).
- This feature may appear to suggest that our system is free in the infinite volume limit (whats going on in one single particle state does not affect the partition function of another single particle state).
- While this suggestion sounds initially reasonable, it is not correct. We can see this by noting that the coefficients of $e^{-\beta(E_n-\mu)}$ in the expansion of Z_n above are not integers. It follows that the different Z_n are not partition functions over independently defined Hilbert Spaces.

Explanation of the Product Structure

- In our paper we have demonstrated that this product structure is, infact, a manifestation of an interesting universality in the fusion rule algebra in the large insertion limit.
- Consider any generic collection of integrable representations of the WZW algebra $R_1 \dots R_n$.
- Let us now sequentially fuse our representations with each other. Once this process is completed let us suppose we are left with n_{R_i} representations of type R_i for each integrable representation R_i.
- In the limit that n is the largest number in the problem, we have demonstrated that

$$\frac{n_{R_i}}{n_{R_j}} = \frac{d_{R_i}}{d_{R_j}}$$

Independent of the details of the participating representations R_i.

Explanation of factorization

- This universality explains the factorization of our partition functions as follows
- The coefficient of $e^{-n\beta(E_n-\mu)}$ in Z_n is actually proportional to the number of sea particles in the representation conjugate to the n box symmetric representation.
- The conjecture of the previous slide explains why this number is independent of the precise state of the 'sea', explaining why the product structure of single particle
- The universality described in our conjecture is tightly connected to the fact that the unitary matrix U localizes on the same universal matrix in the $\mathcal{V}_2 \to \infty$ limit, independent of the temperature, chemical potential and masses together with the fact that $d_R = \chi_R(U)$.

Interpolation between bosons and fermions I

- Recall that

$$Z_B^a = \sum_{m=0}^{k_B} \binom{N_B}{m}_q e^{-r\beta(E_a-\mu)} \qquad (22)$$

- Let us keep κ fixed to a large value and vary N_B. In the limit $\frac{N_B}{\kappa} \to 0$

$$\binom{N_B}{m}_q \to \binom{N_B}{m}$$

  Also in this limit k_B so the upper limit on the summation in (22) $\to \infty$. We obtain the statistics of N_B flavours of bosons in a given energy state.

- On the other hand if $\frac{|k_B|}{\kappa} \to 0$, (22) can be shown to be the partition function of $|k_B|$ free fermions in a single particle state.

Interpolation between free bosons and free fermions, II

- When $\frac{N_B}{\kappa_B}$ are both of order unity, on the other hand, then (22) defines genuinely new statistics.
- In summary, (22) may be thought of as a one parameter generaliation of the formulae of free bosonic and fermionic statistics, that reduce to these special cases at the two ends. From the Bosonic viewpoint this deformation, apart from changing the exact values of occupation probabilities, also imposes the Bose Exclusion Principle.

New Single particle Thermal Statistics

- As a consequence we obtain a one parameter deformation of many of the familiar rules of free statistical physics that we learn about as undergraduates.
- For instance, in the t'Hooft Large N limit we find the following formula for the average occupation number of any given single particle state at temperature T and chemical potential μ

$$\bar{n}_B(\epsilon, \mu) = \frac{1 - |\lambda_B|}{2|\lambda_B|} - \frac{1}{\pi|\lambda_B|} \tan^{-1}\left(\frac{e^{\beta(\epsilon - q\mu)} - 1}{e^{\beta(\epsilon - q\mu)} + 1} \cot \frac{\pi|\lambda_B|}{2} \right),$$

- Generalizing the familiar free boson result

$$\bar{n}_B(\epsilon, \mu) = \frac{1}{e^{\beta(\epsilon - q\mu)} - 1}$$

- Similar results apply for fermions and respect duality

Putting it all together: 1

- Armed with our thorough physical understanding of I_B and I_F, we now return to the comparison of the formulae

$$Z_{RB} = \left[e^{-N_B \mathcal{V}_2 \beta \left(\frac{1}{2\pi}(m_B^2 - c_B^2)\sigma + \frac{b_4 \lambda_B}{\pi}\sigma^2 + \frac{\lambda_B^2}{2\pi}(x_6^B + \frac{4}{3})\sigma^3 \right)} \; I_B(c_B) \right]_{c_B, \sigma, \mathcal{S}}$$

for the RB theory and

$$Z^{eff} = \left[e^{-N_B \mathcal{V}_2 \beta \left(\frac{1}{2\pi}(m_B^2 - c_B^2)\sigma + \frac{\tilde{b}_4 \lambda_B}{\pi}\sigma^2 + \frac{\lambda_B^2}{2\pi}(x_6^B + \frac{4}{3})\sigma^3 \right)} \; \mathrm{tr}_{\mathcal{H}^{Fock}_{c_B}} \left(e^{-\beta H_{c_B}} \right) \right]$$

for the related effective ungauged scalar theory

Putting it all together: 2

- We see that the full partition function of the RB theory is precisely the partition function of Fock Space projected down to WZW singlets, corrected by the mean field or forward scattering interactions of the ungauged effective scalar theory.
- Starting with the RB theory it is possible to integrate out the gauge fields (this is how we originally solved these theories at large N). However this process yields a highly nonlocal (and at first sight extremely ugly) scalar effective action .
- The fact that this highly nonlocal theory was solvable at large N always suggested that the effective scalar theory was simpler than it appeared. We now see in what sense that is true. Atleast as far as the partition function is concerned, the entire effective of the ugly looking nonlocal interactions is to impose (nonlocal but beautiful) WZW constraint on free fock space. The apparent ugliness probably had to do with our choice of gauge.

Conclusions and future directions

- We have demonstrated that the partition function of Large N matter Chern Simons theories is effectively that of a free fock space, constrained to the subspace of WZW singlets, and then renormalized by forward scattering interactions in a mean field like way
- As far as the partition function is concerned, in other words, the entire effect of the nonlocal interactions induced by the gauge fields on matter is captured by projecting Fock Space to WZW singlets.
- An important dynamical consequence is the Bosonic Exclusion principle.
- In the non relativistic limit one might expect that wave functions (states) are 'single-valued section of the bundle of conformal blocks'. Seems like this structure greatly simplifies at large N (only surviving nontriviality is Bosonic Exclusion Principle). Would be very interesting to directly understand how this happens.

Discussion

- It would be interesting if the new 'free thermal statistics' of this talk showed up in a real two dimensional material in an experimental context, though I have no clue of how or where this might happen.
- The WZW singlet condition is intimately connected to the mathematical structure of quantum groups. It would be interesting if this connection had dynamical implications, perhaps in governing the structure of interaction matrix elements on the almost free structures encountered in ths talk.
- It would be very interesting to see how far this understanding persists away from the large N limit and also from the large volume limit. It would be fantastic if we could rewrite the Index of superconformal Chern Simons matter theories in a language similar to this talk (i.e. free theory subject to an effective constraint, which may be a deformation of the WZW singlet condition).

Discussion

- The Bose condensate encountered in our analysis is an extremely simple stabilization of the run away instability of free theory. The sharp cut off at k_F plus the Bose exclusion principle gives this phase all the properties of a Fermi Sea. I would be very intersted to know of any other situation in which such a stabilized Bose condensate has been encountered.
- It would be interesting to investigate the dynamical implications of the Bose condensation principle. Cut off lasers? Connection with quantum groups.
- It would be interesting to better understand how the path integral 'knows' that mixed 'correlators' ofv non integrable and integrable Wilson lines must vanish. Is there a pure Chern Simons theory version of this statement (see remark in red above) or does its origin lie in the spatial wave functions of particles?