Sparse SYK

by

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Introduction

Introduction

(all-to-all) SYK

Quantum mechanical system of N Majorana fermions χ^j with all-to-all random interactions [Kitaev '15]

$$H = i^{q/2} \sum_{1 \le j_1 < \ldots < j_q \le N} \underbrace{J_{j_1 \ldots j_q}}_{\text{Gaussian}} \underbrace{\chi^{j_1} \ldots \chi^{j_q}}_{q\text{-body}}, \qquad \langle \left(J_{j_1 \ldots j_q} \right)^2 \rangle = \frac{(q-1)! J^2}{N^{q-1}}$$

- Analytically solvable at large N
- Emergent conformal symmetry at low energies
- Maximally chaotic $\lambda_L = \frac{2\pi}{\beta}$ [Maldacena, Shenker, Stanford 1503.01409]

SYK is a model of holographic duality

Drawback: computational cost

Quantum simulation ($q = 4$)

$$\text{number of gates} \sim \mathcal{O}(N^{7/2} Jt + ...)$$

Classical:

$$\text{number of terms} \sim N^q$$

state of the art $N = 52$, 7 million terms

Is there an SYK modification that retains all the interesting physics but is more computationally efficient?

Sparse SYK

Sparse SYK

Sparse SYK model

Sparsity: Reduce number of terms in the Hamiltonian summation while preserving original properties, e.g., chaotic behavior

[Xu, Susskind, Su, Swingle 2008.02303]

- Random pruning
- Hypergraphs

$$H = i^{q/2} \sum_{1 \le j_1 < \ldots < j_q \le N} J_{j_1 \ldots j_q} x_{j_1 \ldots j_q} \chi^{j_1} \ldots \chi^{j_q}, \qquad \langle \left(J_{j_1 \ldots j_q} \right)^2 \rangle = \frac{(q-1)! J^2}{p N^{q-1}}$$

where $x_{ijkl} = 0$ with probability $1 - p$ or 1 with probability p

Note that

- Computational cost ✓
  Quantum simulation (any q), number of gates

$$\sim \mathcal{O}(kNJt)$$

  Classical, number of terms

$$\sim kN.$$

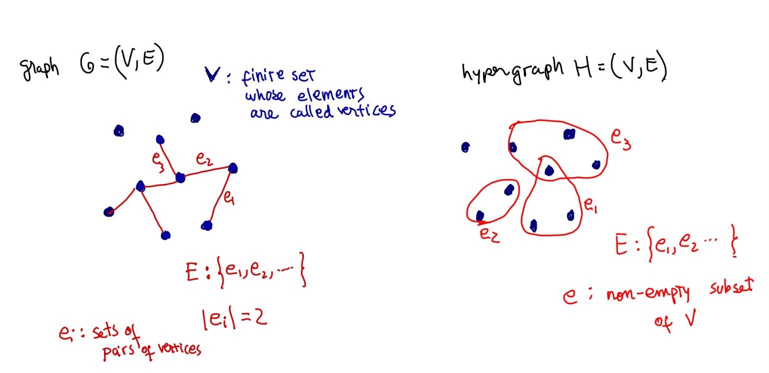
  For $N = 52$, $k = 4$, 208 terms
- Chaos ✓
- Path integral G and Sigma ✓

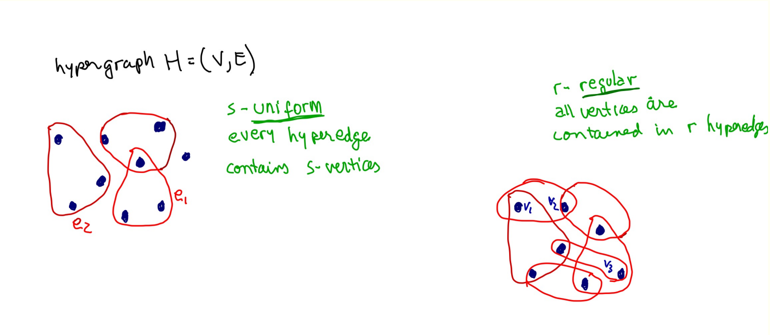
How much sparsity?
Other physics?

Hypergraphs

Hypergraphs

Hypergraphs: Generalization of a graph where hyperedges can connect more than two vertices





(r,s) r-regular, s-uniform

Sparse SYK as (kq, q) Hypergraphs: Majorana fermions are identified with vertices, and each interaction term correspond to a hyperedge connecting q vertices (q-uniform).

kq-regular hypergraphs: Every vertex is contained in exactly kq hyperedges.
q uniform indicates that the Hamiltonian contains q-body interactions

- k quantifies the degree of sparsity in the Hamiltonian

$$k = \frac{p}{N}\binom{N}{q}$$

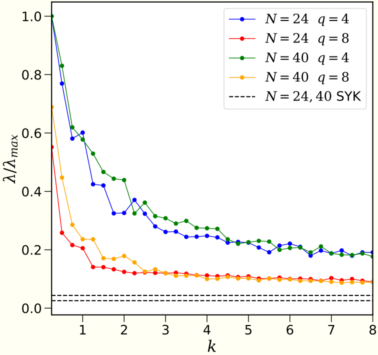
$\Rightarrow$ Sparse Hamiltonian is a sum of exactly kN terms

- Math results for random (r, s) hypergraphs

Adjancency matrix

We want sparse hypergraphs that are highly connected, ***expanders***

$$[A]_{ij} = \begin{cases} \text{\# of hyperedges containing vertices } i \text{ and } j & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$



- Second largest eigenvalue λ → Spectral gap
- Indication that $k \gtrsim 1$ has good connectivity
- Other measures of hypergraph expansion: algebraic entropy and vertex expansion

Other measures of connectivity:

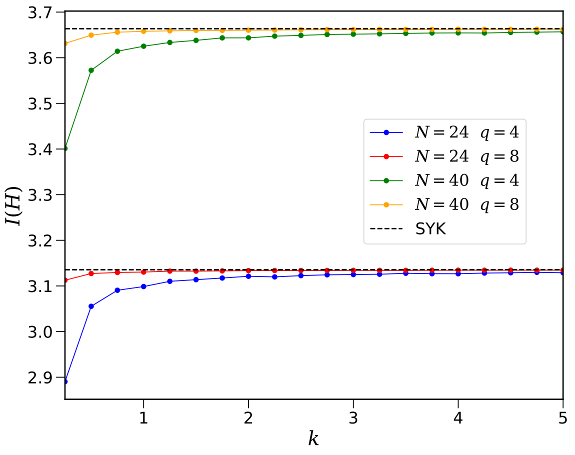
- Algebraic hypergraph entropy
  Consider a hypergraph $H(V,E)$ and its adjancency matrix $A(H)$. Define

$$D = diag(d_1, d_2, ..., d_N)), \qquad d_i = \sum_{j \in V} A_{ij}.$$

and

$$L(H) = \frac{1}{TrD}(D - A(H)) \qquad \text{with eigenvalues } \nu_i$$

$$I(H) = \sum \nu_i \log \nu_i$$



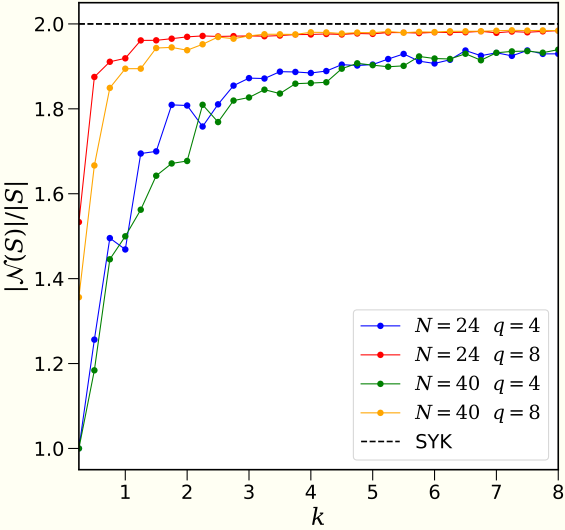
3.7
3.6
3.5
3.4
3.3
3.2
3.1
3.0
2.9
$I(H)$
1
2
3
4
5
k
$N = 24 \quad q = 4$
$N = 24 \quad q = 8$
$N = 40 \quad q = 4$
$N = 40 \quad q = 8$
SYK

- Vertex expansion
  Consider a subset $S \subset V$. Define its neighborhood

  $$\mathcal{N}(S) := \{i : \exists j \in S \text{ such that } \{i,j\} \subseteq e \text{ for some } e \in E\}.$$

  Lower bound on vertex expansion (Dumitriu and Zhu, 2019)

  $$\frac{|\mathcal{N}(S)|}{|S|} \geq \left[1 - \frac{1}{2}\left(1 - \frac{\lambda^2}{r^2(s-1)^2}\right)\right]^{-1}.$$



$|\mathcal{N}(S)|/|S|$
2.0
1.8
1.6
1.4
1.2
1.0
1
2
3
4
5
6
7
8
k
$N = 24 \quad q = 4$
$N = 24 \quad q = 8$
$N = 40 \quad q = 4$
$N = 40 \quad q = 8$
SYK

$q = 4, \quad k = 4$ ✓
$q = 8, \quad k = 2$ ✓
Do they reproduce desired physics?

- Hypergraphs are a useful tool. Much to explore.

Traversable wormholes and sparse SYK

Two coupled sparse SYK

Eternal traversable wormhole with a global AdS_2 geometry can be realized by coupling two copies of SYK in the large N and small coupling limit [Maldacena, Qi 1804.00491]

- Solution can be obtained from JT gravity by adding coupling between boundaries
  [Gao, Jafferis, Wall 1608.05687]
- Traversable for any time
- Same physics can be derived from two coupled SYKs

$$H = H_L^{\text{SYK}} + H_R^{\text{SYK}} + H_{\text{int}}, \quad H_{\text{int}} = i\mu \sum_{j=1}^{N} \chi_L^j \chi_R^j$$

→ Study two coupled sparse SYKs

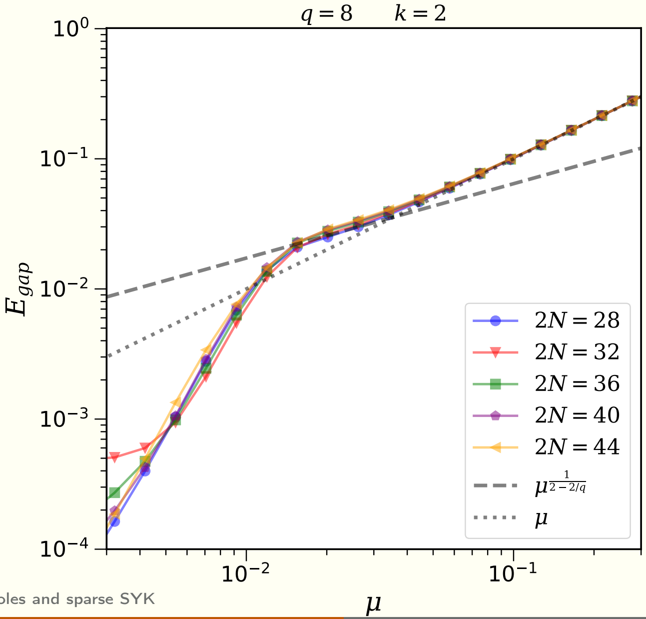
Properties of the two coupled SYK model
[Maldacena, Qi 1804.00491]

$$H = H_L^{\mathrm{SYK}} + H_R^{\mathrm{SYK}} + i\mu \sum_{j=1}^{N} \chi_L^j \chi_R^j$$

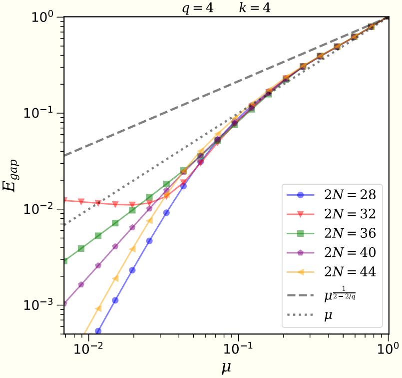
- Ground state $|\Psi_0\rangle$ approximately a TFD state (for some $\beta(\mu)$)
- Energy gap scaling (Derived from large N analysis (gravitational))

$$E_{\mathrm{gap}} \sim \mu^{\frac{1}{2-2/q}} \qquad \text{at weak coupling}$$
$$E_{\mathrm{gap}} \sim \mu \qquad \text{at strong coupling}$$

Energy gap



$q = 8 \quad k = 2$
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
E_{gap}
10^{-2}
10^{-1}
μ
$2N = 28$
$2N = 32$
$2N = 36$
$2N = 40$
$2N = 44$
$\mu^{\frac{1}{2-2/q}}$
μ



$q = 4 \quad k = 4$
E_{gap}
μ
10^0
10^{-1}
10^{-2}
10^{-3}
$2N = 28$
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$2N = 36$
$2N = 40$
$2N = 44$
$\mu^{\frac{1}{2-2/q}}$
μ

- $q = 8$ matches scaling expected from gravity for large N and appropriate range of couplings
- Finite N effects dominate at very small couplings μ

Revival dynamics phenomena

1. Start with ground state $|\Psi_0\rangle$ of the two coupled SYK
2. Create Majorana excitation in Right system

$$|\Psi(t=0)\rangle = \chi_R |\Psi_0\rangle$$

3. Excitation gets scrambled
4. Excitation reassembles and becomes localized in Left system

$$|\Psi(t=t_{\text{rev}})\rangle = \chi_L |\Psi_0\rangle$$

5. Process is repeated with $L \leftrightarrow R \Rightarrow$ 'Revival oscillations'

   [Plugge, Lantagne-Hurtubise, Franz 2003.03914]

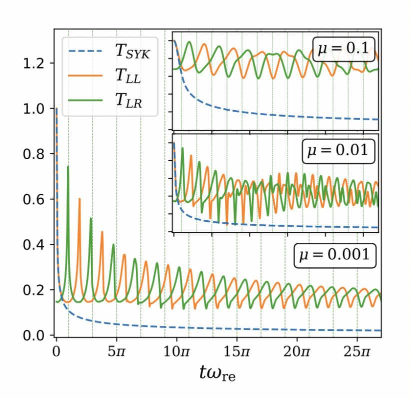
Gravity picture: Perturbation travels through the wormhole

Diagnostic of revivals

Transmission amplitude $\quad T_{ab} = 2|G^{>}_{ab}|$

$$G^{>}_{ab}(t) = -\frac{i\theta(t)}{N}\sum_j \langle \chi^j_a(t)\chi^j_b(0)\rangle = \begin{pmatrix} G^{>}_{LL}(t) & G^{>}_{LR}(t) \\ G^{>}_{RL}(t) & G^{>}_{RR}(t) \end{pmatrix}$$

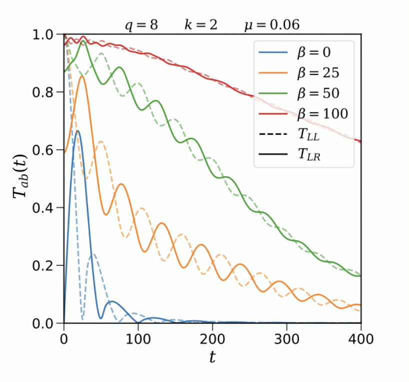
$|T_{ab}(t)|^2$: Probability of recovering χ^j_a at some time t after inserting χ^j_b at $t = 0$.



[Plugge, Lantagne-Hurtubise and Franz, 2020]

Revivals in sparse SYK

Increasing the temperature decreases transmission amplitude but enhances the oscillatory behavior



SYK with $q = 4$ does not follow expected frequency scaling from AdS_2 gravity, but $q = 8$ is compatible for some range of couplings and temperatures

Summary so far: Hypergraphs are a useful mathematical framework that provides information about the connectivity of the sparse model. Sparse SYK is a computationally tractable quantum mechanical system with emergent gravitational behavior and can describe an eternal traversable wormhole.

dynamite: a python library that makes use of PETSc and SLEPc. Krylov subspace methods combined with massive parallelization [Github:GregDMeyer/dynamite]

Texas Advanced Computing Center (TACC): Use of computational resources from Stampede2 supercomputer

Chaos and spectral form factor

Classical vs quantum chaos

Classical chaos: Sensitivity to initial conditions

- Diagnosed by Poisson brackets

$$\frac{\partial x(t)}{\partial x(0)} = \{x(t), p(0)\}_{\text{P.B.}} \sim e^{\lambda t}, \qquad \lambda : \text{Lyapunov exponent}$$

Quantum chaos: First proposal by Larkin and Ovchinnikov (1969). New insights by Shenker, Stanford (2014) and Kitaev (2015)

- Out of Time Order Correlators (OTOC)

$$C(t) = -\langle [W(t), V(0)]^2 \rangle_\beta, \qquad V, W \text{ Hermitian operators}$$

  Basic intuition: How much an early perturbation V affects the later measurement of W. Lyapunov exponent

- Random Matrix Theory (RMT)

- The Hamiltonian of a chaotic system, when considering small energy windows where the density of states is constant, is believed to resemble a random matrix.
- Wigner. By studying statistical properties of random matrices subject to the symmetries of the Hamiltonian, we can understand the statistical properties of energy levels and eigenstates of the system.
- One observable that captures the statistics of energy levels is the spectral form factor (SFF)

Level statistics

Random matrix theory (RMT) provides an alternative diagnostic of quantum chaos

- Quantum chaos encoded in the statistical properties of the spectrum
- Spectra of quantum chaotic systems show the same fluctuation properties as predicted by RMT

Classical ensembles of RMT: Hermitian random matrices whose entries are random variables independently distributed

Level statistics

Level spacing: $s = \frac{E_{i+1} - E_i}{\Delta}$, Δ: mean level spacing

Level spacing distribution: $P(s)$, probability to find consecutive eigenvalues E_i, E_{i+1} at distance s

- For quantum chaotic system:

$$P_W(s) \simeq A_\alpha s^\alpha e^{-B_\alpha s^\alpha}, \quad \alpha = \begin{cases} 1 \text{ GOE} \\ 2 \text{ GUE} \\ 4 \text{ GSE} \end{cases} \qquad \text{(Wigner-surmise)}$$

- For integrable system:

$$P_P(s) = e^{-s} \quad \text{(Poisson)}$$

Quantity sensitive to energy level statistics: Spectral Form Factor

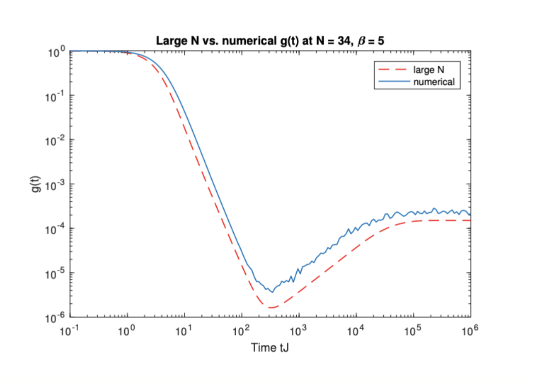
$$g(t,\beta) = \frac{\langle Z(\beta,t)\, Z^*(\beta,t)\rangle_J}{\langle Z(\beta)\rangle_J^2}$$

$$g_d(t,\beta) = \frac{\langle Z(\beta,t)\rangle_J \langle Z^*(\beta,t)\rangle_J}{\langle Z(\beta)\rangle_J^2}$$

$$g_c(t,\beta) = g(t,\beta) - g_d(t,\beta)$$

Spectral form factor

The late time behavior of the spectral form factor in the all-to-all SYK is governed by random matrix theory, just as expected from a chaotic system.



Slope, dip, ramp, plateaux. [Cotler et. al.]

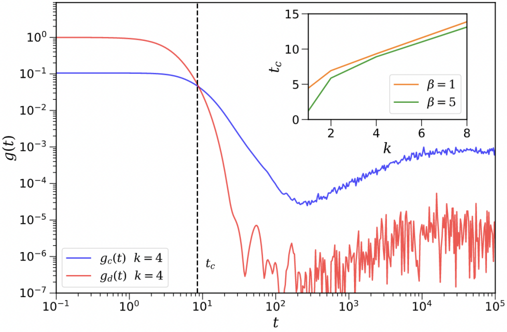
• Berkooz, Bruckner, Noravlansky and Raz, 2020: connected contributions to the moments can dominate at early times.

$$\mathcal{Z}(\beta_1,\beta_2) = \langle Tr(e^{-\beta_1 H})\, Tr(e^{-\beta_2 H})\rangle_J$$
$$= \sum_{m_1,m_2=0}^{\infty} \langle Tr(H^{m_1})\, Tr(H^{m_2})\rangle_J \frac{\beta_1^{m_1}}{m_1!} \frac{\beta_2^{m_2}}{m_2!} (-1)^{m_1+m_2}$$

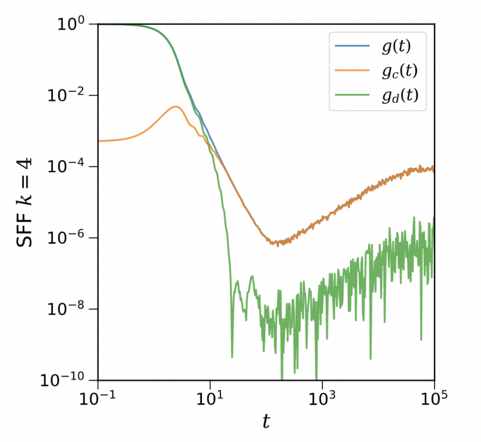
• But a large value of N, beyond the state of the art of numerical simulations, is needed to see this effect in the all-to-all SYK

→ Sparse SYK !

SFF in Sparse SYK



Connected and disconnected parts, exchange of dominance. $N = 30$, $k = 4\, q = 4$



Spectral form factor $k = 4$, $N = 30$, $q = 4$

Work in progress: OTOCs

Another way of diagnosing chaos: Out of TIme Order Correlators (OTOCs)

- $C(t) = \langle [W(t), V(0]^2 \rangle = 2 - 2F(t)$
- $$F(t) \equiv \langle W(t)V(0)W(t)V(0) \rangle_\beta$$

Work in progress: OTOCs

Intuition:

$$\langle W(t)V(0)W(t)V(0)\rangle = \langle \psi_2|\psi_1\rangle$$

$$|\psi_2\rangle = V(0)W(t)|\beta\rangle \quad \text{and} \quad |\psi_1\rangle = W(t)V(0)|\beta\rangle$$

In non-chaotic system: early measurement of V has no effect on the later measurement of W, $|\psi_1\rangle \sim |\psi_2\rangle$, so overlap $\langle \psi_1|\psi_2\rangle = 1 \to C(t) \sim 0$

In a chaotic system: Pertubation V makes $|\psi_1\rangle$ and $|\psi_2\rangle$ distinguishable, so overlap $\langle \psi_1|\psi_2\rangle$ is small, $C(t) \to 2$ for late times

Scrambling time t_*

Lyapunov exponent λ_L

Work in progress: chaos bound

Maldacena, Shenker, Stanford '15

$$\lambda_L \leq \frac{2\pi k_B T}{\hbar}$$

- Valid for generic quantum systems, with the assumption of analyticity and factorization at large times of thermal correlation functions
- A large class of black holes saturate this bound
- Bound is also saturated in the SYK model
- Sparse SYK? k dependence?

Future directions

Future directions

- Collisions behind the horizon [Haehl and Zhao, 2105.12755, 2202.04661]

$$\mathscr{F}_6 \sim \frac{\langle W_1 W_1 \mathscr{O}_j \mathscr{O}_j W_2 W_2 \rangle}{\langle W_1 W_1 \rangle \langle \mathscr{O}_j \mathscr{O}_j \rangle \langle W_2 W_2 \rangle}$$

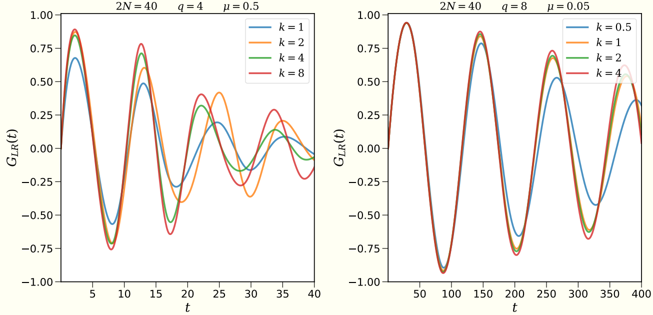
- Operator growth, complexity, etc. Hyergraphs?
- .....

Thanks!

Gracias!

Green's functions

$$G_{ab}(t) = \frac{1}{N}\sum_j 2\mathrm{Re}\langle \chi_a^j(t)\chi_b^j(0)\rangle, \qquad a,b = L,R.$$

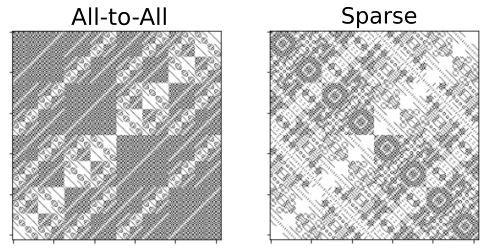


- Sparse SYK similar to original SYK using k of order 1
- Larger q allows us to choose smaller k

Numerical methods

SYK maps to $N/2$-qubit system via **Jordan-Wigner transformation**

$$\chi_{2n} = \left(\prod_{j=1}^{n-1} \sigma_j^x\right)\sigma_n^z, \quad \chi_{2n-1} = \left(\prod_{j=1}^{n-1} \sigma_j^x\right)\sigma_n^y, \qquad \{\chi_i, \chi_j\} = 2\delta_{ij}$$



Numerical methods

Krylov subspace

$$\mathcal{K}_m = \mathrm{span}\{|\psi(t)\rangle, H|\psi(t)\rangle, H^2|\psi(t)\rangle, \ldots, H^{m-1}|\psi(t)\rangle\}$$

Get approximation for time evolution

$$e^{-iH\Delta t}|\psi(t)\rangle \simeq V_m e^{-iV_m H V_m \Delta t} e_1$$

Typicality:

$$\langle \chi_a^j(t)\chi_b^j(0)\rangle = \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}\chi_a^j(t)\chi_b^j(0)\right] \simeq \frac{\langle\beta|\chi_a^j(t)\chi_b^j(0)|\beta\rangle}{\langle\beta|\beta\rangle}$$

$|\beta\rangle = e^{-\frac{\beta}{2}H}|\psi\rangle$, $\quad |\psi\rangle$ random state