Central Charges of Two-Dimensional Boundaries and Defects

Andy O'Bannon

UNIVERSITY OF Southampton

THE ROYAL SOCIETY

Science & Technology Facilities Council

Holotube
November 30, 2021

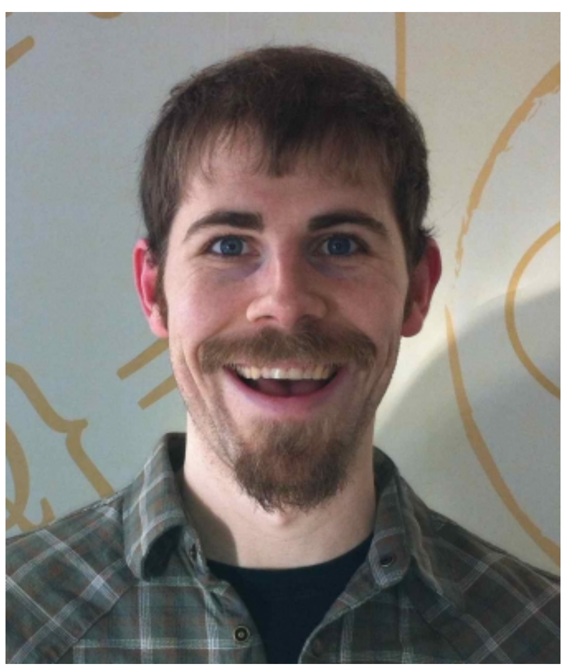
Based on 1509.02160, 1812.00923, 1812.08745, 2003.02857, 2111.14713

Adam Chalabi
Southampton

John Estes
SUNY Old Westbury

Chris Herzog
KCL

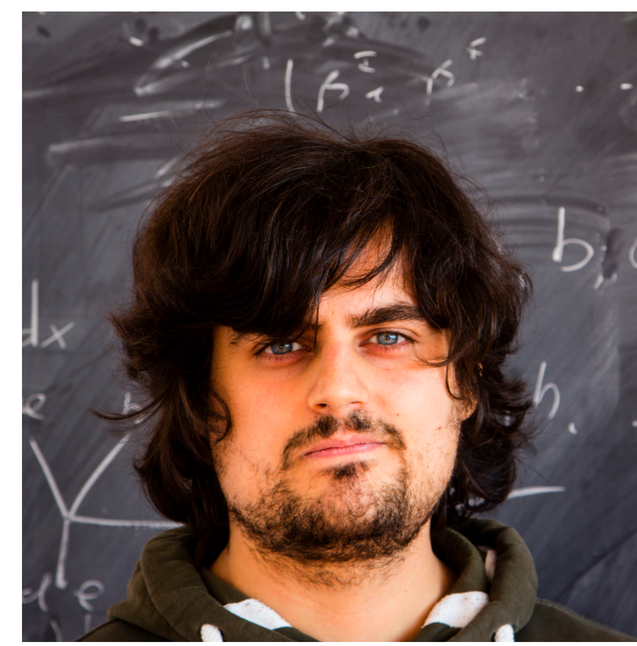
Darya Krym
NYC College of Tech.



Kristan Jensen
Victoria

Brandon Robinson
Leuven

Ronnie Rodgers
Nordita



Jacopo Sisti
Uppsala

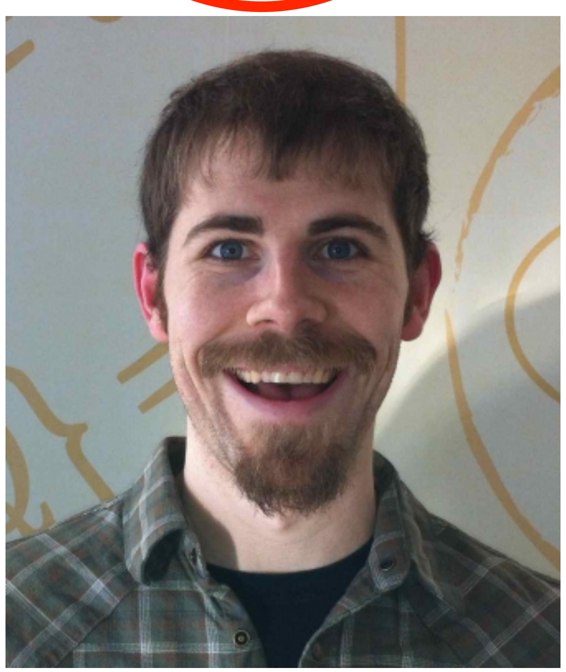
Based on 1509.02160, 1812.00923, 1812.08745, 2003.02857, 2111.14713

Adam Chalabi
Southampton

John Estes
SUNY Old Westbury

Chris Herzog
KCL

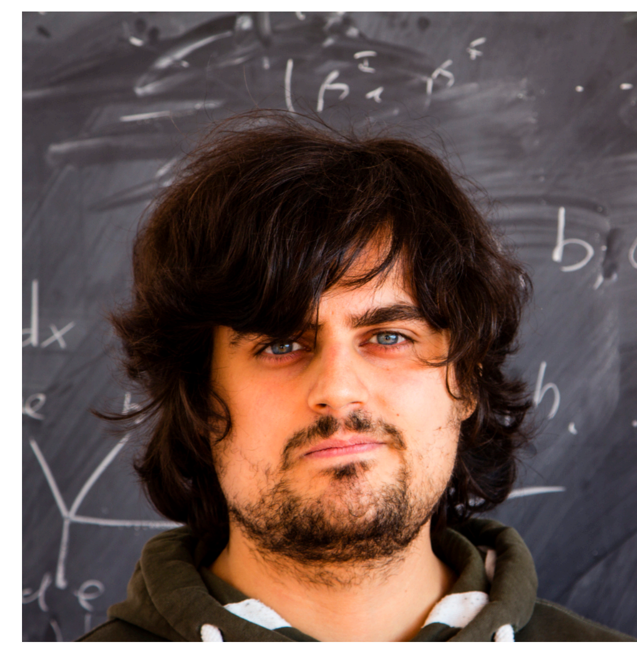
Darya Krym
NYC College of Tech.



Kristan Jensen
Victoria

Brandon Robinson
Leuven

Ronnie Rodgers
Nordita



Jacopo Sisti
Uppsala

Outline:

- Motivation
- 2d CFT Central Charge
- The Systems
- Boundary/Defect Central Charges
- Examples
- Summary and Outlook

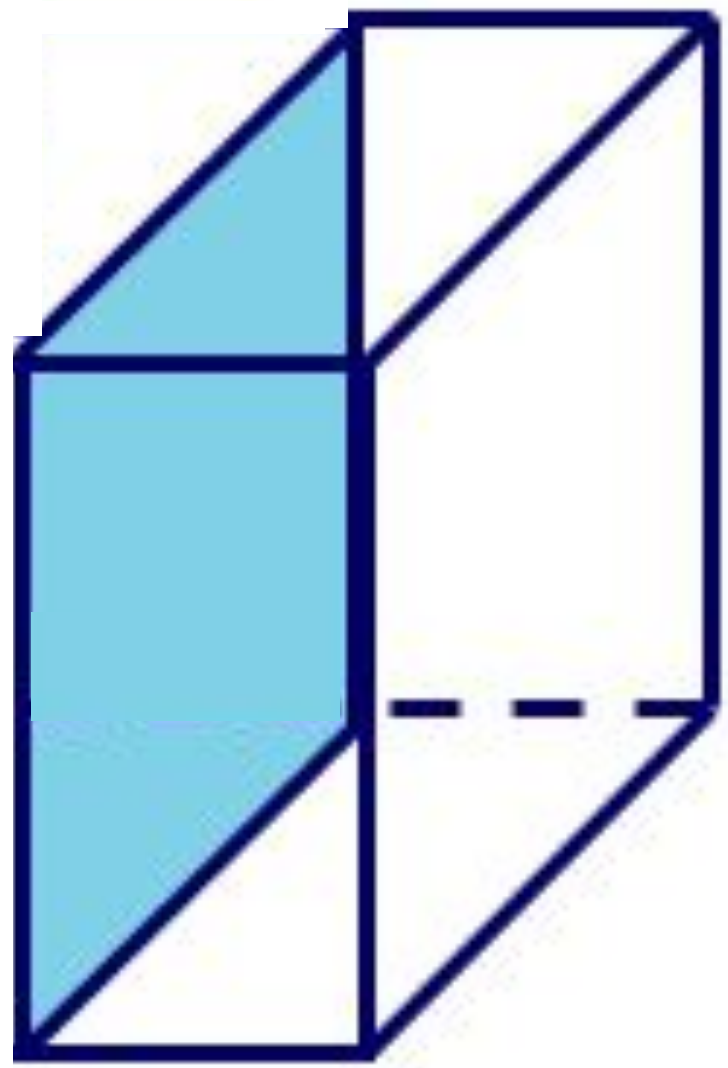
Outline:

- Motivation
- 2d CFT Central Charge
- The Systems
- Boundary/Defect Central Charges
- Examples
- Summary and Outlook

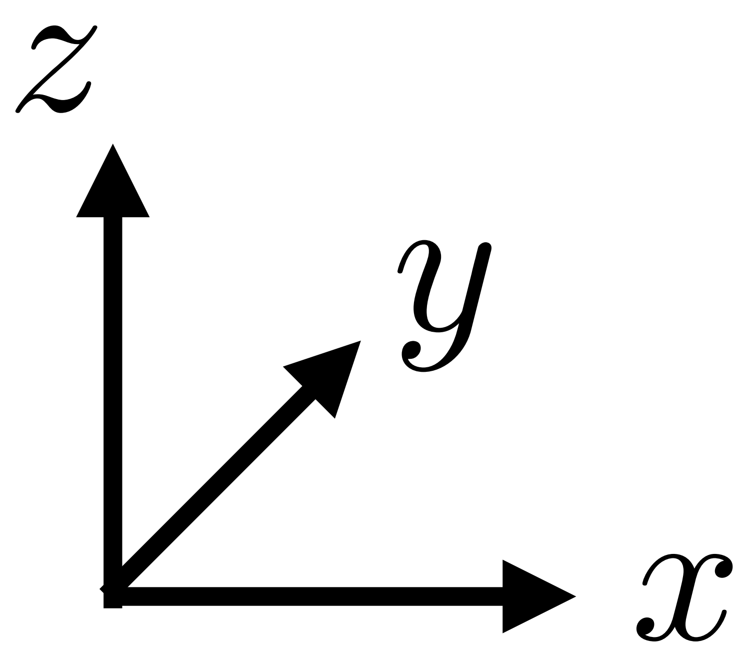
Central Charges of Two-Dimensional Boundaries and Defects

conformal field theory (CFT) with a boundary
+ conformally-invariant boundary conditions
possibly + massless degrees of freedom at the boundary

Boundary Conformal Field Theory (BCFT)



$d = 3$ BCFT



Examples

critical Ising model in $d = 3$ with a boundary

graphene with a boundary

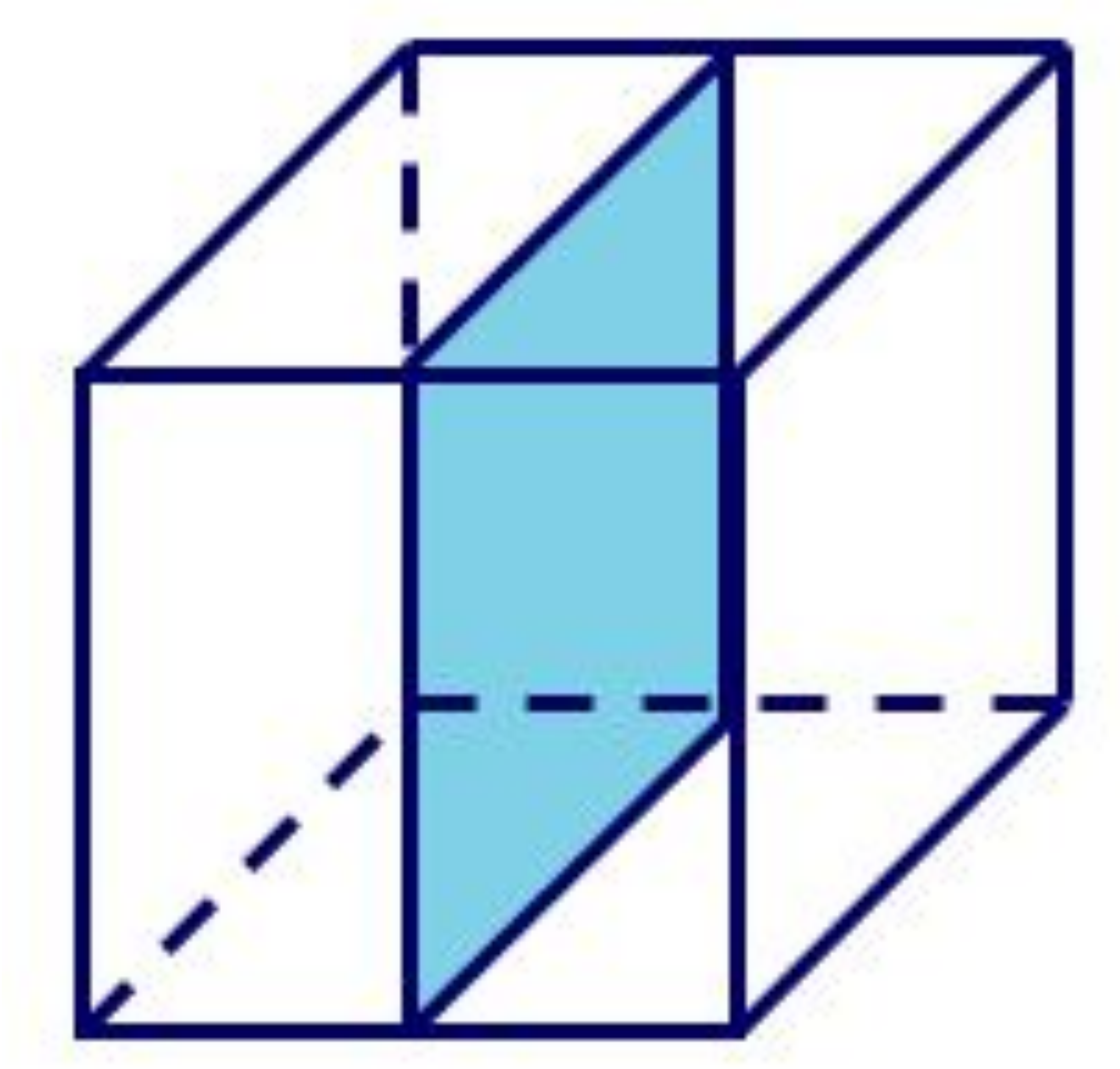
M-theory: M2s ending on M5s

string theory: various branes ending on branes

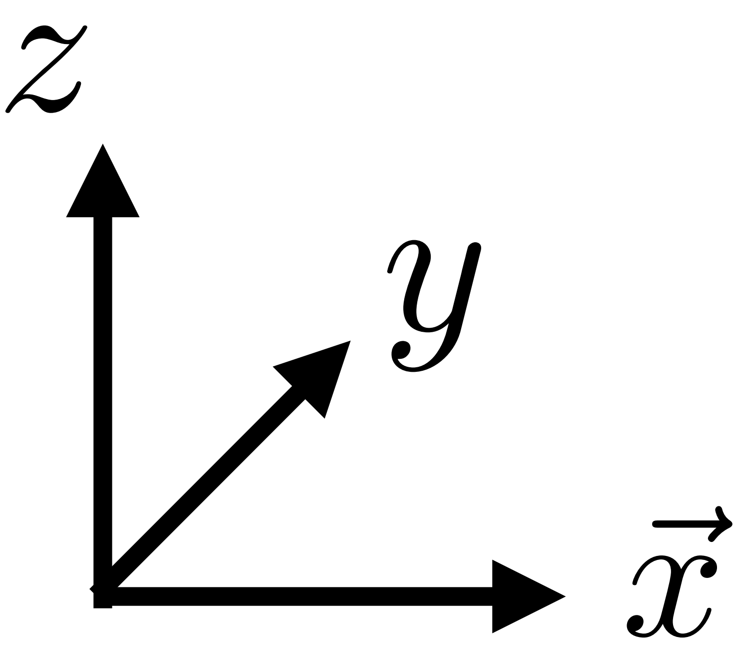
holographic examples

CFT with a conformally-invariant defect:
conformally-invariant boundary conditions along a submanifold
and/or massless degrees of freedom supported along a submanifold

Defect Conformal Field Theory (DCFT)



$d \geq 3$ DCFT



Examples

critical Ising model in $d = 3$ with a domain wall

graphene with a line defect

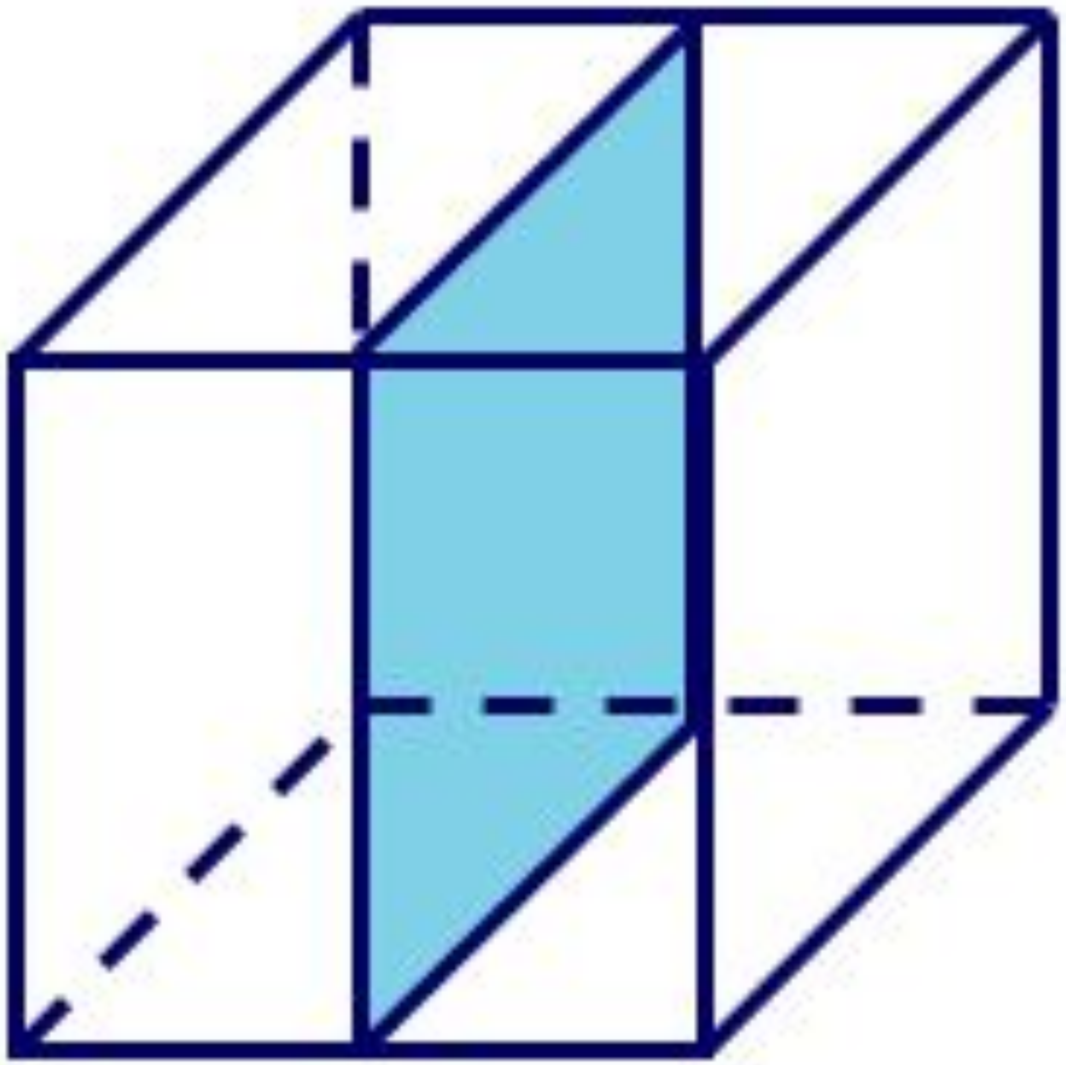
M-theory: M2s ending on M5s

string theory: various branes intersecting branes

holographic examples

Question

Can we define central charges
for conformal boundaries and defects?



Central Charge

$d = 2$ CFT

Virasoro algebra $\Rightarrow$ central charge c

Stress tensor 2-pt. function

$$\langle T_{zz}(z) T_{zz}(0) \rangle = \frac{c/2}{z^4}$$

Thermodynamic entropy

$$S_{\text{thermo}} = \frac{\pi}{3} c L T + \ldots$$

Trace Anomaly

$$T_\mu{}^\mu = \frac{c}{24\pi} R$$

Entanglement Entropy

$$S_{\text{EE}} = \frac{c}{3} \ln(\ell/\varepsilon) + \ldots$$

Central Charge

$d = 2$ CFT

Virasoro algebra $\Rightarrow$ central charge c

The c-theorem

A.B. Zamolodchikov
JETP Vol. 43 No. 12 p. 565, 1986

RG flow from UV CFT to IR CFT

$$c_{\rm UV} \geq c_{\rm IR}$$

Euclidean symmetry (Poincaré symmetry)
Locality
Reflection positivity (Unitarity)

Counts the number of degrees of freedom

Central Charge

$d = 2$ CFT

Virasoro algebra $\Rightarrow$ central charge c

Stress tensor 2-pt. function	$\langle T_{zz}(z) T_{zz}(0) \rangle = \frac{c/2}{z^4}$
Thermodynamic entropy	$S_{\text{thermo}} = \frac{\pi}{3} c L T + \ldots$
Trace Anomaly	$T_\mu{}^\mu = \frac{c}{24\pi} R$
Entanglement Entropy	$S_{\text{EE}} = \frac{c}{3} \ln(\ell/\varepsilon) + \ldots$

Central Charge

$d = 2$ boundary or defect in $d \geq 3$ BCFT or DCFT

~~Virasoro algebra $\Rightarrow$ central charge c~~

Stress tensor 2-pt. function ??????

Thermodynamic entropy ??????

Trace Anomaly ??????

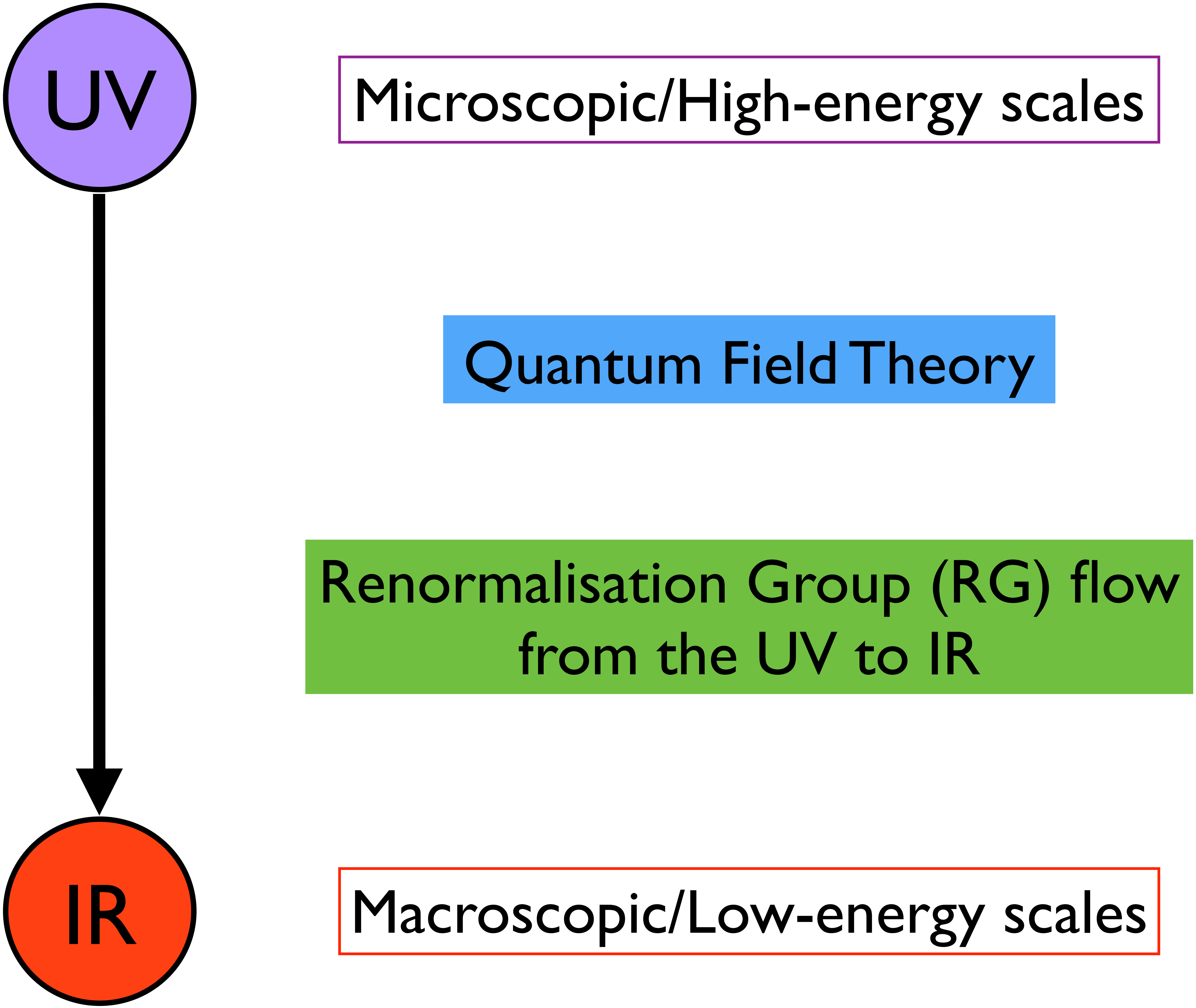
Entanglement Entropy ??????

Outline:

- Motivation
- 2d CFT Central Charge
- The Systems
- Boundary/Defect Central Charges
- Examples
- Summary and Outlook

Outline:

- Motivation
- 2d CFT Central Charge
- The Systems
- Boundary/Defect Central Charges
- Examples
- Summary and Outlook


UV
Microscopic/High-energy scales
Quantum Field Theory
Renormalisation Group (RG) flow
from the UV to IR
IR
Macroscopic/Low-energy scales

Quantum Field Theory

Generating functional Z

Non-dynamical background metric $g_{\mu\nu}(x)$

Stress-Energy Tensor

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} \ln Z[g_{\mu\nu}]$$

$$T_{\mu\nu} = T_{\nu\mu}$$

Quantum Field Theory

Generating functional Z

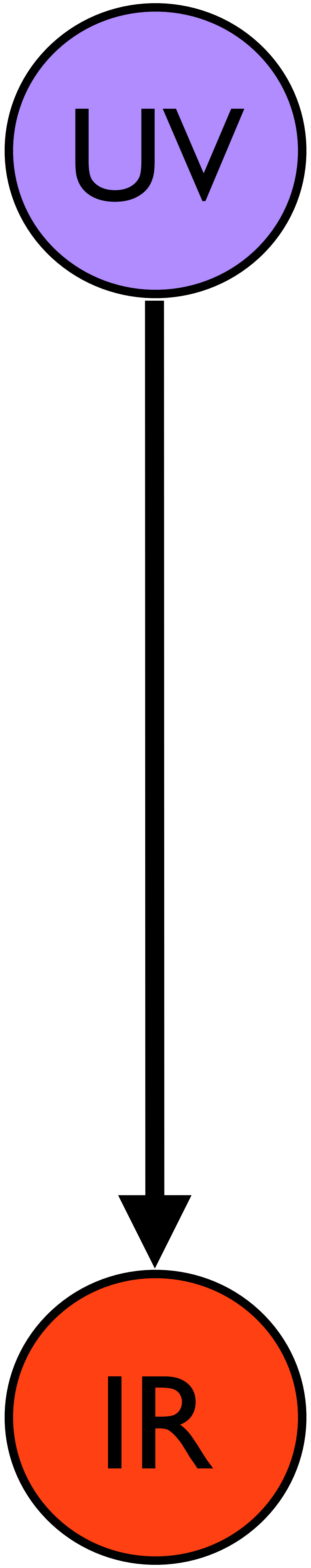
Non-dynamical background metric $g_{\mu\nu}(x)$

Stress-Energy Tensor

$$g_{\mu\nu} \to \delta_{\mu\nu}$$

Translational and Rotational Symmetry

$$\partial_\mu T_{\mu\nu} = 0$$



UV CFT

Conformal Field Theories (CFTs)

RG fixed points

IR CFT

Conformal Field Theory

Generating functional Z

Non-dynamical background metric $g_{\mu\nu}(x)$

Conformal Transformation

Diffeomorphism

$$x^\mu \to x'^\mu(x)$$

such that

$$g_{\mu\nu}(x) \to e^{2\Omega(x)} g_{\mu\nu}(x)$$

Conformal Field Theory

$$g_{\mu\nu}(x) \to e^{2\Omega(x)} g_{\mu\nu}(x)$$

$$T_\mu{}^\mu = \frac{1}{\sqrt{g}} \frac{\delta}{\delta \Omega} \ln Z$$

Conformal invariance

$$T_\mu{}^\mu = 0$$

Conformal Field Theory

$$g_{\mu\nu}(x) \to \delta_{\mu\nu}$$

$$d > 2$$

Rotations	$x^\mu \to \Lambda^\mu{}_\nu \, x^\nu$
Translations	$x^\mu \to x^\mu + c^\mu$
Dilatations	$x^\mu \to \lambda \, x^\mu$
Special Conformal	$x^\mu \to \dfrac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2}$

$$SO(d+1,1)$$

Conformal Field Theory

$$g_{\mu\nu}(x) \to \delta_{\mu\nu}$$

$$d = 2$$

complex coordinates $z\,,\,\bar{z}$

Conformal Transformation

$$z \to w(z) \qquad \bar{z} \to \bar{w}(\bar{z})$$

Conformal Field Theory

$$g_{\mu\nu}(x) \to \delta_{\mu\nu}$$

$$d = 2$$

Holomorphic and anti-holomorphic sectors DECOUPLE

$$[T_{\mu\nu}] = \begin{pmatrix} T_{zz} & T_{z\bar{z}} \\ T_{\bar{z}z} & T_{\bar{z}\bar{z}} \end{pmatrix}$$

$$T_{z\bar{z}} = T_{\bar{z}z}$$

$$T_{\mu}{}^{\mu} = T_{z\bar{z}} = T_{\bar{z}z}$$

Conformal Field Theory

$$g_{\mu\nu}(x) \to \delta_{\mu\nu}$$

$$d = 2$$

Holomorphic and anti-holomorphic sectors DECOUPLE

$$\partial_\mu T_{\mu\nu} = 0$$

$$\partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}} T_{z\bar{z}} = 0$$

$$T_\mu{}^\mu = T_{z\bar{z}} = T_{\bar{z}z} = 0$$

$$\partial_{\bar{z}} T_{zz} = 0 \qquad \partial_z T_{\bar{z}\bar{z}} = 0$$

Conformal Field Theory

$$T_{zz}(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \qquad T_{\bar{z}\bar{z}}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}}$$

Virasoro algebra

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

$$[L_m, \bar{L}_n] = 0$$

$$m, n \in \mathbb{Z}$$

An infinite number of symmetry generators!

Conformal Field Theory

$$T_{zz}(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \qquad T_{\bar{z}\bar{z}}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}}$$

Virasoro algebra

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

$$[L_m, \bar{L}_n] = 0$$

$SO(d+1,1) = SO(3,1)$ **subgroup** $L_{\pm 1}$ and L_0

$\bar{L}_{\pm 1}$ and $\bar{L}_0$

Conformal Field Theory

$$T_{zz}(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \qquad T_{\bar{z}\bar{z}}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}}$$

Virasoro algebra

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{c}{12}n(m^2-1)\delta_{m+n,0}$$

$$[L_m, \bar{L}_n] = 0$$

$SO(d+1,1) = SO(3,1)$ **subgroup** $L_{\pm 1}$ and L_0

$\bar{L}_{\pm 1}$ and $\bar{L}_0$

Central Charge

Counts the number of degrees of freedom

The c-theorem

RG flow from UV CFT to IR CFT

$$c_{\rm UV} \geq c_{\rm IR}$$

Reflection positivity (unitarity)
vacuum state normalizability

$$c \geq 0$$

Add a single, free, massless, real scalar field
or single, free, massless Dirac fermion

$$c \rightarrow c + 1$$

Central Charge

$d = 2$ CFT

Virasoro algebra $\Rightarrow$ central charge c

Stress tensor 2-pt. function

$$\langle T_{zz}(z) T_{zz}(0) \rangle = \frac{c/2}{z^4}$$

Thermodynamic entropy

$$S_{\rm thermo} = \frac{\pi}{3} c L T + \ldots$$

Trace Anomaly

$$T_\mu{}^\mu = \frac{c}{24\pi} R$$

Entanglement Entropy

$$S_{\rm EE} = \frac{c}{3} \ln(\ell/\varepsilon) + \ldots$$

Central Charge

Counts the number of degrees of freedom

Stress tensor 2-pt. function

$$\langle T_{zz}(z) T_{zz}(0) \rangle = \frac{c/2}{z^4}$$

The c-theorem

A.B. Zamolodchikov
JETP Vol. 43 No. 12 p. 565, 1986

RG flow from UV CFT to IR CFT

$$c_{\rm UV} \geq c_{\rm IR}$$

Central Charge

Counts the number of degrees of freedom

Thermodynamic entropy

Cardy NPB 270 (186) 1986

System size L Temperature T

$$T \gg 1/L$$

$$S_{\text{thermo}} = \frac{\pi}{3} c L T + \ldots$$

Central Charge

Counts the number of degrees of freedom

Trace Anomaly

$$g_{\mu\nu} = \delta_{\mu\nu}$$

Conformal invariance

$$T_{\mu}{}^{\mu} = 0$$

Central Charge

Counts the number of degrees of freedom

Trace Anomaly

$$g_{\mu\nu} \neq \delta_{\mu\nu}$$

Quantum Effects
Break Conformal Invariance

$$T_{\mu}{}^{\mu} \neq 0$$

Central Charge

Counts the number of degrees of freedom

Trace Anomaly

CFT in any d

d odd $\qquad T_\mu{}^\mu = 0$

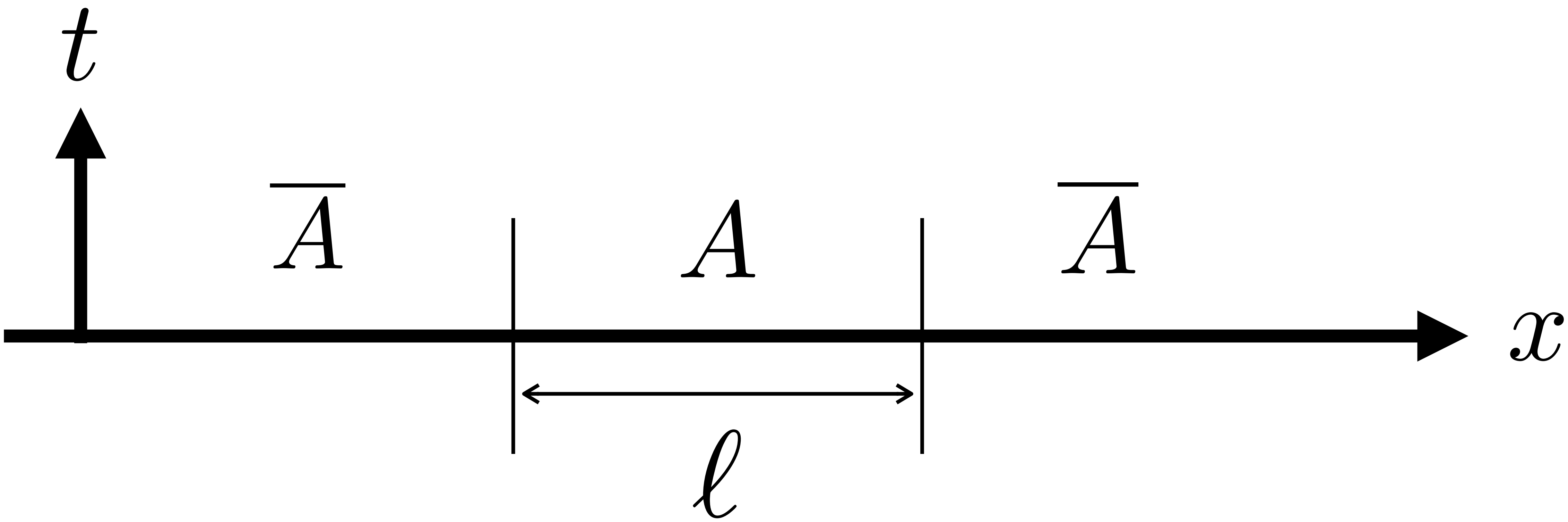
d even $\qquad T_\mu{}^\mu \neq 0$

$d = 2 \qquad T_\mu{}^\mu = \dfrac{c}{24\pi} R$

Central Charge

Counts the number of degrees of freedom

Entanglement Entropy (EE)



$$\rho_{\rm A} \equiv {\rm tr}_{\overline{A}}\, \rho$$

$$S_{\rm EE} = -{\rm tr}\rho_{\rm A} \ln \rho_{\rm A}$$

Central Charge

Counts the number of degrees of freedom

Entanglement Entropy (EE)

$$d = 2 \text{ CFT}$$

Holzhey, Larsen, Wilczek hep-th/9403108   Calabrese + Cardy hep-th/0405152

interval of length ℓ

short-distance cutoff ε

$$S_{\text{EE}} = \frac{c}{3} \ln \frac{\ell}{\varepsilon} + \ldots$$

$$3\,\ell\,\frac{d}{d\ell} S_{\text{EE}} = c$$

Central Charge

$d = 2$ CFT

Virasoro algebra $\Rightarrow$ central charge c

Stress tensor 2-pt. function	$\langle T_{zz}(z) T_{zz}(0) \rangle = \frac{c/2}{z^4}$
Thermodynamic entropy	$S_{\text{thermo}} = \frac{\pi}{3} c L T + \ldots$
Trace Anomaly	$T_\mu{}^\mu = \frac{c}{24\pi} R$
Entanglement Entropy	$S_{\text{EE}} = \frac{c}{3} \ln(\ell/\varepsilon) + \ldots$

Central Charge

$d=2$ boundary or defect in $d \geq 3$ BCFT or DCFT

~~Virasoro algebra $\Rightarrow$ central charge c~~

Stress tensor 2-pt. function ??????

Thermodynamic entropy ??????

Trace Anomaly ??????

Entanglement Entropy ??????

Outline:

- Motivation
- 2d CFT Central Charge
- The Systems
- Boundary/Defect Central Charges
- Examples
- Summary and Outlook

Outline:

- Motivation
- 2d CFT Central Charge
- The Systems
- Boundary/Defect Central Charges
- Examples
- Summary and Outlook

CFT

$d \geq 3$ flat space

Rotations	$x^\mu \to \Lambda^\mu{}_\nu \, x^\nu$
Translations	$x^\mu \to x^\mu + c^\mu$
Dilatations	$x^\mu \to \lambda \, x^\mu$
Special Conformal	$x^\mu \to \dfrac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2}$

$$SO(d+1,1)$$

BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

Rotations $\quad x^\mu \to \Lambda^\mu{}_\nu \, x^\nu$

Translations $\quad x^\mu \to x^\mu + c^\mu$

Dilatations $\quad x^\mu \to \lambda \, x^\mu$

Special Conformal $\quad x^\mu \to \dfrac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2}$

$$SO(d+1,1)$$

Broken to subgroup
that preserves the boundary or defect

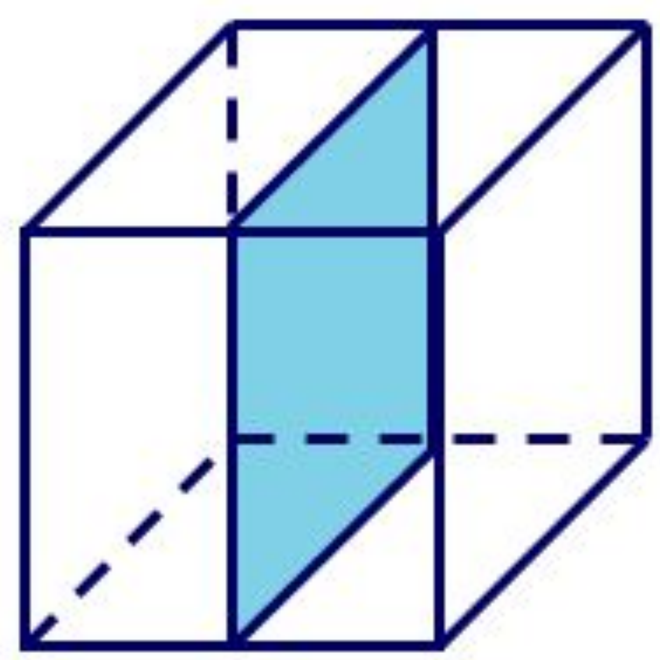
BCFT and DCFT

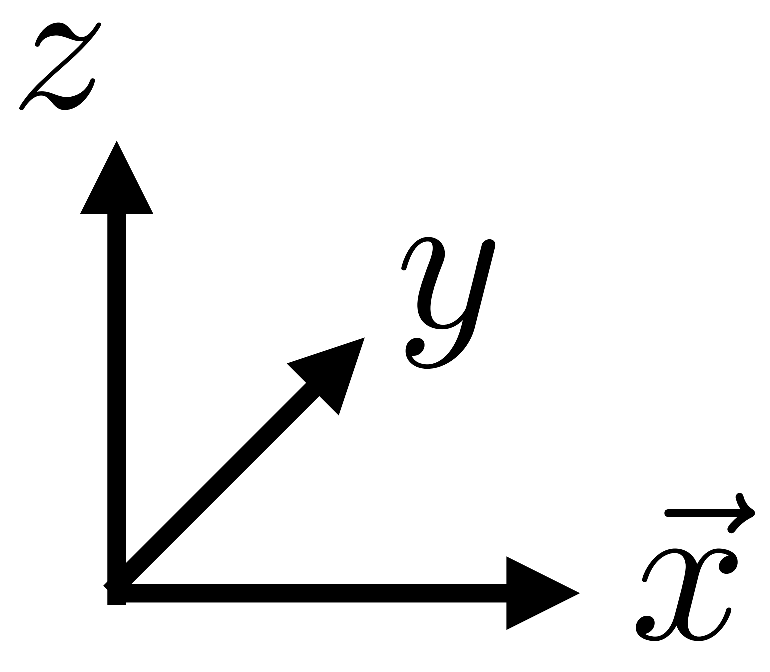
$d=2$ boundary or defect in $d \geq 3$ flat space

Rotations

$$x^\mu \to \Lambda^\mu{}_\nu \, x^\nu$$

Broken to rotations in (y, z)
+ rotations in transverse directions



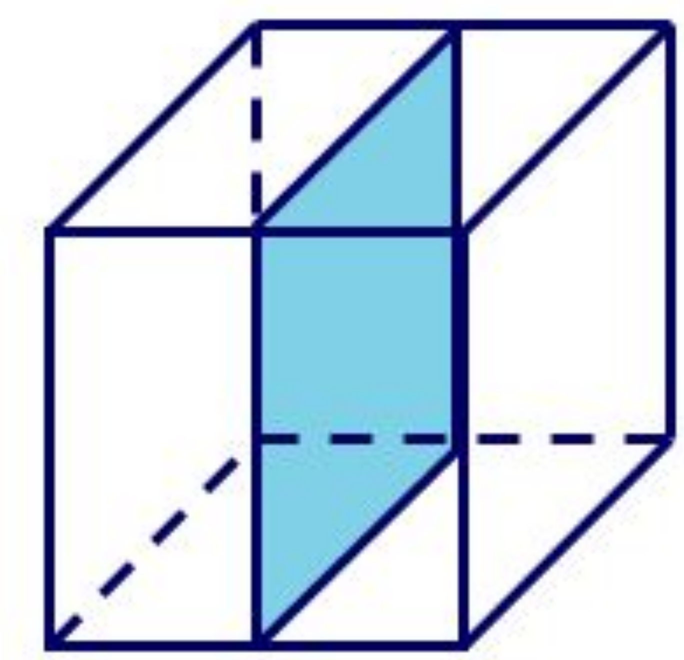


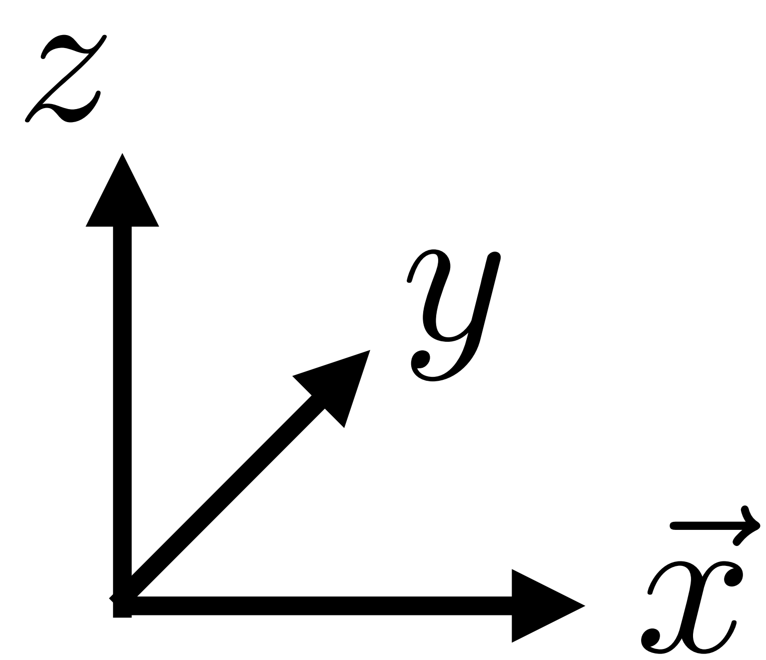
BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

Translations $x^\mu \to x^\mu + c^\mu$

Broken to translations along (y, z)



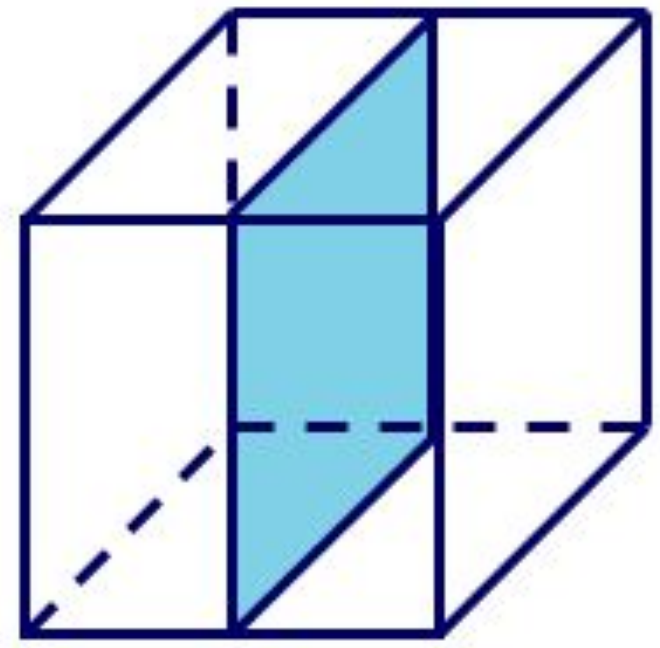


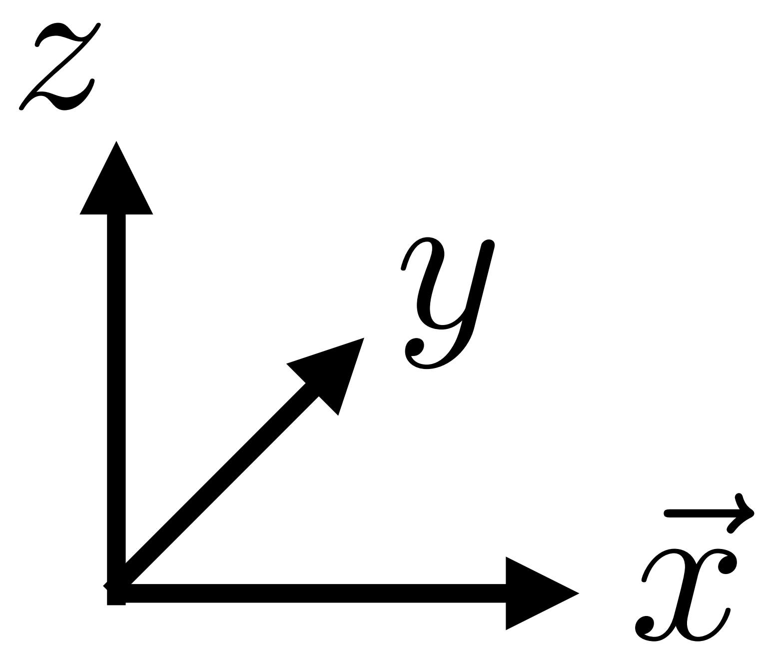
BCFT and DCFT

$d=2$ boundary or defect in $d \geq 3$ flat space

Dilatations $\qquad x^\mu \to \lambda\, x^\mu$

Unbroken





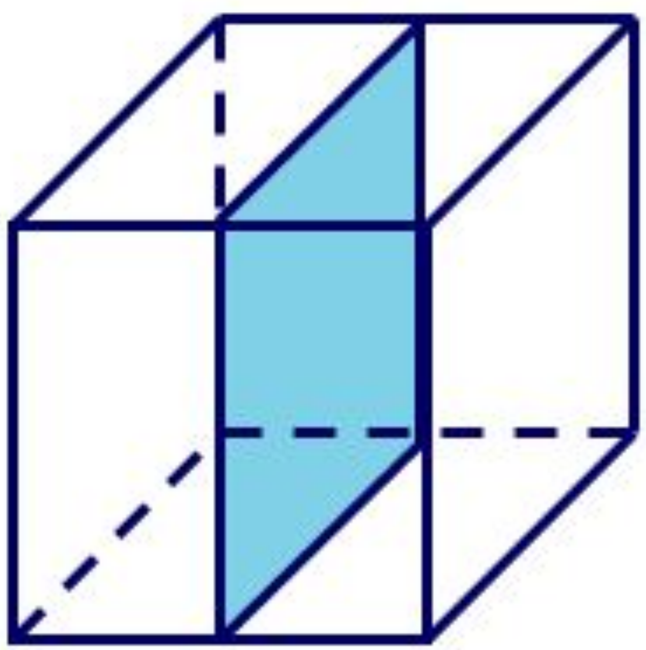
BCFT and DCFT

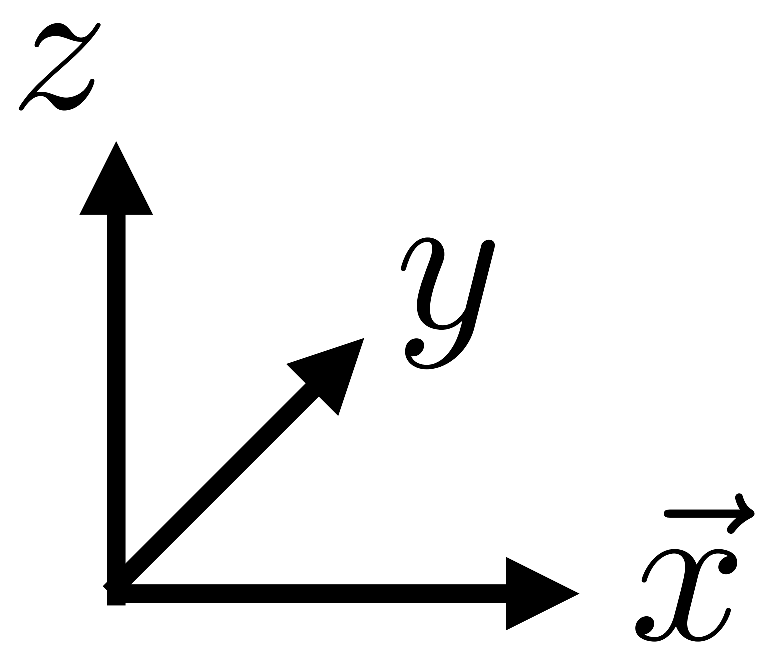
$d = 2$ boundary or defect in $d \geq 3$ flat space

Special Conformal

$$x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2x^\nu b_\nu + b^2 x^2}$$

Broken to $b^\perp = 0$





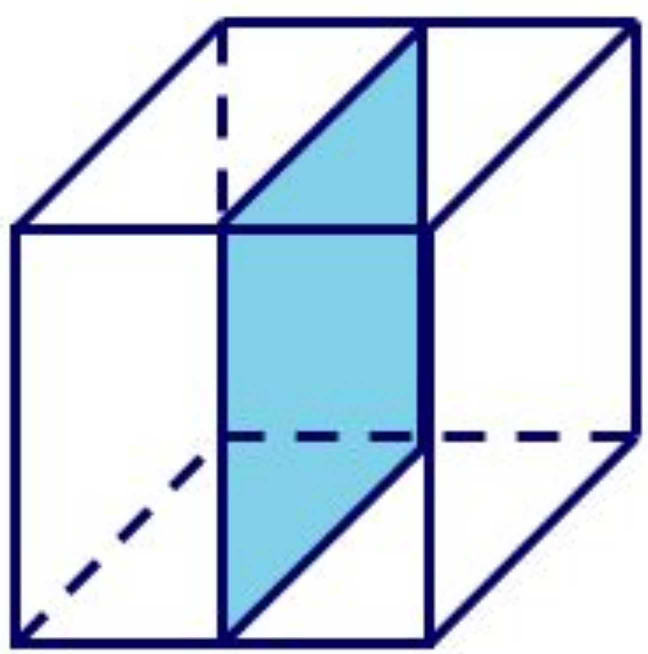
BCFT and DCFT

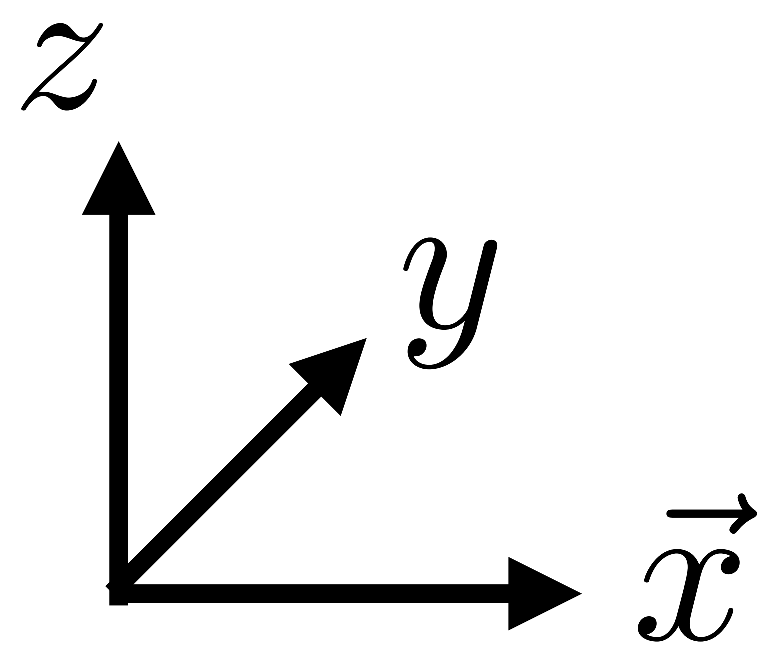
$d = 2$ boundary or defect in $d \geq 3$ flat space

$$SO(d+1,1) \to SO(3,1) \times SO(d-2)$$

conformal transformations preserving the boundary/defect

rotations around the defect





BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

$$SO(d+1,1) \to SO(3,1) \times SO(d-2)$$

conformal transformations preserving the boundary/defect → $SO(3,1)$

rotations around the defect → $SO(d-2)$

Generically NOT Virasoro!

A FINITE number of symmetry generators!

BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

$$SO(d+1,1) \to SO(3,1) \times SO(d-2)$$

conformal transformations preserving the boundary/defect

rotations around the defect

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

$$[\overline{L}_m, \overline{L}_n] = (m-n)\overline{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

$SO(3,1)$ **subgroup**

$L_{\pm 1}$ and L_0

$\bar{L}_{\pm 1}$ and $\bar{L}_0$

BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

Ward Identities

x^μ with $\mu = 1, 2, 3, \ldots, d$

along the boundary/defect

x^a with $a = 1, 2$

transverse to the boundary/defect

x^i with $i = 3, 4, \ldots, d$

BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

Ward Identities

$$\partial_\mu T^{\mu i} = \delta^{d-2}(x^j) D^i$$

Displacement operator D^i

$$T^{\mu\nu} = [T^{\mu\nu}]_{\text{bulk}} + \delta^{d-2} \left[T^{ab}\right]_{2d}$$

Gaussian pillbox integration around boundary/defect

Boundary case:

$$D \propto \left[T^{\perp\perp}\right]^{\partial}_{\text{bulk}}$$

$$\partial_a \left[T^{ab}\right]_{2d} \propto \left[T^{b\perp}\right]^{\partial}_{\text{bulk}}$$

BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

Ward Identities

$$\partial_\mu T^{\mu i} = \delta^{d-2}(x^j) D^i$$

Displacement operator D^i

$$T^{\mu\nu} = [T^{\mu\nu}]_{\text{bulk}} + \delta^{d-2} \, [T^{ab}]_{2d}$$

Conformal invariance

$$T_\mu{}^\mu = 0 \qquad \Rightarrow \qquad [T_\mu{}^\mu]_{\text{bulk}} = 0 \qquad [T_a{}^a]_{2d} = 0$$

BCFT and DCFT

$d = 2$ boundary or defect in $d \geq 3$ flat space

Ward Identities

$$\partial_\mu T^{\mu i} = \delta^{d-2}(x^j) D^i$$

Displacement operator D^i

$$T^{\mu\nu} = [T^{\mu\nu}]_{\text{bulk}} + \delta^{d-2} \left[T^{ab}\right]_{2d}$$

Conformal invariance

$$\langle D^i(x^a) D^j(0) \rangle \propto \frac{\delta^{ij}}{|x^a|^4}$$

Outline:

- Motivation
- 2d CFT Central Charge
- The Systems
- Boundary/Defect Central Charges
- Examples
- Summary and Outlook

Outline:

- Motivation
- 2d CFT Central Charge
- The Systems
- Boundary/Defect Central Charges
- Examples
- Summary and Outlook

Central Charge

$d = 2$ CFT

Virasoro algebra $\Longrightarrow$ central charge c

Stress tensor 2-pt. function

$$\langle T_{zz}(z) T_{zz}(0) \rangle = \frac{c/2}{z^4}$$

Thermodynamic entropy

$$S_{\text{thermo}} = \frac{\pi}{3} c L T + \dots$$

Trace Anomaly

$$T_\mu{}^\mu = \frac{c}{24\pi} R$$

Entanglement Entropy

$$S_{\text{EE}} = \frac{c}{3} \ln(\ell/\varepsilon) + \dots$$

Central Charge

$d = 2$ boundary or defect in $d \geq 3$ BCFT or DCFT

~~Virasoro algebra $\Rightarrow$ central charge c~~

Stress tensor 2-pt. function ??????

Thermodynamic entropy ??????

Trace Anomaly ??????

Entanglement Entropy ??????

Central Charge

$d = 2$ boundary or defect in $d \geq 3$ BCFT or DCFT

~~Virasoro algebra $\Rightarrow$ central charge c~~

Stress tensor 2-pt. function

??????

Billo, Goncalves, Lauria, Meineri
1601.02883

Prochazka
1804.01974

Herzog + Shrestha
2010.04995

Thermodynamic entropy

??????

Central Charge

Counts the number of degrees of freedom

Trace Anomaly

$$g_{\mu\nu} = \delta_{\mu\nu}$$

Conformal invariance

$$T_{\mu}{}^{\mu} = 0$$

Central Charge

Counts the number of degrees of freedom

Trace Anomaly

$$g_{\mu\nu} \neq \delta_{\mu\nu}$$

Quantum Effects
Break Conformal Invariance

$$T_{\mu}{}^{\mu} \neq 0$$

Central Charge

Counts the number of degrees of freedom

Trace Anomaly

CFT in any d

d odd $\qquad T_\mu{}^\mu = 0$

d even $\qquad T_\mu{}^\mu \neq 0$

$d = 2 \qquad T_\mu{}^\mu = \frac{c}{24\pi} R$

Central Charge

$d = 2$ boundary or defect in $d \geq 3$ BCFT or DCFT

Trace Anomaly

$$T_\mu{}^\mu = \left[T_\mu{}^\mu\right]_{\text{bulk}} + \delta^{d-2}\left[T_\mu{}^\mu\right]_{2d}$$

What is $\left[T_\mu{}^\mu\right]_{2d}$?

Berenstein, Corrado, Fischler, Maldacena hep-th/9809188

Graham + Witten hep-th/9901021

Henningson + Skenderis hep-th/9905163

Gustavsson hep-th/0310037, 0404150

Asnin 0801.1469

Schwimmer + Theisen 0802.1017

Trace Anomaly

What is the general form of $T_\mu{}^\mu$?

Step #1

Write down all curvature invariants built from $g_{\mu\nu}$ with the correct dimension

$d = 4$ CFT

$$T_\mu{}^\mu = c_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R^2 + c_4 \Box R$$

Trace Anomaly

What is the general form of $T_{\mu}{}^{\mu}$?

Step #2

Wess-Zumino consistency

$$g_{\mu\nu} \to e^{2\Omega_1} e^{2\Omega_2} g_{\mu\nu} \quad = \quad g_{\mu\nu} \to e^{2\Omega_2} e^{2\Omega_1} g_{\mu\nu}$$

Fixes some coefficients

$$T_{\mu}{}^{\mu} = c_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R^2 + c_4 \Box R$$

Trace Anomaly

What is the general form of $T_\mu{}^\mu$?

Step #3

Add local counterterms

Determine how they enter $T_\mu{}^\mu$

Fixes more coefficients

$$T_\mu{}^\mu = c_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R^2 + c_4 \Box R$$

0

Trace Anomaly

CFT in any d

d odd $\quad T_\mu{}^\mu = 0$

d even $\quad T_\mu{}^\mu \neq 0$

$d = 2 \quad T_\mu{}^\mu = \frac{c}{24\pi} R$

Trace Anomaly

$d = 4$ CFT

$$T_\mu{}^\mu = a\,E - c\,W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$$

Euler density

$$E = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

Weyl tensor

$$W_{\mu\nu\rho\sigma}$$

central charges a and c

Trace Anomaly

Wess-Zumino Consistency

$\int d^dx \sqrt{g}\, T_\mu{}^\mu$ is conformally invariant

$$T_\mu{}^\mu = a\, E - c\, W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$$

Type A

Type B

$\sqrt{g}\, E$

$\sqrt{g}\, W^2$

Changes by a total derivative

Invariant

Trace Anomaly

Wess-Zumino Consistency

Type A central charges

Independent of marginal couplings

Osborn NPB 363 (1991) 486

Obey c-theorems

Cardy PLB 215 (1988) 749 Jack and Osborn NPB 343 (1990) 647

Komargodski and Schwimmer JHEP 12 (2011) 099

$d = 2$ $$T_\mu{}^\mu = \frac{c}{24\pi} R$$

$d = 4$ $$T_\mu{}^\mu = a\, E - c\, W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$$

Central Charge

$d=2$ boundary or defect in $d \geq 3$ BCFT or DCFT

Trace Anomaly

$$T_{\mu}{}^{\mu} = \left[T_{\mu}{}^{\mu}\right]_{\text{bulk}} + \delta^{d-2}\left[T_{\mu}{}^{\mu}\right]_{2d}$$

What is $\left[T_{\mu}{}^{\mu}\right]_{2d}$?

Berenstein, Corrado, Fischler, Maldacena hep-th/9809188

Graham + Witten hep-th/9901021

Henningson + Skenderis hep-th/9905163

Gustavsson hep-th/0310037, 0404150

Asnin 0801.1469

Schwimmer + Theisen 0802.1017

Geometry of Submanifolds

"worldsheet" σ^1, σ^2

"target space" x^μ

embedding $x^\mu(\sigma^a)$

induced metric

$$\hat{g}_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b}$$

Intrinsic curvature $\hat{R}_{abcd} \Rightarrow \hat{R}_{ab} \Rightarrow \hat{R}$

Extrinsic curvature K^μ_{ab}

Mean curvature $K^\mu \equiv K^\mu_{ab}\, \hat{g}^{ab}$

Central Charge

$$[T_\mu{}^\mu]_{2d} = \frac{b}{24\pi}\hat{R} + \frac{d_1}{24\pi}(K^\mu_{ab}K^{ab}_\mu - \frac{1}{2}K^\mu K_\mu) - \frac{d_2}{24\pi}W^{ab}{}_{ab}$$

boundary/defect central charges

$b \qquad d_1 \qquad d_2$

Central Charge

$$[T_\mu{}^\mu]_{2d} = \frac{b}{24\pi}\hat{R} + \frac{d_1}{24\pi}\left(K^\mu_{ab}K^{ab}_\mu - \frac{1}{2}K^\mu K_\mu\right) - \frac{d_2}{24\pi}W^{ab}{}_{ab}$$

Type A

Type B

$$\sqrt{\hat{g}}\,\hat{R}$$

$$\sqrt{\hat{g}}\left(K^\mu_{ab}K^{ab}_\mu - \frac{1}{2}K^\mu K_\mu\right)$$

$$\sqrt{\hat{g}}\,W_{ab}{}^{ab}$$

Changes by a total derivative

Invariant

Central Charge

$$[T_\mu{}^\mu]_{2d} = \frac{b}{24\pi}\hat{R} + \frac{d_1}{24\pi}(K^\mu_{ab}K^{ab}_\mu - \frac{1}{2}K^\mu K_\mu) - \frac{d_2}{24\pi}W^{ab}{}_{ab}$$

Wess-Zumino Consistency

b is independent of boundary/defect marginal couplings

in general, b can depend on bulk marginal couplings

$d = 2 \quad \mathcal{N} = (2,0)$ supersymmetry

b is also independent of bulk marginal couplings

Herzog and Shamir 1906.11281, 1907.04952
Bianchi 1907.06193

Central Charge

$$[T_\mu{}^\mu]_{2d} = \frac{b}{24\pi}\hat{R} + \frac{d_1}{24\pi}(K^\mu_{ab}K^{ab}_\mu - \frac{1}{2}K^\mu K_\mu) - \frac{d_2}{24\pi}W^{ab}{}_{ab}$$

b-theorem

Jensen and O'Bannon 1509.02160

Casini, Landea, Torroba 1812.08183

RG flow on the boundary/defect

$$b_{\rm UV} \geq b_{\rm IR}$$

Euclidean symmetry (Poincaré symmetry) along the boundary/defect

Locality

Reflection positivity (Unitarity)

Counts the number of degrees of freedom

Central Charge

$$[T_\mu{}^\mu]_{2d} = \frac{b}{24\pi}\hat{R} + \frac{d_1}{24\pi}(K^\mu_{ab}K^{ab}_\mu - \frac{1}{2}K^\mu K_\mu) - \frac{d_2}{24\pi}W^{ab}{}_{ab}$$

$d = 2$ $\mathcal{N} = (2,0)$ supersymmetry

$U(1)_R$ current J^a_R

anomaly

$$\partial_a J^a_R = \frac{k}{4\pi}F$$

$$b = 3k$$

RG flow on the boundary/defect

b-extremization

Wang 2012.06574

Central Charge

$$[T_\mu{}^\mu]_{2d} = \frac{b}{24\pi}\hat{R} + \frac{d_1}{24\pi}(K^\mu_{ab}K^{ab}_\mu - \frac{1}{2}K^\mu K_\mu) - \frac{d_2}{24\pi}W^{ab}{}_{ab}$$

Bianchi, Meineri, Myers, Smolkin 1511.06713
Herzog + Huang 1707.06224
Herzog, Huang, Jensen 1709.07431

Displacement operator

$$\nabla_\mu T^{\mu i} = \delta^{d-2}(x^j)\, D^i$$

$$\langle D^i(x^a) D^j(0)\rangle \propto \frac{d_1\,\delta^{ij}}{|x^a|^4}$$

Reflection positivity (unitarity)

$$d_1 \geq 0$$

Central Charge

$$[T_\mu{}^\mu]_{2d} = \frac{b}{24\pi}\hat{R} + \frac{d_1}{24\pi}(K^\mu_{ab}K^{ab}_\mu - \frac{1}{2}K^\mu K_\mu) - \frac{d_2}{24\pi}W^{ab}{}_{ab}$$

Weyl tensor

$$d = 3 \;\Rightarrow\; W_{\mu\nu\rho\sigma} \equiv 0$$

d_2 is only defined for $d > 3$

$\Rightarrow$ defect co-dimension > 1

Central Charge

$$[T_\mu{}^\mu]_{2d} = \frac{b}{24\pi}\hat{R} + \frac{d_1}{24\pi}\left(K^\mu_{ab}K^{ab}_\mu - \frac{1}{2}K^\mu K_\mu\right) - \frac{d_2}{24\pi}W^{ab}{}_{ab}$$

$SO(3,1) \times SO(d-2)$ symmetry allows for

$$\langle T^{ab} \rangle = -\frac{h}{2\pi}\frac{\eta^{ab}}{|x^i|^d} \qquad \langle T^{ai} \rangle = 0$$

$$\langle T^{ij} \rangle = \frac{h}{2\pi(d-3)}\frac{3\delta^{ij}|x^k|^2 - d\,x^i x^j}{|x^l|^{d+2}}$$

$$h \equiv \frac{1}{3\,\mathrm{vol}(S^{d-3})}\frac{d-3}{d-1}\,d_2$$

Bianchi, Meineri, Myers, Smolkin 1511.06713

Jensen, O'Bannon, Robinson, Rodgers 1812.08745

Central Charge

$$[T_\mu{}^\mu]_{2d} = \frac{b}{24\pi}\hat{R} + \frac{d_1}{24\pi}(K^\mu_{ab}K^{ab}_\mu - \frac{1}{2}K^\mu K_\mu) - \frac{d_2}{24\pi}W^{ab}{}_{ab}$$

Average Null Energy Condition (ANEC)

Faulkner, Leigh, Parrikar, Wang 1605.08072

Hartman, Kundu, Tajdini 1610.05308

along any null ray

$$\int_{-\infty}^{\infty} du \langle T_{uu} \rangle \geq 0$$

Central Charge

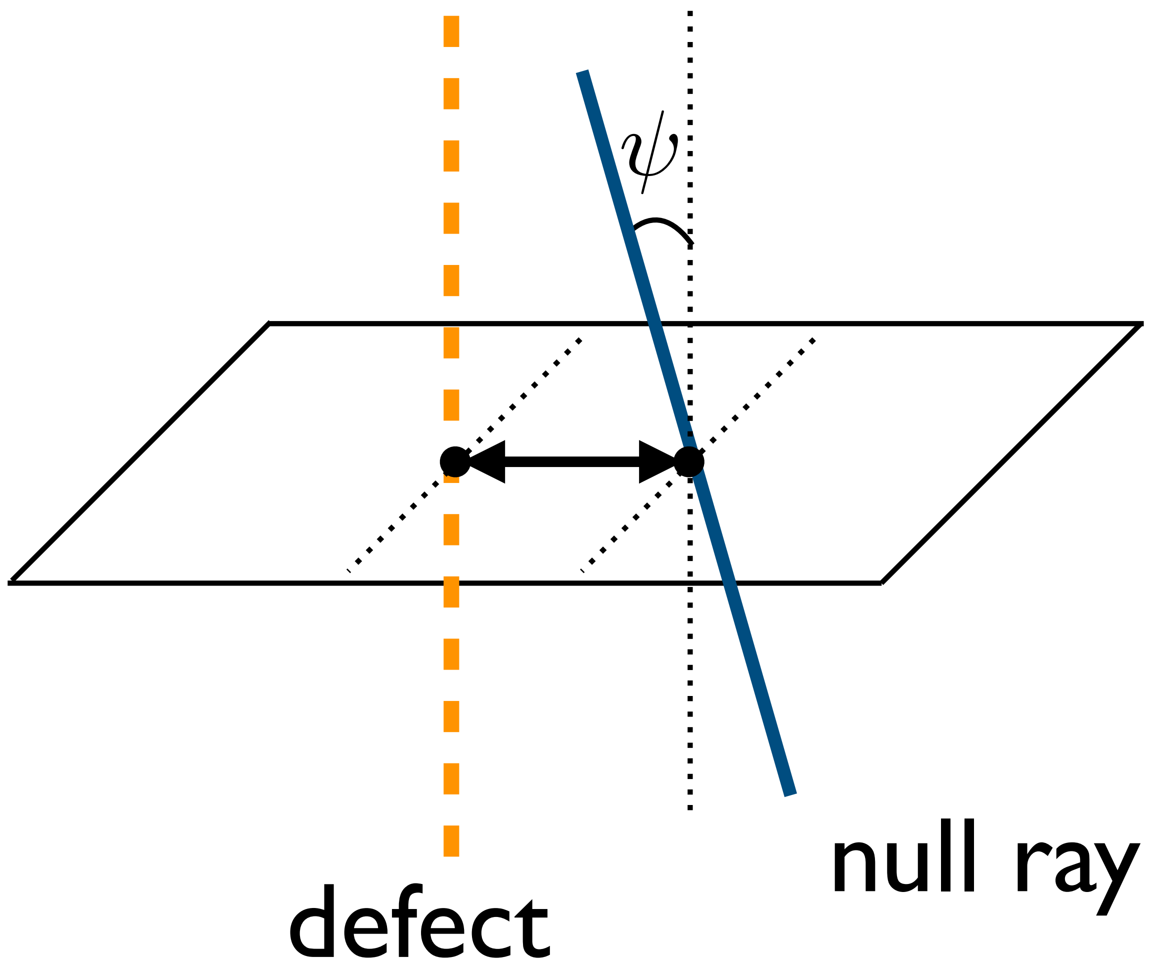
$$[T_\mu{}^\mu]_{2d} = \frac{b}{24\pi}\hat{R} + \frac{d_1}{24\pi}\left(K^\mu_{ab}K^{ab}_\mu - \frac{1}{2}K^\mu K_\mu\right) - \frac{d_2}{24\pi}W^{ab}{}_{ab}$$

Average Null Energy Condition (ANEC)

Jensen, O'Bannon, Robinson, Rodgers 1812.08745

$$\int_{-\infty}^{\infty} du \langle T_{uu} \rangle \geq 0$$

$$d_2 \geq 0$$



Central Charge

$$[T_\mu{}^\mu]_{2d} = \frac{b}{24\pi}\hat{R} + \frac{d_1}{24\pi}(K^\mu_{ab}K^{ab}_\mu - \frac{1}{2}K^\mu K_\mu) - \frac{d_2}{24\pi}W^{ab}{}_{ab}$$

$d = 4$

$d = 2 \quad \mathcal{N} = (2,0)$ supersymmetry

$d_1 = d_2$

Bianchi and Lemos 1911.05082

conjectured for $d > 4$

Central Charge

$d=2$ boundary or defect in $d \geq 3$ BCFT or DCFT

~~Virasoro algebra $\Rightarrow$ central charge c~~

Stress tensor 2-pt. function ??????

Thermodynamic entropy ??????

Trace Anomaly ??????

Entanglement Entropy ??????

Central Charge

Entanglement Entropy

$d = 2$ CFT

Holzhey, Larsen, Wilczek hep-th/9403108 Calabrese + Cardy hep-th/0405152

interval of length ℓ

short-distance cutoff ε

$$S_{\mathrm{EE}} = \frac{c}{3} \ln \frac{\ell}{\varepsilon} + \ldots$$

$$3\,\ell\,\frac{d}{d\ell} S_{\mathrm{EE}} = c$$

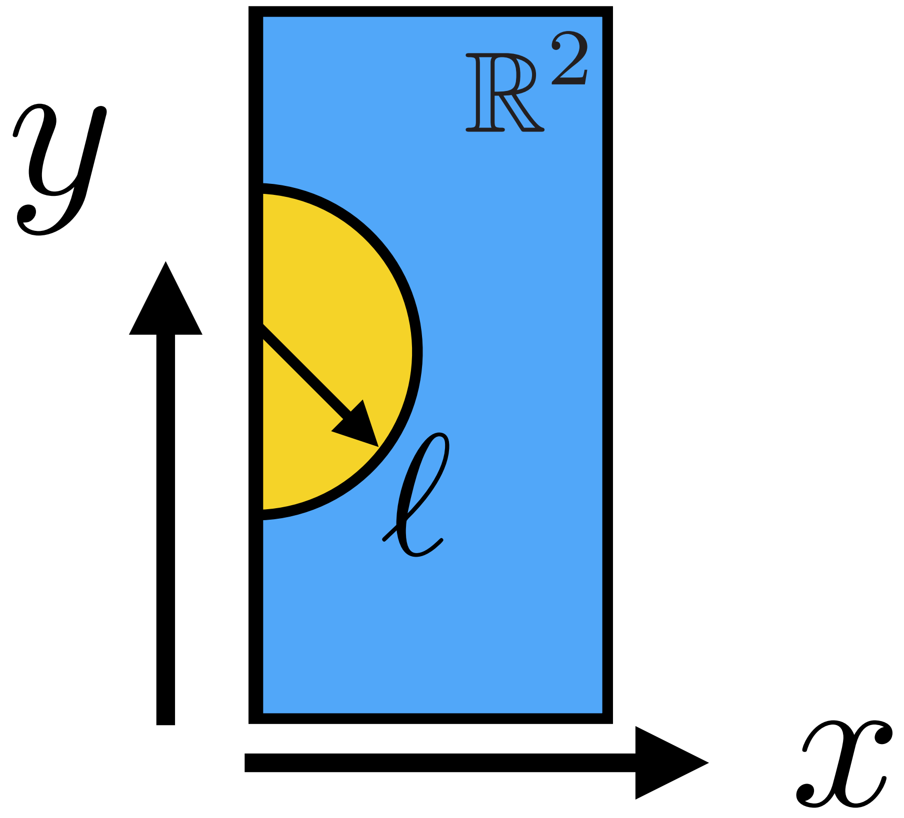
Central Charge

Entanglement Entropy

$d = 3$ BCFT

semi-circle of radius ℓ

centered on the boundary



Central Charge

Entanglement Entropy

$d = 3$ BCFT

Fursaev and Solodukhin 1305.2334, 1601.06418 Bethiere + Solodukhin 1604.07571

$$S_{\text{EE}} = \frac{1}{2} S_{\text{EE}}^{\text{CFT}} + S_{\text{EE}}^{2d}$$

$$S_{\text{EE}}^{\text{CFT}} = \# \frac{\text{Length}}{\varepsilon} + \# + \ldots$$

$$S_{\text{EE}}^{2d} = \frac{b}{3} \ln(\ell/\varepsilon) + \# + \ldots$$

$$3\,\ell \frac{d}{d\ell} \left[S_{\text{EE}} - \frac{1}{2} S_{\text{EE}}^{\text{CFT}} \right] = b$$

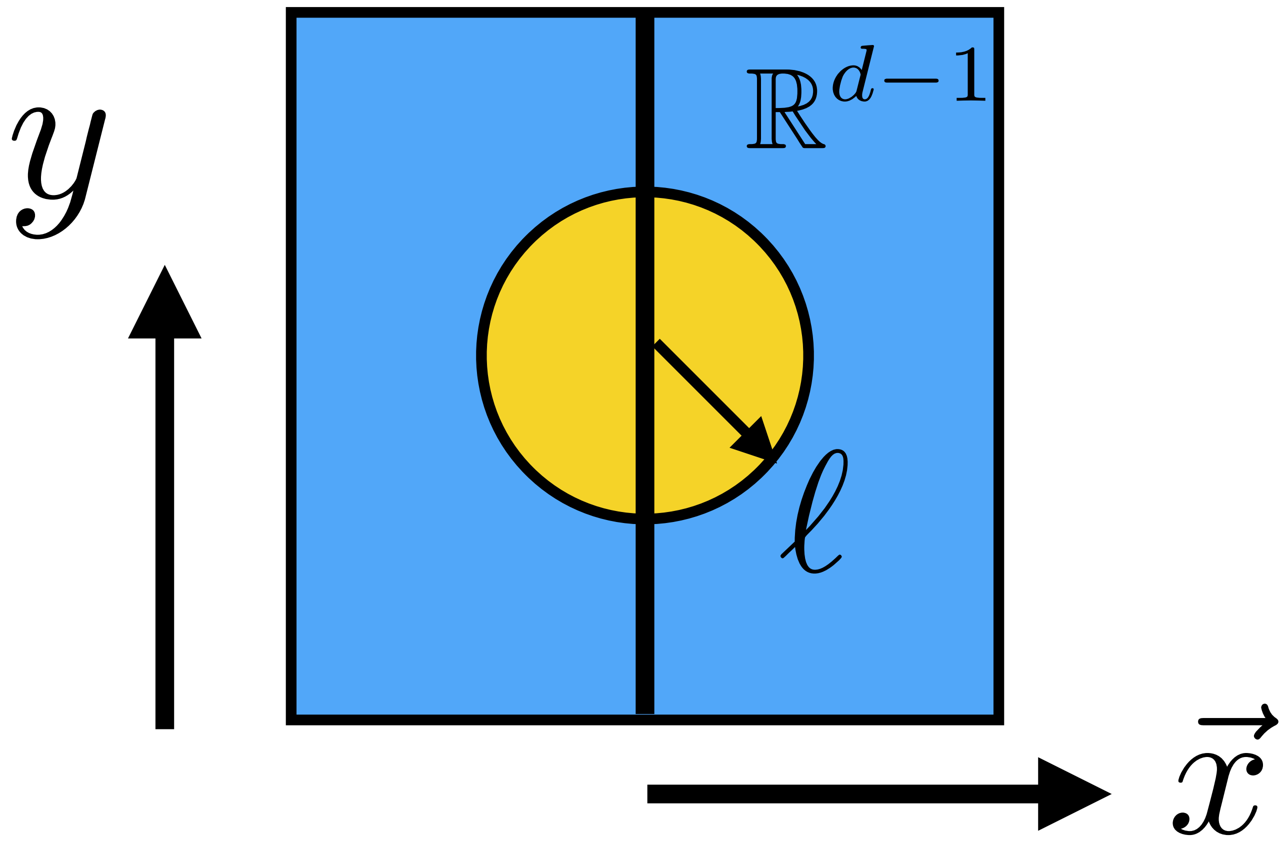
Central Charge

Entanglement Entropy

$d > 3$ DCFT

sphere of radius ℓ

centered on the defect



Central Charge

Entanglement Entropy

$d > 3$ DCFT

Kobayashi, Nishioka, Sato, Watanabe 1810.06995

Jensen, O'Bannon, Robinson, Rodgers 1812.08745

$$S_{\text{EE}} = S_{\text{EE}}^{\text{CFT}} + S_{\text{EE}}^{2d}$$

$$S_{\text{EE}}^{\text{CFT}} = \# \frac{\text{Area}}{\varepsilon^{d-2}} + \frac{\#}{\varepsilon^{d-4}} + \ldots + \# a \ln\left(\ell/\varepsilon\right) + \# + \ldots$$

$$S_{\text{EE}}^{2d} = \frac{1}{3}\left(b - \frac{d-3}{d-1} d_2\right) \ln\left(\ell/\varepsilon\right) + \# + \ldots$$

$$3\,\ell\,\frac{d}{d\ell}\left[S_{\text{EE}} - S_{\text{EE}}^{\text{CFT}}\right] = b - \frac{d-3}{d-1} d_2$$

Central Charge

$d=2$ boundary or defect in $d \geq 3$ BCFT or DCFT

~~Virasoro algebra $\rightarrow$ central charge c~~

Stress tensor 2-pt. function ??????

Thermodynamic entropy ??????

Trace Anomaly ??????

Entanglement Entropy ??????

Outline:

- Motivation
- 2d CFT Central Charge
- The Systems
- Boundary/Defect Central Charges
- Examples
- Summary and Outlook

Outline:

- Motivation
- 2d CFT Central Charge
- The Systems
- Boundary/Defect Central Charges
- Examples
- Summary and Outlook

Examples

$d = 3$ BCFTs

free massless real scalar or Dirac fermion

Theory	BC	b	d_1	d_2
Scalar	Dirichlet	$-1/16$	$3/32$	N/A
Scalar	Neumann	$1/16$	$3/32$	N/A
Fermion	Mixed	0	$3/16$	N/A

Nozaki, Takayanagi, Ugajin 1205.1573

Jensen and O'Bannon 1509.02160

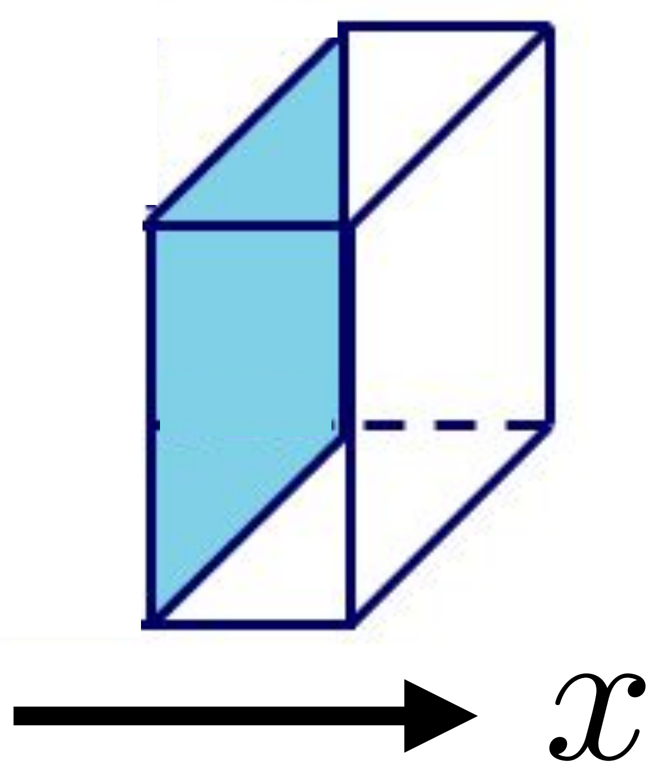
Fursaev and Solodukhin 1601.06418

Examples

$d = 3$ BCFTs

free massless real scalar or Dirac fermion

Theory	BC	b	d_1	d_2
Scalar	Dirichlet	$-1/16$	$3/32$	N/A
Scalar	Neumann	$1/16$	$3/32$	N/A
Fermion	Mixed	0	$3/16$	N/A



“mixed”

$$\Psi_\pm \equiv \frac{1}{2}\left(1 \pm \gamma^x\right)\Psi$$

$$\text{Dirichlet for } \Psi_+ \text{ or } \Psi_-$$

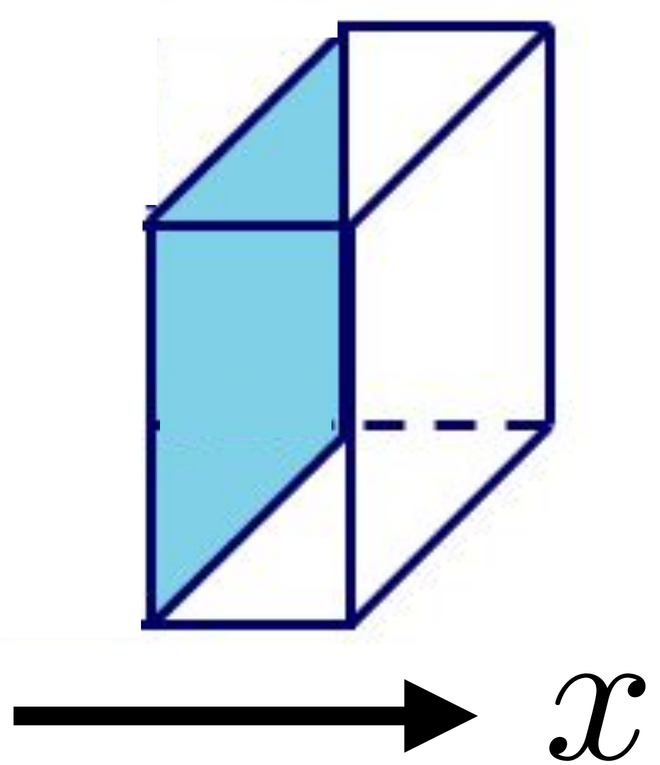
$$+ \text{ Neumann for } \Psi_- \text{ or } \Psi_+$$

Examples

$d = 3$ BCFTs

free massless real scalar or Dirac fermion

Theory	BC	b	d_1	d_2
Scalar	Dirichlet	$-1/16$	$3/32$	N/A
Scalar	Neumann	$1/16$	$3/32$	N/A
Fermion	Mixed	0	$3/16$	N/A



"mixed"

$$\Psi_\pm \equiv \frac{1}{2}\left(1 \pm \gamma^x\right)\Psi$$

Dirichlet for Ψ_+ or Ψ_-

$+$ Neumann for Ψ_- or Ψ_+

Examples

$d = 3$ BCFTs

free massless real scalar or Dirac fermion

Theory	BC	b	d_1	d_2
Scalar	Dirichlet	$-1/16$	$3/32$	N/A
Scalar	Neumann	$1/16$	$3/32$	N/A
Fermion	Mixed	0	$3/16$	N/A

$$\text{“fermion} = \text{Scalar Dirichlet} + \text{Scalar Neumann”}$$

$b = 0$ because:

b is independent of boundary marginal couplings

$m\bar{\Psi}\Psi\big|_{\partial}$ is marginal

Examples

$d = 3$ BCFTs

free massless real scalar or Dirac fermion

Theory	BC	b	d_1	d_2
Scalar	Dirichlet	$-1/16$	$3/32$	N/A
Scalar	Neumann	$1/16$	$3/32$	N/A
Fermion	Mixed	0	$3/16$	N/A

$b < 0$ in unitary theory

Does b have a lower bound?

$d_1 \geq 0$

Examples

$d = 3$ BCFTs

free massless real scalar or Dirac fermion

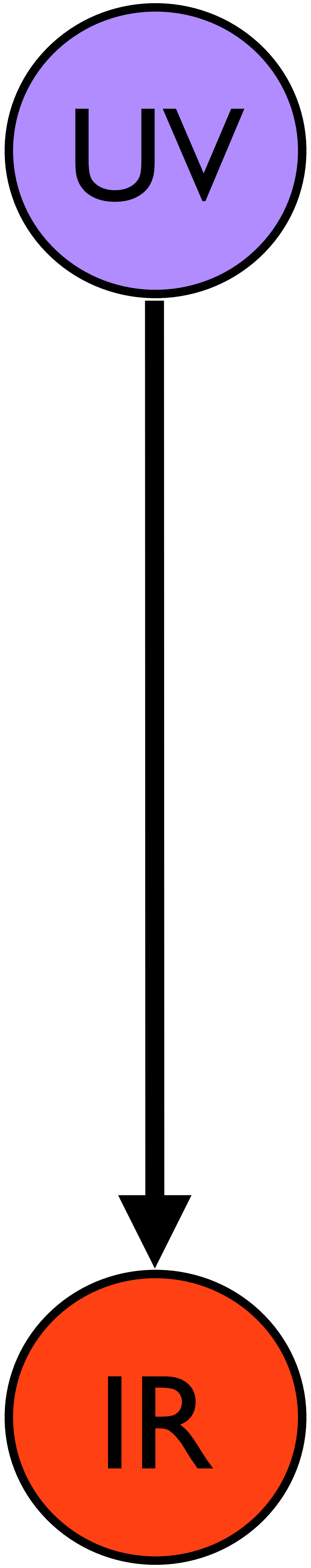
Theory	BC	b	d_1	d_2
Scalar	Dirichlet	$-1/16$	$3/32$	N/A
Scalar	Neumann	$1/16$	$3/32$	N/A
Fermion	Mixed	0	$3/16$	N/A

b-theorem

Jensen and O'Bannon 1509.02160
Casini, Landea, Torroba 1812.08183

RG flow on the boundary

$$b_{\text{UV}} \geq b_{\text{IR}}$$



UV BCFT

Free, massless, real scalar

Neumann B.C.

Boundary RG Flow

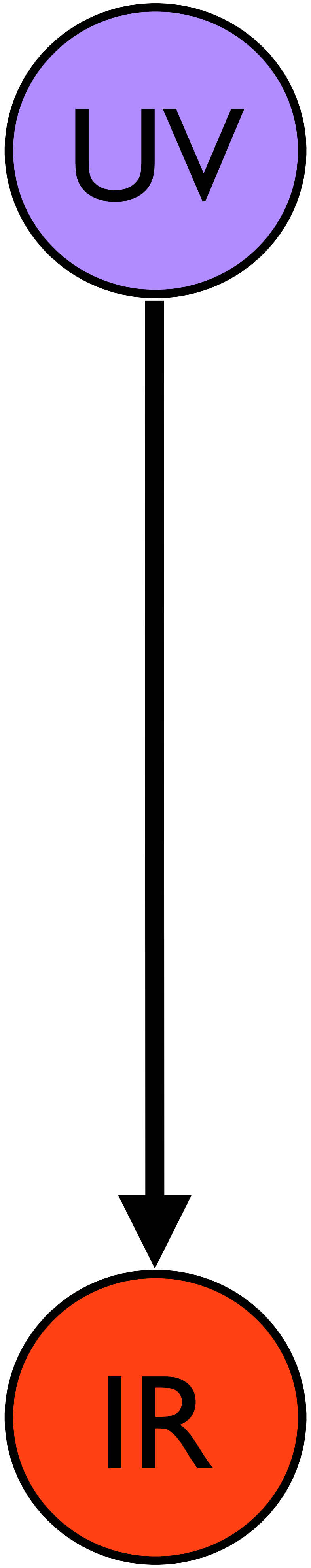
$$S_{\mathrm{BCFT}}^{\mathrm{UV}} \to S_{\mathrm{BCFT}}^{\mathrm{UV}} + \int d^3x\, \delta(x)\, m^2 \Phi^2(\vec{x})$$

$$\Delta_{\Phi^2} = 1 < 2$$

IR BCFT

Free, massless, real scalar

Dirichlet B.C.



$$b_{\mathrm{UV}} = \frac{1}{16}$$

Neumann B.C.

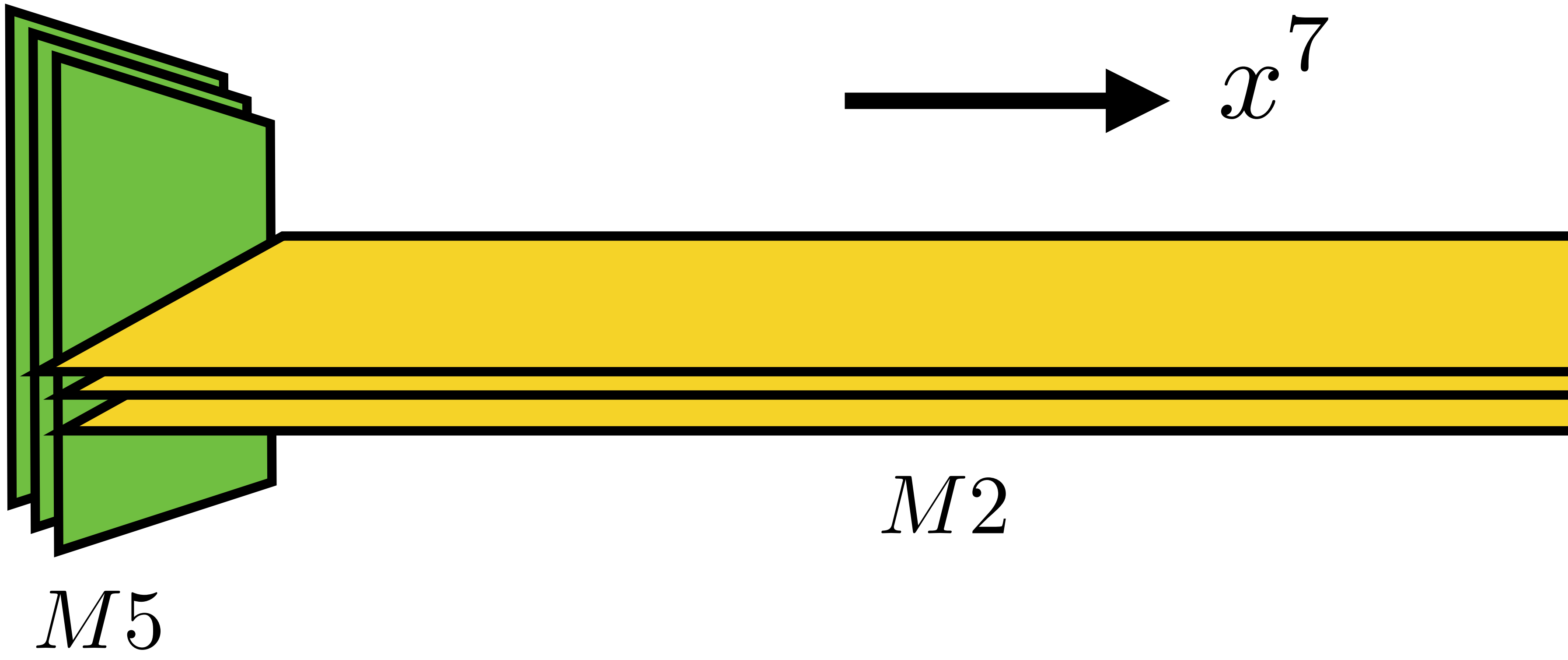
$$b_{\mathrm{UV}} \geq b_{\mathrm{IR}}$$

$$b_{\mathrm{IR}} = -\frac{1}{16}$$

Dirichlet B.C.

Examples

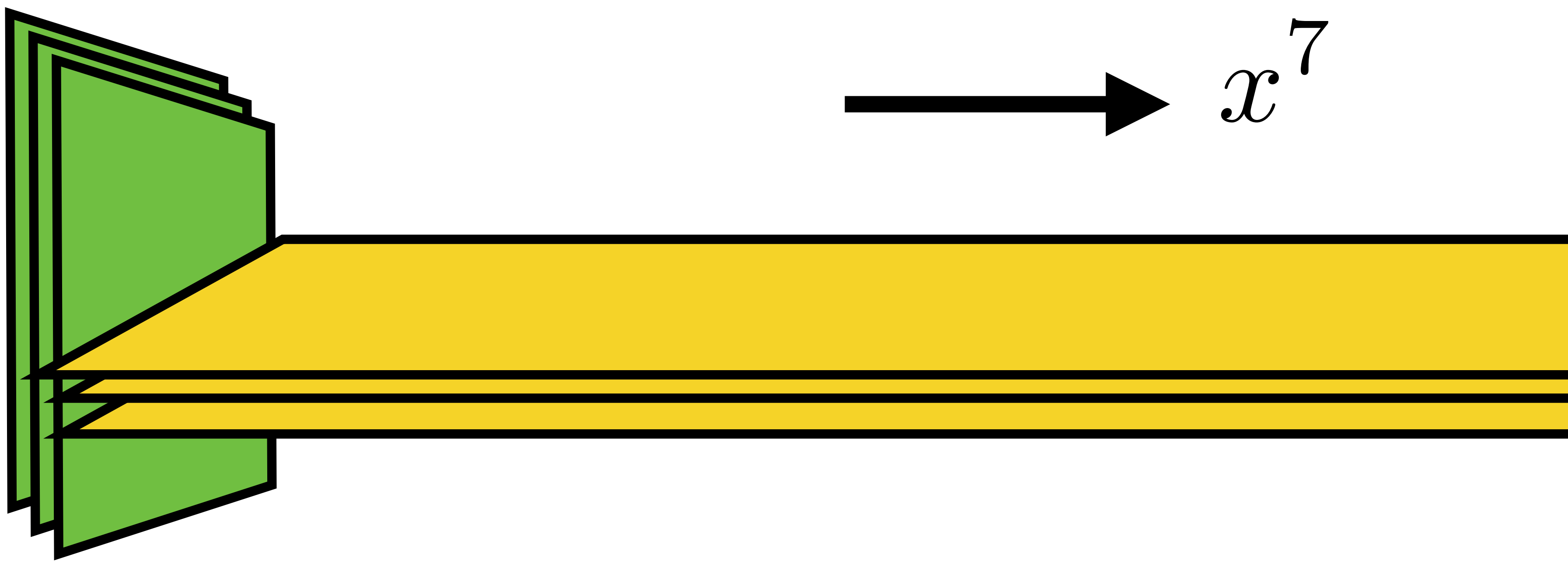
M2-branes ending on M5-branes



	1	2	3	4	5	6	7	8	9	10	11
M5	●	●	●	●	●	●	—	—	—	—	—
M2	●	●	—	—	—	—	●	—	—	—	—

Examples

M2-branes ending on M5-branes



N coincident M5-branes' worldvolume theory

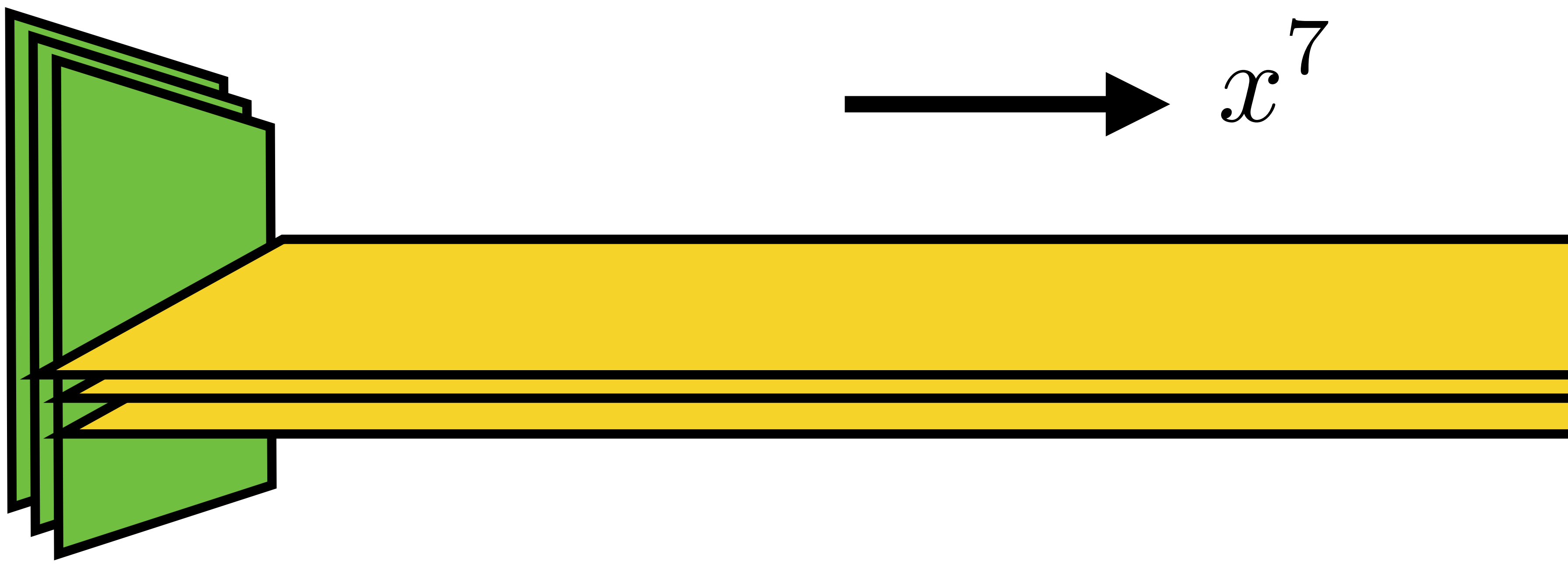
$d = 6$ $\mathcal{N} = (2,0)$ supersymmetric CFT

Gauge algebra $U(N)$

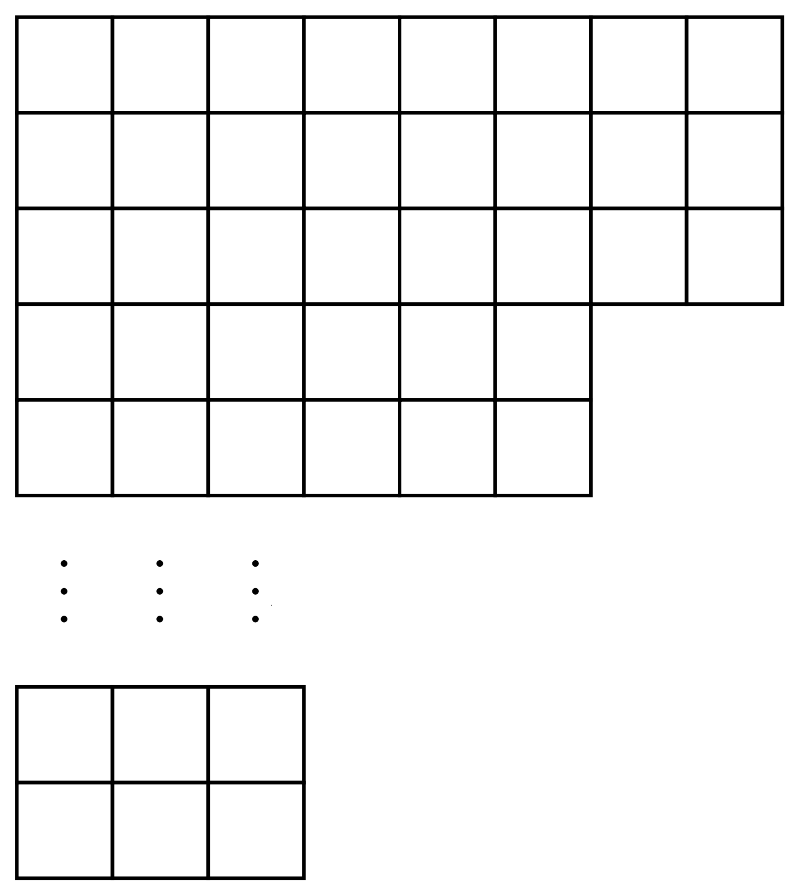
M2-branes = "Wilson surface"

Examples

M2-branes ending on M5-branes



$\mathcal{R} \leftrightarrow$



Examples

M2-branes ending on M5-branes

Estes, Krym, O'Bannon, Robinson, Rodgers 1812.00923
Jensen, O'Bannon, Robinson, Rodgers 1812.08745
Chalabi, O'Bannon, Robinson, Sisti 2003.02857
Drukker, Probst, Trépanier 2003.12372, 2009.10732, 2012.11087
Drukker, Giombi, Tseytlin, Zhou 2004.04562

$$b = 24(\lambda, \rho) + 3(\lambda, \lambda)$$

$$d_2 = 24(\lambda, \rho) + 6(\lambda, \lambda)$$

$\lambda \equiv \text{highest weight}$ $\qquad$ $\rho \equiv \text{Weyl vector}$

Calculated both using holography

However a SUSY index calculation shows that this result for d_2 is exact!

Examples

M2-branes ending on M5-branes

Estes, Krym, O'Bannon, Robinson, Rodgers 1812.00923
Jensen, O'Bannon, Robinson, Rodgers 1812.08745
Chalabi, O'Bannon, Robinson, Sisti 2003.02857
Drukker, Probst, Trépanier 2003.12372, 2009.10732, 2012.11087
Drukker, Giombi, Tseytlin, Zhou 2004.04562

$$b = 24(\lambda, \rho) + 3(\lambda, \lambda)$$

$$d_2 = 24(\lambda, \rho) + 6(\lambda, \lambda)$$

$\lambda \equiv \text{highest weight}$ $\qquad$ $\rho \equiv \text{Weyl vector}$

$b \geq 0 \text{ and } d_2 \geq 0$

Both are invariant under the Weyl group and complex conjugation of the representation $\mathcal{R} \to \mathcal{R}^*$

Examples

AdS_{d+1} radius of curvature L

AdS_3 probe brane

$$S_{\text{brane}} = \mathcal{T} \int d^3\xi \sqrt{\det P[G_{MN}]}$$

$$b = d_1 = d_2 = 6\pi L^3 \mathcal{T}$$

Graham + Witten hep-th/9901021

Jensen, O'Bannon, Robinson, Rodgers 1812.08745

Similar to holographic CFTs in $d = 4$, which have $a = c$

Outline:

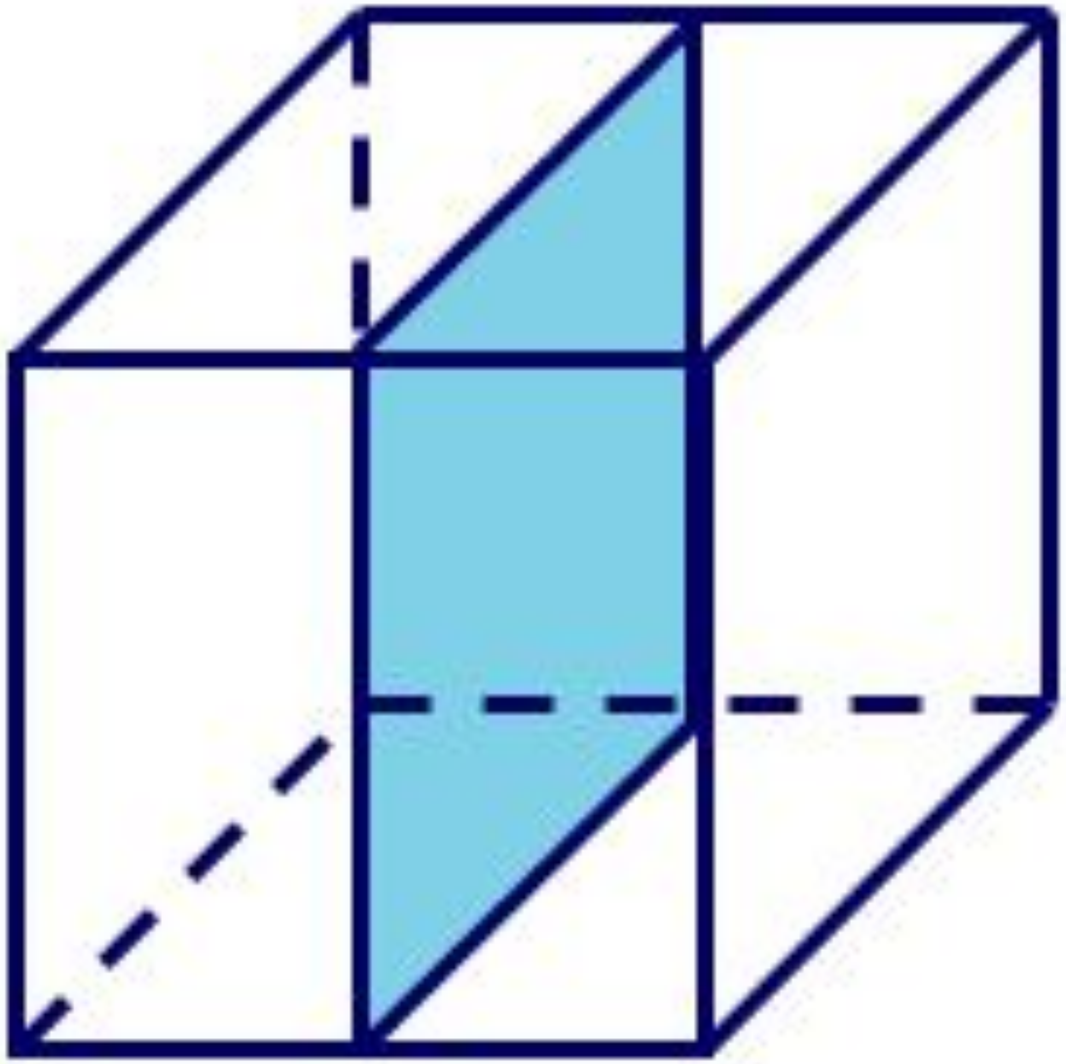
- Motivation
- 2d CFT Central Charge
- The Systems
- Boundary/Defect Central Charges
- Examples
- Summary and Outlook

Outline:

- Motivation
- 2d CFT Central Charge
- The Systems
- Boundary/Defect Central Charges
- Examples
- Summary and Outlook

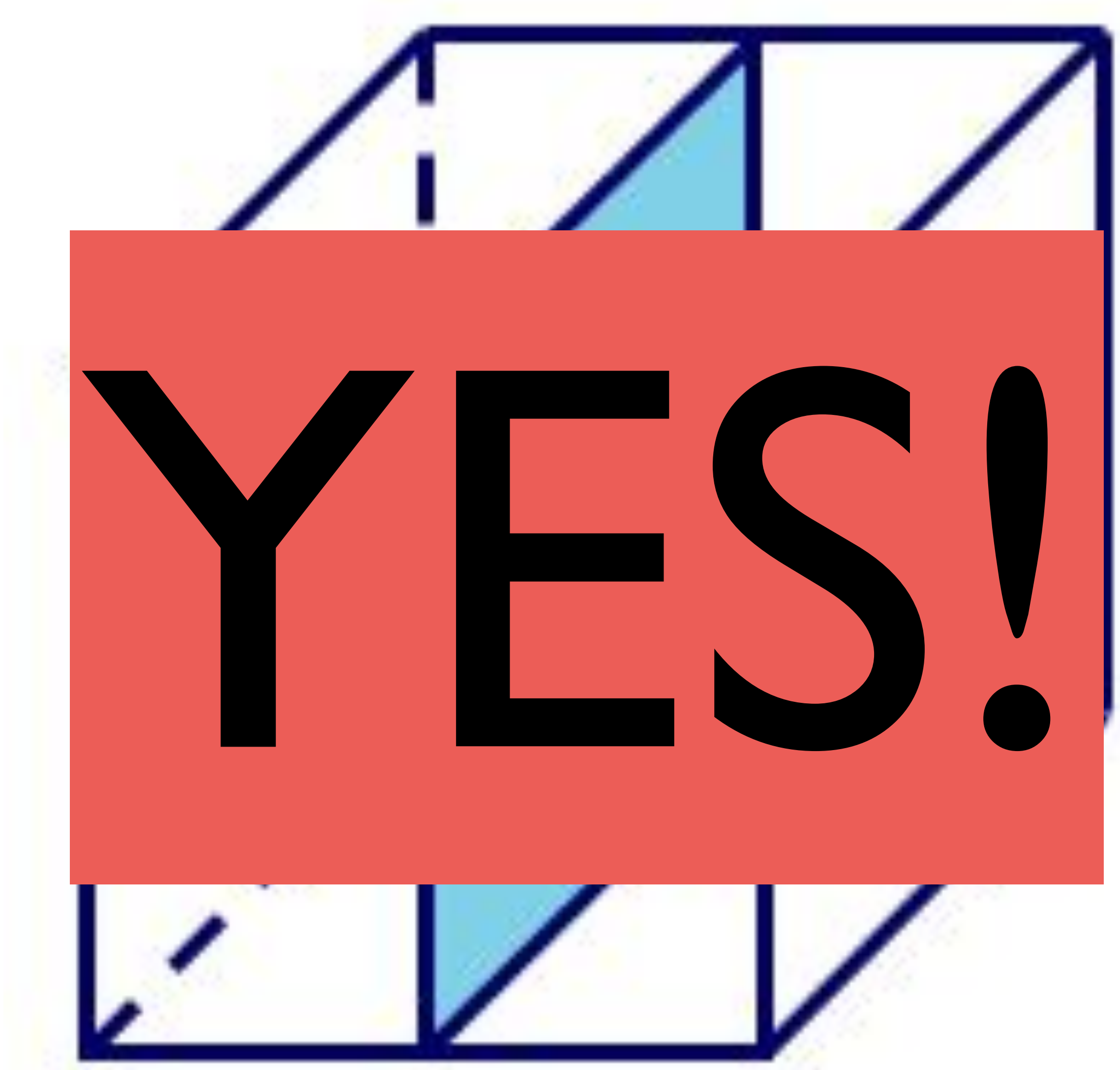
Question

Can we define a central charge
for $d = 2$ boundaries and defects?



Question

Can we define a central charge
for $d = 2$ boundaries and defects?



Summary

$d=2$ boundary or defect in $d \geq 3$ BCFT or DCFT

~~Virasoro algebra $\Rightarrow$ central charge c~~

Stress tensor 2-pt. function ??????

Thermodynamic entropy ??????

Trace Anomaly ??????

Entanglement Entropy ??????

Summary

$d=2$ boundary or defect in $d \geq 3$ BCFT or DCFT

~~Virasoro algebra $\Rightarrow$ central charge c~~

Stress tensor 2-pt. function ??????

Thermodynamic entropy ??????

Trace Anomaly ✓

Entanglement Entropy ✓

Outlook

More examples?

Stress-energy tensor and thermal entropy?

Constraints? Bounds? c-theorems?

Applications?

Boundaries and defects of other dimensions?

$[T_a{}^a]_{3d}$

Herzog, Huang, Jensen
1510.00021, 1709.07431

$[T_a{}^a]_{4d}$

Chalabi, Herzog, O'Bannon, Robinson, Sisti
2111.14713

Thank You.