HoloTube Seminar, July 2021

JT gravity with defects and the Aharonov-Bohm effect

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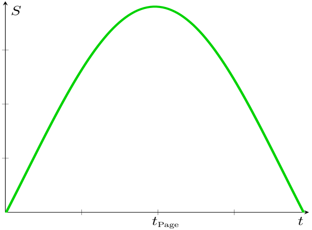
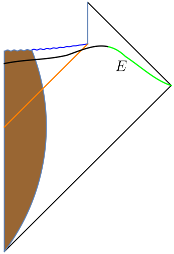
E. Mefford & KS; 2011.04695 [hep-th]

MOTIVATIONS

• In recent years, some exactly solvable systems of quantum gravity in lower dimension have led to a major breakthrough in the black hole information problem.

• The "island" proposal correctly produced the Page curve of an evaporating black hole from a low-energy gravity theory:





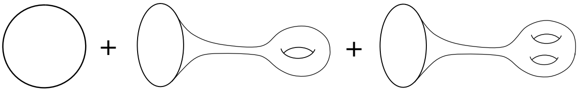
• But there is also another important development for quantum gravity theory: a "new" type of AdS/CFT correspondence including ensemble average.

AdS/CFT with ensemble average

• Jackiw-Teitelboim (JT) gravity is remarkable, especially when it is defined on Euclidean negatively curved backgrounds:

$$I = \underbrace{-\frac{S_0}{2\pi}\left[\frac{1}{2}\int_{\mathcal{M}}\sqrt{g}R + \int_{\partial\mathcal{M}}\sqrt{h}K\right]}_{\text{topological term } = S_0\,\chi(\mathcal{M})} - \left[\underbrace{\frac{1}{2}\int_{\mathcal{M}}\sqrt{g}\phi(R+2)}_{\text{sets } R=-2} + \underbrace{\int_{\partial\mathcal{M}}\sqrt{h}\phi(K-1)}_{\text{gives Schwarzian action}}\right]$$

• There is a strong evidence that if the partition function is defined as sum over all higher genus topologies:

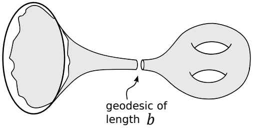
$\langle Z(\beta)\rangle =$  $+ \cdots$

the dual of JT gravity is a random ensemble of some quantum mechanical systems. [Saad, Shenker & Stanford '19]

Genus expansion

• For JT gravity, the physical d.o.f. is localized at the boundary, which is given by Schwarzian theory.

• The leading topology (Poincare disk) is described by the $SL(2,\mathbb{R})$ Schwarzian theory, while all higher genus topology is described by the $U(1)$ hyperbolic Schwarzian theory:



• Therefore, we can formally write down the genus expansion as

$$\langle Z(\beta)\rangle \simeq e^{S_0}\, Z^{\text{disk}}_{\text{Sch}}(\beta) + \sum_{g=1}^{\infty} e^{(1-2g)S_0} \int_0^{\infty} db\, b\, V_{g,1}(b)\, Z^{\text{trumpet}}_{\text{Sch}}(\beta, b)$$

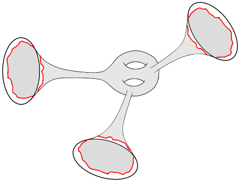
Multi-boundaries

- More generically we consider connected n-point functions

$$\langle Z(\beta_1)...Z(\beta_n)\rangle_{\text{conn}} \simeq \sum_{g=0}^{\infty} \frac{Z_{g,n}(\beta_1,...,\beta_n)}{(e^{S_0})^{2g+n-2}}$$

where each component is like

$$Z_{g=2,n=3}(\beta_1,\beta_2,\beta_3) \quad =$$



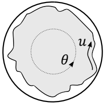
- Again, we can formally write down the decomposition as

$$Z_{g,n}(\beta_1,\cdots,\beta_n) = \int_0^\infty b_1 db \cdots \int_0^\infty b_n db_n \, V_{g,n}(b_1,...,b_n) \times Z^{\text{trumpet}}_{\text{Sch}}(\beta_1,b_1)...Z^{\text{trumpet}}_{\text{Sch}}(\beta_n,b_n)$$

JT gravity on disk/trumpet

- Let us first study the Poincare disk/trumpet case:

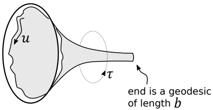
$$I_{\mathsf{JT}} = -\frac{1}{2}\int_{\mathcal{M}} \sqrt{g}\,\phi(R+2) + \int_{\partial\mathcal{M}} \sqrt{h}\phi(K-1)$$



- Variation of ϕ leads to $R=-2$, so that the EH Action is reduced to the Euler characteristic χ.

- The two simplest on-shell geometries are the disk D and the trumpet $T(b)$:

$$\begin{aligned} ds_D^2 &= d\rho^2 + \sinh^2\rho\, d\theta^2\,, \qquad (0 \le \theta < 2\pi) \\ ds_T^2 &= d\rho^2 + \cosh^2\rho\, d\tau^2\,, \qquad (0 \le \tau < b) \end{aligned}$$



The periodicity of τ breaks $SL(2,\mathbb{R}) \to U(1)$.

- The solution of the dilaton field for each geometry is given by

$$\phi_D = \gamma_D \cosh\rho\,, \qquad \phi_T = \gamma_T \sinh\rho$$

Schwarzian action

- On-shell actions are given by the Schwarzian action [Maldacena, Stanford & Yang '16]

$$I_{\mathsf{Sch}}^{\mathsf{disk}} = -\gamma_D \int_0^\beta du \left\{ \tan\left(\frac{\theta(u)}{2}\right), u \right\}$$

$$I_{\mathsf{Sch}}^{\mathsf{trumpet}} = -\gamma_T \int_0^\beta du \left\{ \tanh\left(\frac{\tau(u)}{2}\right), u \right\}$$

where

$$\{F(u), u\} \equiv \frac{F'''(u)}{F'(u)} - \frac{3}{2}\left(\frac{F''(u)}{F'(u)}\right)^2$$

- Therefore, the partition functions are

$$Z_{\mathsf{Sch}}^{\mathsf{disk}}(\beta) = \int \frac{d\mu[\theta]}{SL(2,\mathbb{R})} \exp\left[-\frac{\gamma}{2}\int_0^\beta du \left(\frac{\theta''^2}{\theta'^2} - \theta'^2\right)\right]$$

$$Z_{\mathsf{Sch}}^{\mathsf{trumpet}}(\beta, b) = \int \frac{d\mu[\tau]}{U(1)} \exp\left[-\frac{\gamma}{2}\int_0^\beta du \left(\frac{\tau''^2}{\tau'^2} + \tau'^2\right)\right]$$

Topological Recursion

- We now consider the Weil-Petersson volume $V_{g,n}$.

$$Z_{g,n}(\beta_1,\cdots,\beta_n) = \int_0^\infty b_1 db \cdots \int_0^\infty b_n db_n \, V_{g,n}(b_1,...,b_n) \times Z^{\text{trumpet}}_{\text{Sch}}(\beta_1,b_1)...Z^{\text{trumpet}}_{\text{Sch}}(\beta_n,b_n)$$

- It is know that they satisfy a topological recursion relation, which is, in terms of resolvent, written as [Eynard, '04] [Mirzakhani, '07]

$$W_{g,n}(z_1,\overbrace{z_2,\ldots,z_n}^{J}) = \operatorname*{Res}_{z\to 0}\left\{\frac{1}{(z_1^2-z^2)}\frac{1}{4y(z)}\left[W_{g-1,n+1}(z,-z,J) + \sum'_{I\cup I'=J;h+h'=g} W_{h,1+|I|}(z,I)\,W_{h',1+|I'|}(-z,I')\right]\right\}$$

where $W_{0,1} = 2z\,y(z)$ is the input.

A few solutions

- For JT $y(z) = \sin(2\pi z)/4\pi$, and the recursion relation can be solved explicitly:

$$W_{0,1} = 2z_1 \frac{\sin(2\pi z_1)}{4\pi}, \qquad W_{0,2} = \frac{1}{(z_1 - z_2)^2}, \qquad W_{0,3} = \frac{1}{z_1^2 z_2^2 z_3^2}$$

$$W_{1,1} = \frac{3 + 2\pi^2 z_1^2}{24 z_1^4}, \qquad W_{2,1} = \left(\frac{105}{128 z_1^{10}} + \frac{203\pi^2}{192 z_1^8} + \frac{139\pi^4}{192 z_1^6} + \frac{169\pi^6}{480 z_1^4} + \frac{29\pi^8}{192 z_1^2} \right)$$

$$W_{1,2} = \frac{5(z_1^4 + z_2^4) + 3z_1^2 z_2^2 + 4\pi^2(z_1^4 z_2^2 + z_2^4 z_1^2) + 2\pi^4 z_1^4 z_2^4}{8 z_1^6 z_2^6}$$

- The Weil-Petersson volume $V_{g,n}$ can be found by

$$W_{g,n}(z_1, ..., z_n) = \int_0^\infty db_1\, b_1\, e^{-b_1 z_1} ... \int_0^\infty db_n\, b_n\, e^{-b_n z_n}\, V_{g,n}(b_1, ..., b_n)$$

Topological expansion of one-hermitian matrix integral

• In fact, the same recursion relation is also satisfied by n-point functions of the loop operators

$$Z_n(\ell_1, \ldots, \ell_n) \equiv \left\langle \mathrm{Tr}\left(e^{-\ell_1 H}\right) \cdots \mathrm{Tr}\left(e^{-\ell_n H}\right) \right\rangle$$

in the one-hermitian matrix integral [Eynard, '04]

$$\mathcal{Z} = \int dH \, e^{-N \mathrm{Tr} V(H)}$$

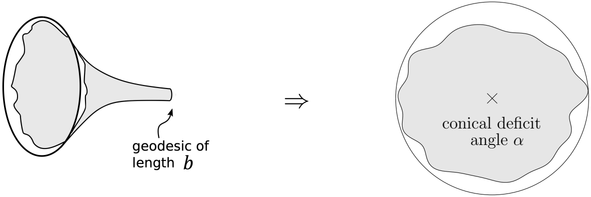
• In the large N limit, this also leads to a genus expansion:

$$Z_n(\ell_1, \ldots, \ell_n) = \sum_{g=0}^{\infty} N^{-2g} Z_{g,n}(\ell_1, \ldots, \ell_n)$$

• One can check that all $Z_{g,n}$ match with the JT gravity results. Therefore, JT gravity defined by summing over all higher genus topologies is dual to the one-hermitian matrix integral!

Trumpet & punctured disk geometries

• The trumpet geometry is related to the punctured disk (i.e. the hyperbolic disk with a conical defect) by a simple analytical continuation by $b \to i\alpha$:



• Also Riemann surfaces with conical singularities naturally arise when we generalize the dilaton potential of JT gravity to some exponential form.

Outline

1. Deformed JT Gravity

- We study the generalized JT gravity in Euclidean signature ($\phi_0 \gg \phi$): [Maxfield & Turiaci '20] [Witten '20]

$$I = \underbrace{-\frac{\phi_0}{2}\left(\int_M dx^2\sqrt{g}R + 2\int_{\partial M}\sqrt{h}K\right)}_{\text{Einstein-Hilbert Action}}$$
$$\underbrace{-\frac{1}{2}\left(\int_M dx^2\sqrt{g}\Big(\phi R + W(\phi)\Big) + 2\int_{\partial M}\sqrt{h}\phi_b(K-1)\right)}_{\text{Modified JT Action}}$$

- We consider the following form of the potential:

$$W(\phi) = 2\phi + 2\epsilon\, e^{-\alpha\phi} + \mathcal{O}(\epsilon^2)$$

- $\mathcal{O}(\epsilon^0)$ corresponds to the original JT gravity.

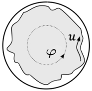
Expansion in ϵ

- Expanding in ϵ for the partition function

$$\exp\left(-I\right) = \exp\left(-I_{(0)}\right)\left(1 + \epsilon\int d^2x_1\,\sqrt{g(x_1)}\,e^{-\alpha\phi(x_1)} + \mathcal{O}(\epsilon^2)\right)$$

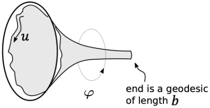
- For $\mathcal{O}(\epsilon^0)$, variation of ϕ leads to $R = -2$, so that the EH Action is reduced to the Euler characteristic χ: [Figs taken from SSS '19]

$$I_{(0)} = -2\pi\phi_0\chi(M) - \phi_b\int_{\partial M}(K-1)$$



- The two simplest on-shell geometries are the disk D and the trumpet $T(b)$:

$$ds_D^2 = d\rho^2 + \sinh^2\rho\,d\varphi^2\,, \qquad (0 \le \varphi < 2\pi)$$
$$ds_T^2 = d\rho^2 + \cosh^2\rho\,d\varphi^2\,, \qquad (0 \le \varphi < b)$$



- The solution of the dilaton field for each geometry is given by

$$\phi_D = \gamma_D\cosh\rho\,, \qquad \phi_T = \gamma_T\sinh\rho$$

Order $\mathcal{O}(\epsilon)$

• Pulling the integral over x_1 outside of the path integral, the corresponding action is

$$I_{(1)} = -\frac{\phi_0}{2}\left(\int_M R + 2\int_{\partial M} K\right) - \left(\frac{1}{2}\int_M \phi(R+2) - \alpha\phi(x_1) + \int_{\partial M}\phi_b K\right)$$

• Now the variation of ϕ leads to $R(x) + 2 = 2\alpha\delta^2(x - x_1)$. Besides the point $x = x_1$, this is still described by (Euclidean) AdS$_2$, but it has a conical singularity at $x = x_1$:

$$ds^2_{D_\alpha} = d\rho^2 + \sinh^2\rho\, d\varphi^2\,, \qquad (0 \leq \varphi < 2\pi - \alpha)$$

and $\phi_{D_\alpha} = \gamma_{D_\alpha}\cosh\rho$, so that

$$I_{(1)} = -(2\pi - \alpha)\phi_0\chi(M) - \phi_b\int_{\partial M}(K-1)$$

Schwarzian action

• We introduce a boundary cutoff at $\rho = \rho_0$ and at the boundary we fix the metric along the boundary and the value of the dilaton field:

$$\varphi\Big|_{\rho_0} = \phi_b\, u\,, \qquad \text{with} \qquad \phi_b = \phi(\rho_0)$$

where u is the time of boundary theory.

• The gravitational dynamics solely comes from the boundary term:

$$I_{\mathsf{bdy}} = -\phi_b \int_{\partial M} d\varphi \sqrt{g}\,(K-1)$$

• If we regard the bulk angular coordinate as a function of the boundary time $\varphi = \varphi(u)$, evaluating the extrinsic curvature in the $\rho_0 \to \infty$ limit, one gets the Schwarzian action [Maldacena, Stanford & Yang '16]

$$I_{\mathsf{Sch}} = -C \int_0^\beta du \left\{ \tan\left(\frac{\varphi(u)}{2}\right), u \right\}$$

2. Charged Particle Picture

- We can also write the effective actions in the charged particle picture:

$$\boxed{I_{\partial M} = -q\Big(\Delta\varphi\,\chi(M) + A_M - L_M\Big)}$$

where we used the Gauss-Bonnet theorem

$$\int_{\partial M} du\sqrt{g}\,K = \Delta\varphi\,\chi(M) + A_M$$

with the area and boundary circumference

$$A_M \equiv \int_M d^2x\sqrt{g}\,, \qquad L_M \equiv \int_{\partial M} d\varphi\sqrt{g}$$

and

$$\beta = C \lim_{\rho_0\to\infty} \frac{L_M}{\phi_b}\,, \qquad q \equiv \phi_b$$

Constant background magnetic field (for D)

• The area term $A_D = 2\pi(\cosh\rho_0 - 1)$ can be expressed in terms the corresponding gauge field $\mathbf{A}_D = (\cosh\rho - 1)d\varphi$ as [Kitaev & Suh '18] [Yang '18]

$$q\,A_D \;=\; q\int_{\partial D}\mathbf{A}_D \;=\; q\int_D \mathbf{B}_D$$

where $\mathbf{B}_D = d\mathbf{A}_D = \sinh\rho\, d\rho\wedge d\varphi$.

• Therefore $\mathbf{B}_D$ has the *constant* $\times$ *volume* form and indeed interpreted as a constant magnetic field.

• It is also useful to express

$$A_D \;=\; -\int_D d^2x\sqrt{g}\,\frac{R}{2} \;=\; -\int_D d\omega$$

where ω is the spin-connection.

Aharonov-Bohm gauge field (for D_α)

- In order to see the topological contribution for D_α, it is useful to rescale $\varphi \to \zeta\,\varphi$, so that

$$d\tilde{s}^2 = d\rho^2 + \zeta^2 \sinh^2 \rho\, d\varphi^2 , \qquad (0 \le \varphi < 2\pi) , \qquad \left(\zeta = \frac{2\pi - \alpha}{2\pi} \right)$$

- In terms of the new coordinates, the effective action reads

$$I_{D_\alpha} \supset -q_\alpha A_{D_\alpha} + q_\alpha L_{D_\alpha} = -q\tilde{A}_{D_\alpha} + q\tilde{L}_{D_\alpha} - q \int \mathbf{A}^{(\alpha)}$$

where the pure gauge Aharonov-Bohm (AB) gauge field is

$$\mathbf{A}^{(\alpha)} = -\frac{\alpha\zeta}{2\pi} \big(\cosh \rho_0 - 1 - \sinh \rho_0 \big) d\varphi$$

3. Quantum Mechanical System

• We need to study the charged particle with a condition: L_M = constant. Inserting a delta function imposing this condition into the path-integral, relativistic line-element is reduced to the non-relativistic one. [Kitaev & Suh '18] [Yang '18]

$$\mathcal{L}_D = \frac{1}{2}\left(\dot{\rho}^2 + \sinh^2\rho\,\dot{\varphi}^2\right) + q\cosh\rho\dot{\varphi}$$
$$\mathcal{H}_D = \frac{1}{2}\left[p_\rho^2 + \frac{1}{\sinh^2\rho}\left(p_\varphi - iq\cosh\rho\right)^2\right]$$

• The propagator is obtained by the Schrödinger equation

$$(\partial_u + \hat{\mathcal{H}})G(x,\varphi;x',\varphi';u) = 0$$

and the partition function is given by the vacuum diagram

$$Z_D = A_D^{-1}\int_D dx^2\sqrt{g(x)}\,G(x,\varphi;x,\varphi;\beta) = \int_0^\infty ds\, e^{-\beta\frac{s^2}{2}}\rho(s)$$

Partition function for D

- It is convenient to define the resolvent by

$$G(x,\varphi;x',\varphi';u) \;=\; \int dE\, e^{-Eu}\, \Pi(x,\varphi;x',\varphi';E)$$

where $b = iq$, $2E = j(1-j)$ and $x = \tanh^2\frac{\rho}{2}$ [Comtet '87] [Kitaev '17]

$$\Pi(x,\varphi;0,\varphi';E) \;=\; \left|a^b_{j,b}\right|^2 e^{-ib(\varphi-\varphi')}(1-x)^j\, {}_2F_1(j-b,\, j+b;\, 1;\, x)$$

- Hence the exact spectral density is given by [Kitaev & Suh '18] [Yang '18]

$$\rho(s) \;=\; \frac{s\sinh(2\pi s)}{\cos(2\pi b)+\cosh(2\pi s)}$$

- The Schwarzian limit is defined by setting $b = iq$ and taking $q\to\infty$ with s fixed. In this limit the spectral density becomes

$$\rho(s) \;\approx\; 2s\sinh(2\pi s)$$

Partition function for $D(\alpha)$

- The effect of the AB field is just to shift the angular momentum [Lisovyy '07]

$$\mathcal{H}_{D(\alpha)} = \frac{1}{2}\left[p_\rho^2 + \frac{1}{\sinh^2\rho}\left(p_\varphi + \xi - iq\cosh\rho\right)^2\right]$$

where

$$\xi = -\frac{iq}{2\pi}\oint_{\partial M}\mathbf{A}^{(\alpha)}$$

- The exact spectral density is given by

$$\rho^{(\xi)}(s,b) \propto \frac{1}{s}\left[\frac{s\sinh 2\pi s + (\frac{1}{2} - b + \xi)\sin 2\pi(b-\xi)}{\cosh 2\pi s + \cos 2\pi(b-\xi)} - \frac{s\sinh 2\pi s + (\frac{1}{2} - b + \xi)\sin 2\pi b}{\cosh 2\pi s + \cos 2\pi b}\right]$$

and the Schwarzian limit is

$$\rho^{(\alpha)}(s) \approx \cosh\left[(2\pi-\alpha)s\right]$$

4. Topological Entropies

- Entanglement entropy can be computed by the Euclidean replica trick

$$S \equiv -\partial_n \left(\frac{Z_n}{Z_1^n} \right) \Big|_{n=1}$$

with the n-th Rényi partition function Z_n.

- For disk D, the Rényi partition function is just $Z_n(\beta) = Z_1(n\beta)$ so that

$$Z_n = \int_0^\infty ds\, s\, e^{-n\beta \frac{s^2}{2}} \rho(s)$$

This leads to the entanglement entropy for D [Lin '18]

$$S_D = \int_0^\infty ds\, s\, p_s \Big[-\log p_s + \log(\mathrm{dim} R) \Big]$$

where

$$p_s = Z_1^{-1} \mathrm{dim} R\, e^{-\beta \frac{s^2}{2}} , \qquad \mathrm{dim} R = \rho(s)$$

Entanglement entropy for $D(\alpha)$

- For punctured disk $D(\alpha)$, the spectral density also depends on n:

$$Z_n^{(\mathrm{AB})} = \int_0^\infty ds\, s\, e^{-n\beta_\alpha \frac{s^2}{2}} \rho_n^{(\alpha)}(s)$$

where

$$\rho_n^{(\alpha)}(s) \propto \frac{n}{2j-1} \int_0^1 dv \Big(v^{j-1-b} I_+(v) - v^{j-1+b} I_-(v) \Big)$$

with $j = is + 1/2$ and

$$I_\pm(v) = \frac{2n - (1-v)(n \pm 1 \pm 2\xi)}{2(1-v)^2} \pm \frac{v^{\pm \frac{\xi}{n}}}{(1 - v^{\mp \frac{1}{n}})(1-v)}$$

Topological Contribution

- This leads to the entanglement entropy (in the Schwarzian limit $q \to \infty$)

$$S^{(\mathrm{AB})} = S_1 + S_2 = \int_0^\infty ds\, s\, p_s \left[-\log p_s + \log(\mathrm{dim}R) \right] + \frac{1+\zeta}{2}$$

where now

$$p_s = Z_1^{-1}\, \mathrm{dim}R\; e^{-\beta_\alpha \frac{s^2}{2}}\,, \qquad \mathrm{dim}R = \rho_1^{(\alpha)}(s)$$

- S_1 contribution in the semi-classical limit

$$S_1 = \log\cosh\left[(2\pi-\alpha)s\right]_{s=\frac{2\pi-\alpha}{\beta}} = \frac{(2\pi-\alpha)^2}{\beta} = (1-\beta\partial_\beta)\log Z_{D_\alpha}(\beta)$$

agrees with the Bekenstein-Hawking entropy. [Lin '18]

- On the other hand, the constant contribution from S_2 does not depend on β and is therefore a global IR contribution which becomes dominant at low temperatures.

5. Conclusions

• We saw JT gravity defined by summing over all higher genus topologies is dual to the one-hermitian matrix integral.

• We formulated JT gravity on a punctured disk as the ordinary quantum mechanics of a charged particle on a hyperbolic disk in the presence of a constant background magnetic field plus a pure gauge Aharonov-Bohm field.

• This picture allowed us to exactly (not only in the Schwarzian limit) calculate the propagators and spectral density of the corresponding gravitational dynamics.

• We also computed the entanglement entropies for the Hartle-Hawking state and found an extra topological contribution that becomes dominant at low temperatures.

Thank you!