Mean string field theory

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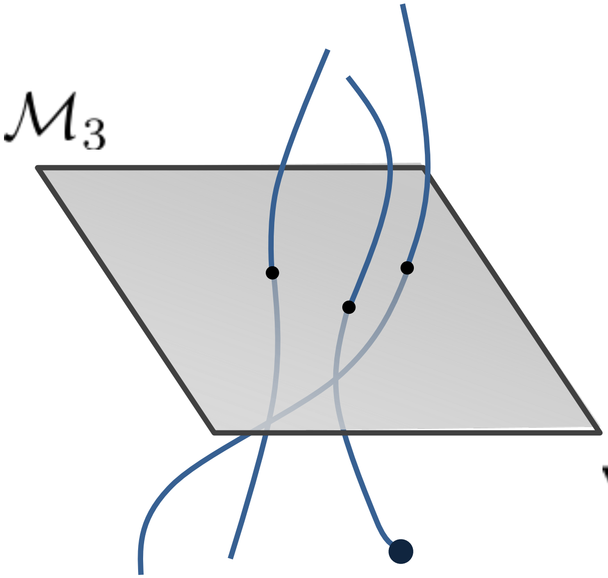
1. Motivation
2. Higher form global symmetries
3. Mean string field theory

Global symmetries

Symmetries are important. Let's review ordinary U(1) 1-index currents:

$$\nabla_\mu j^\mu = 0 \qquad d \star j = 0$$

An ordinary current counts particles; "catch them all" by integrating on a co-dimension 1 subspace (a "time-slice"):



$$Q = \int_{\mathcal{M}_3} \star j$$

Local operators are charged under it:

$$\nabla_\mu j^\mu(x)\mathcal{O}(y) \sim iq\delta(x-y)\mathcal{O}(y)$$

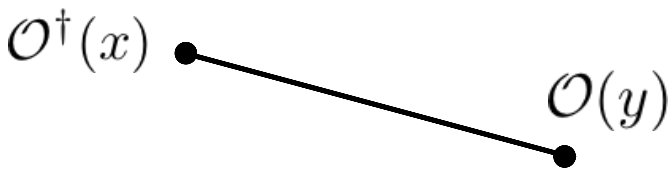
Because local operators are 0-dimensional, we call these 0-form symmetries.

Global symmetries

What are global symmetries good for?

Landau told us: use them to classify phases of matter. Two cases:

1. Unbroken: in this phase, all charged excitations are gapped, and charged correlation functions decay exponentially:



$$\langle \mathcal{O}^\dagger(x)\mathcal{O}(y)\rangle \sim \exp(-m|x-y|)$$

2. Spontaneously broken: something has condensed, and charged correlation functions factorize and saturate at large distances:

$$\langle \mathcal{O}^\dagger(x)\mathcal{O}(y)\rangle \sim \langle \mathcal{O}^\dagger\rangle\langle \mathcal{O}\rangle$$

CFTs describe phase transitions: they sit in between these two cases.

Landau-Ginzburg theory

At the transition, we can often use a universal "Landau-Ginzburg" field theory:

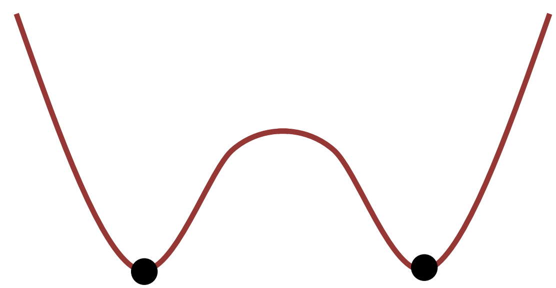
$$S[\phi] = \int d^dx \left(\partial_\mu \phi \partial^\mu \phi^\dagger - m^2|\phi|^2 - \lambda|\phi|^4\right)$$

$\phi(x)$ Map from points to $\mathbb{C}$ $\qquad \phi(x) \to \phi(x)e^{i\alpha}$

$V(\phi) =$

Disordered

$\langle\phi\rangle = 0$



Ordered

$\langle\phi\rangle \neq 0$

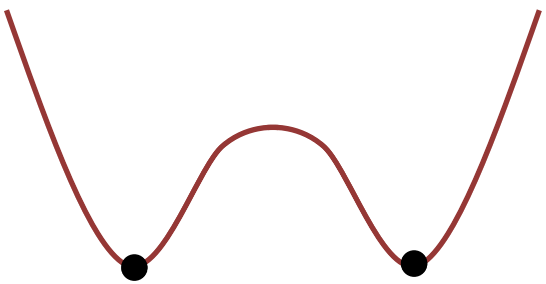
At critical point, $m^2 = 0$: properties of phase transition (divergence of correlation length, thermodynamic singularities, etc.) described by this field theory.

The condensed phase

Also provides low-energy description of the condensed phase:

$$S[\phi] = \int d^d x \left(\partial_\mu \phi \partial^\mu \phi^\dagger - m^2 |\phi|^2 - \lambda |\phi|^4\right)$$

$V(\phi) =$



$$\phi(x) = v e^{i\theta(x)}$$

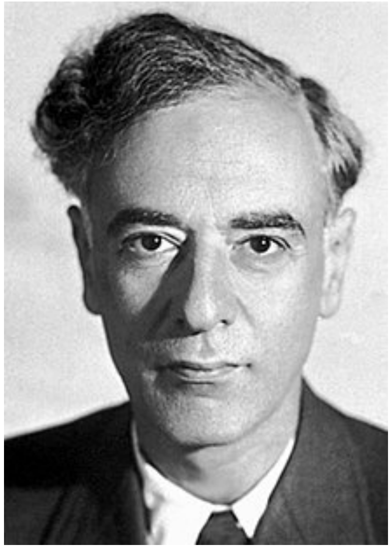
At low energies, reduces to action for gapless Goldstone mode.

$$S[\phi] = \int d^d x \left(v^2 (\partial\theta)^2 + \cdots\right)$$

Low energy physics is completely determined by the pattern of symmetry breaking.

The (old) Landau paradigm

These ideas define the Landau paradigm:



1. Phases of matter are classified by their patterns of broken and unbroken symmetries. (U(1) case described is superfluid).
2. Critical points between phases can be studied by universal theories of the “order parameter”.

This works spectacularly well for many phases of matter, and is the foundation of textbook CMT.

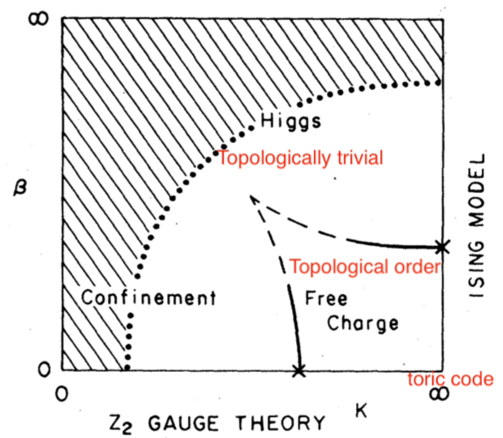
However….

Beyond the (old) Landau paradigm

Much modern work in CMT involves phases and transitions between them that do not fit into this paradigm, e.g.

1. Topological order (e.g. deconfined phases of lattice gauge theory; fractional quantum Hall states).
2. Many others… (deconfined criticality; (non)-Fermi liquids; etc. etc… ask me after!)

Can we do better?



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Higher form global symmetries

Claim (Gaiotto, Kapustin, Seiberg, Willet):

There are other kinds of global symmetries.

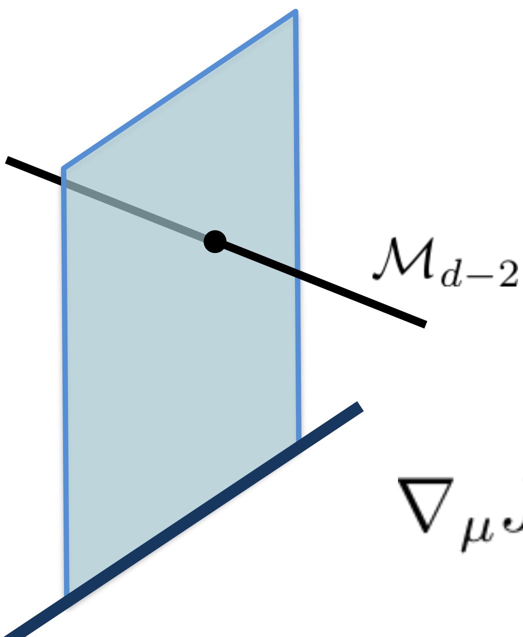
$$J^{\mu} \rightarrow J^{\mu\nu}$$

Higher form global symmetries

In the rest of this talk, we study 2-index currents, i.e.

$$\nabla_\mu J^{\mu\nu} = 0 \qquad d \star J = 0$$

A 2-index current counts strings; as they don't end in space or time, "catch them all" by integrating on a co-dimension-2 subspace:



$$Q = \int_{\mathcal{M}_{d-2}} \star J$$

Line operators are charged under it:

$$\nabla_\mu J^{\mu\nu}(x) W_C \sim iq \int_C dx^\nu \delta(x - X_C) W_C$$

Because lines are 1-dimensional, these are called 1-form symmetries.

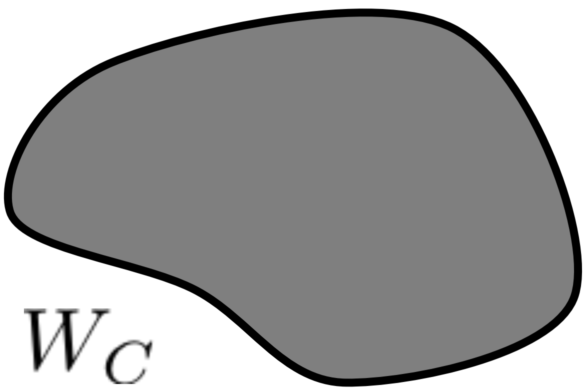
(can also have Z_k discrete symmetry; then strings are conserved modulo k).

Higher form global symmetries

What are higher form global symmetries good for?

Everything that normal symmetries are! (anomalies, hydrodynamics, etc.)
Follow Landau: use them to classify phases of matter. Two cases:

1. Unbroken: in this phase, all charged excitations (strings) are gapped (have tension). Charged line operators have an area law.



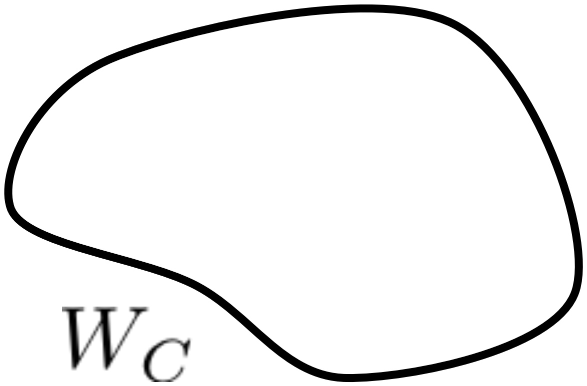
W_C

$$\langle W_C \rangle \sim \exp\left(-T_{p+1} \mathrm{Area}[C]\right)$$

This is the line-like analogue of an exponentially decaying correlation function of local operators.

Higher form global symmetries

2. Spontaneously broken: in this phase, strings have "condensed", i.e. have no tension. More precisely: charged line operators develop perimeter laws at large distances:



$$\langle W_C \rangle \sim \exp(-T_p \text{Perimeter}[C])$$

Depends only locally on the line operator: thus this is the line-like analogue of a factorized correlation function of local operators.

Can prove a higher-form Goldstone theorem (Hofman, NI; Lake): perimeter law implies gapless mode in the spectrum.

Examples

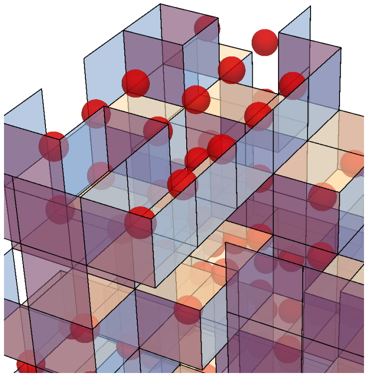
1. Maxwell electrodynamics with matter has a single U(1) 1-form symmetry, associated with conservation of magnetic flux.

$$J^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \qquad \nabla_\mu J^{\mu\nu} = 0$$

Charged line operator is the t'Hooft line (photon is a Goldstone!)

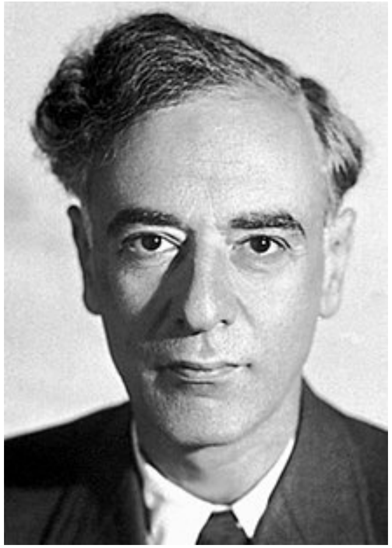
2. Pure SU(N) gauge theory has a Z_N 1-form symmetry ("center symmetry"). Charged line operator is the fundamental Wilson line.

2. 3d Ising model has a Z_2 1-form symmetry associated with the integrity of domain walls. (Makes the most sense in the gauge formulation of the model). Charged line operator is the 3d Ising defect.



The new Landau paradigm

It is tempting to define a new Landau paradigm:



1. Phases of matter are classified by their patterns of broken and unbroken symmetries, which might be higher form.
2. Critical points between phases can be studied by universal theories of the "order parameter".

#1 works for many examples of topological order, gauge theories, etc. Topological phases are those with spontaneously broken higher form symmetries.

But what about #2? Condensation of strings?

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Towards mean string field theory

To describe such a transition, we would need a framework that is the analogue of the Landau-Ginzburg theory below.

$$S[\phi] = \int d^d x \left(\partial_\mu \phi \partial^\mu \phi^\dagger - m^2|\phi|^2 - \lambda|\phi|^4\right)$$

It should allow us to describe the condensation of strings. Notoriously hard problem. Can higher form symmetries help?

Normal symmetries	Higher-form symmetries
Mean field theory	?

Mean field theory ingredients

We will construct everything by analogy with ordinary symmetry:

$$S[\phi] = \int d^d x \left(\partial_\mu \phi \partial^\mu \phi^\dagger - m^2|\phi|^2 - \lambda|\phi|^4\right)$$

Basic degree of freedom is a map from points to ℂ : $\phi(x)$

Transforms linearly under symmetry: $\phi(x) \to \phi(x)e^{i\alpha} \;\; d\alpha = 0$

Can couple this global U(1) symmetry to an external gauge field *A;* this is often useful:

$$D_\mu \phi = \partial_\mu \phi - iA_\mu \phi$$

$$A \to A + d\alpha$$

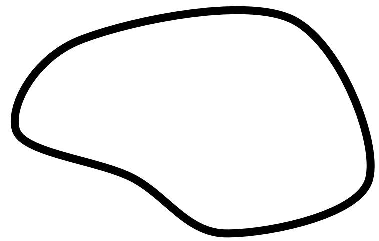
$$\phi \to \phi(x)e^{i\alpha(x)}$$

What are the analogues of these ideas for a 1-form symmetry?

String field

A 1-form symmetry acts on line operators; our basic degree of freedom is a functional: map from the space of closed curves to $\mathbb{C}$

$$\psi[C]$$



$$\longrightarrow \mathbb{C}$$

(Imagine this creates a string along *C*)

1-form symmetry acts linearly: $\psi[C] \to \psi[C] e^{i\int_C \Gamma}$ $d\Gamma = 0$

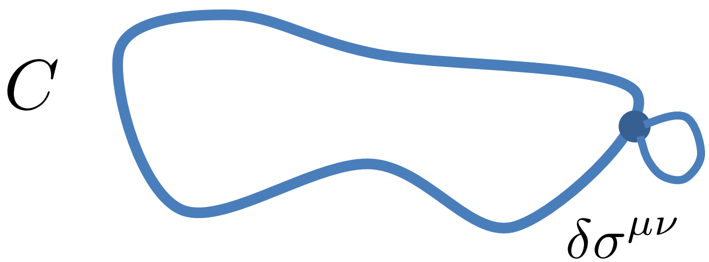
Γ is a closed 1-form

We want to write down an invariant action for the string field.
Want to do so in a way that lets us couple an external 2-form gauge field source:

$$b \to b + d\Gamma \quad \psi[C] \to \psi[C] e^{i\int_C \Gamma}$$ (**Γ** arbitrary).

Area derivative

To differentiate string field, use the area derivative, constructed by Migdal, Polyakov. Beautiful geometric construction that compares field between two nearby curves.



$$\psi[C \cup \delta C] = \psi[C] + \frac{\delta \psi[C]}{\delta \sigma^{\mu\nu}(\lambda)} \delta\sigma^{\mu\nu}(\lambda)$$

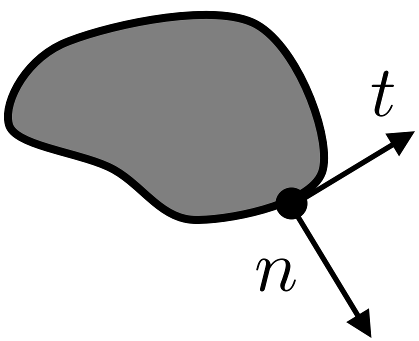
Definition can be used (in principle) to compute derivative of any functional. Examples:

Coupling of curve to gauge field:

$$\frac{\delta}{\delta \sigma^{\mu\nu}(\lambda)} \int_C a_\mu dx^\mu = f_{\mu\nu}(x(\lambda))$$

Minimal area functional:

$$\frac{\delta}{\delta \sigma^{\mu\nu}(\lambda)} A[C] = t_{[\mu} n_{\nu]}(\lambda)$$



Area derivative

We can now construct an area derivative that lets us couple an external gauge field source *b* to our string field:

$$\frac{D\psi[C]}{\delta\sigma^{\mu\nu}(s)} = \left(\frac{\delta}{\delta\sigma^{\mu\nu}(s)} - ib_{\mu\nu}(x(s))\right)\psi[C]$$

Transforms nicely under $b \to b + d\Gamma \qquad \psi[C] \to \psi[C]e^{i\int_C \Gamma}$

(Area derivative is exactly the right thing to make this higher-form gauge-covariant-derivative possible!) Now that we know its possible, set b to 0 for simplicity (can always restore it if we want).

Finally: replace integral over points with an integral over curves.

$$\int d^d x \longrightarrow \int [dC]$$

Mean string field theory action

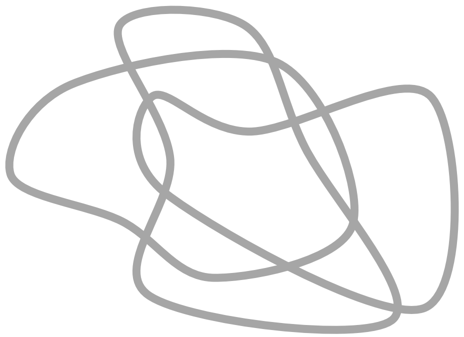
Present the action of mean string field theory:

$$S = \int [dC] \left(\frac{1}{L[C]} \oint ds \frac{\delta \psi^\dagger[C]}{\delta \sigma^{\mu\nu}(s)} \frac{\delta \psi[C]}{\delta \sigma_{\mu\nu}(s)} + M^2 \psi^\dagger[C]\psi[C] + \lambda |\psi[C]|^4 \right)$$

Idea: like normal Landau-Ginzburg, write down most general* dynamics for string field that respects the symmetries…

$$\int [dC] = \int [dX] e^{-mL[C]}$$

Basic degree of freedom Ψ[C] is a functional; even evaluating the action requires a functional integral over the space of closed curves.



Difficult but well-posed problem: what does this theory describe?

Easier than "real" string field theory; not UV complete (similar actions: Rey 1989). *This is not the most general.

Phases of mean string field theory

$$S = \int [dC] \left(\frac{1}{L[C]} \oint ds \frac{\delta \psi^\dagger [C]}{\delta \sigma^{\mu\nu}(s)} \frac{\delta \psi[C]}{\delta \sigma_{\mu\nu}(s)} + M^2 \psi^\dagger[C] \psi[C] + \lambda |\psi[C]|^4 \right)$$

Unbroken phase: $V(\psi) =$

Classical equations of motion are complicated equations in functional space. Hard to solve linear equations (both conceptual and calculational issues).

$$\oint ds \frac{\delta}{\delta \sigma^{\mu\nu}(s)} \left(\frac{1}{L[C]} \frac{\delta \psi[C]}{\delta \sigma^{\mu\nu}(s)} \right) = M^2 \psi[C]$$

Try for limit when curve C is large…

Phases of mean string field theory

$$S = \int [dC] \left(\frac{1}{L[C]} \oint ds \frac{\delta \psi^\dagger[C]}{\delta \sigma^{\mu\nu}(s)} \frac{\delta \psi[C]}{\delta \sigma_{\mu\nu}(s)} + M^2 \psi^\dagger[C]\psi[C] + \lambda |\psi[C]|^4 \right)$$

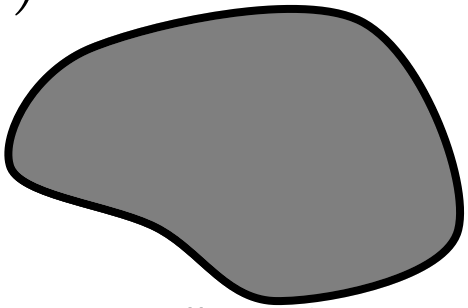
Try WKB-inspired ansatz where answer depends only on minimal area A[C] that fills in curve:

$$\psi[C] = \exp(-S(A[C]))$$

Because of simplicity of area derivative acting on minimal area, obtain:

$$S'(A)^2 = M^2 + \mathcal{O}(A^{-\frac{1}{2}})$$

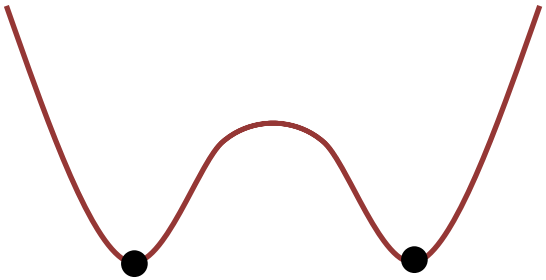
$$\psi[C] = \exp(-MA[C])$$



Expected behavior in the unbroken phase! "Confinement" without gauge bosons (expected for unbroken center symmetry. Relation to loop formulation of QCD by Migdal, Makeenko?)

Broken phase of mean string field theory

$$S = \int [dC] \left(\frac{1}{L[C]} \oint ds \frac{\delta \psi^\dagger[C]}{\delta \sigma^{\mu\nu}(s)} \frac{\delta \psi[C]}{\delta \sigma_{\mu\nu}(s)} + M^2 \psi^\dagger[C]\psi[C] + \lambda |\psi[C]|^4 \right)$$

Broken phase: $V(\psi) =$ 

Action minimized at: $\psi[C] = v$

Expect gapless Goldstone mode, parametrized by space-time dependent 1-form symmetry transformation:

$$\psi[C; a] = v e^{i \int_C a}$$

Effective action?

Broken phase of mean string field theory

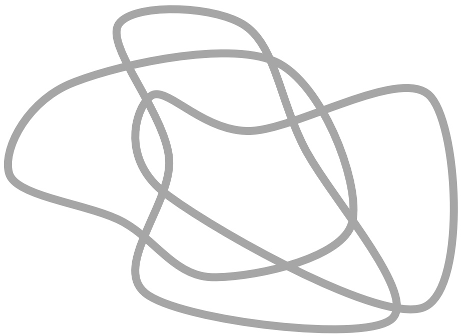
$$S = \int [dC] \left(\frac{1}{L[C]} \oint ds \frac{\delta \psi^\dagger[C]}{\delta \sigma^{\mu\nu}(s)} \frac{\delta \psi[C]}{\delta \sigma_{\mu\nu}(s)} + \cdots \right)$$

Plug in: $\psi[C;a] = ve^{i\int_C a}$

$$S = v^2 \int [dC] \left(\frac{1}{L[C]} \oint ds\, f_{\mu\nu}(x(s)) f^{\mu\nu}(x(s)) + \cdots \right)$$

Do the integral over curves using worldline QFT technology (cf Strassler, etc.). Find:

$$S = \# v^2 \int d^d x \, (f_{\mu\nu}(x) f^{\mu\nu}(x))$$



Gapless Goldstone mode! (Origin of massless photon).

Other gapless modes?

Can imagine doing more general modulations of the phase, e.g.

$$\psi[C;a,t] = v\exp\left(i\int ds(\dot{x}^\mu a_\mu(x) + t(x) + \cdots\right)$$

Same procedure results in an action for t(x), gapped at UV cutoff scale.

$$S = \# v^2 \int d^d x\,(f^2 + (\partial t)^2 + \mu^2 t^2 + \cdots)$$

Happens because there is no symmetry protecting it.
See no evidence for any other gapless modes (no gravity).

This is not "real" string field theory.

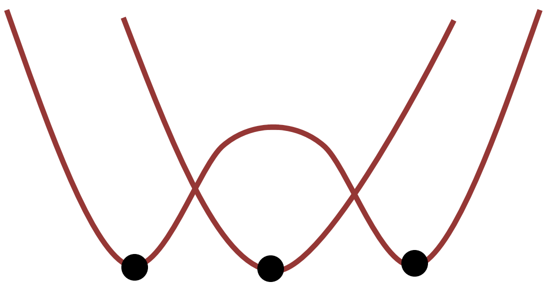
(In discrete symmetry case, reproduce expected TQFT in broken phase).

Transition?

$$S = \int [dC] \left(\frac{1}{L[C]} \oint ds \frac{\delta \psi^\dagger[C]}{\delta \sigma^{\mu\nu}(s)} \frac{\delta \psi[C]}{\delta \sigma_{\mu\nu}(s)} + M^2 \psi^\dagger[C]\psi[C] + \lambda |\psi[C]|^4 \right)$$

There is a "mean-field" transition at $M_c = 0$.

$$\psi[C] \sim \exp\left(-\sqrt{M^2 - M_c^2} A\right)$$



Mean-field scaling of string tension at critical point! Compare with data?

3d Ising numerics ('80s): same exponent is **1.26 ≠ 0.5.**

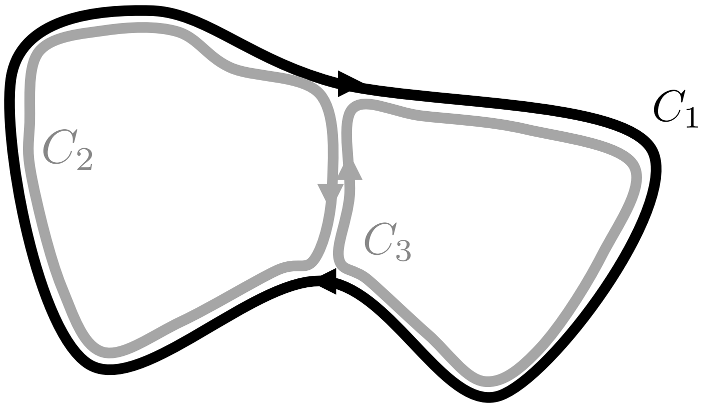
Upper critical dimension of MSFT is (at least) **8** (intriguing connection to old arguments of Parisi…). We do not expect it to be quantitatively accurate in 3d anyway.

Issues with universality

The action discussed is not the most general:

1. Intrinsically stringy (topology-changing) terms exist:

$$S = g\int [dC_{1,2,3}]\delta[C_1 - (C_2 + C_3)]\psi^\dagger[C_1]\psi[C_2]\psi[C_3]$$



Likely important, but hard to treat! (Generated along RG? Confusion about meaning of locality in loop space) May affect upper critical dimension.

2. Couplings could depend on proper length of curve, e.g.

$$M^2(L[C]) = M_0^2 + M_1^2\frac{1}{L[C]\Lambda} + \cdots$$

Such issues are likely important near phase transition (especially if you want to compare with lattice…). Need to build a framework for RG in loop space…

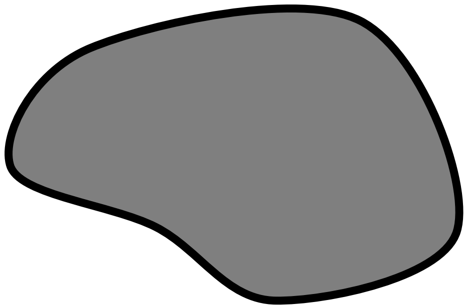
A little philosophy

Grand existing tradition since '70s to reformulate Yang-Mills theory in terms of gauge-invariant loop equations (Migdal, Makeenko, Polyakov…)

But what is Yang-Mills theory, really? Gauge symmetry is not a symmetry but a redundancy.

Candidate answer: pure Yang-Mills theory is a theory that has a Z_N 1-form global symmetry. It is gratifying that the simplest way to build dynamics around that organizing principle gives us an area law in the unbroken phase.

$$\int d^4x F^a_{\mu\nu} F^{\mu\nu a}$$

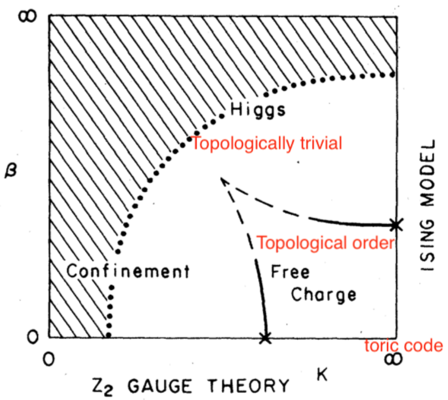


Future directions

Solve linear equations of motion, construct off-shell string propagator, build perturbation theory.

$$\langle \psi^\dagger[C_1]\psi[C_2]\rangle$$

Connect to data? Order of transition of our theory is still a bit unclear. In literature most seem 1st order above d = 3. New motivation to search for continuous transitions. Do numerics on lattice.

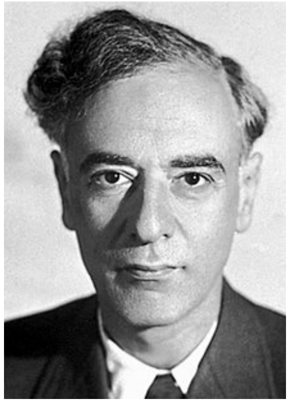


Upper critical dimension is D – can we do (D – ε) expansion? Describe new critical points without Lagrangian description?

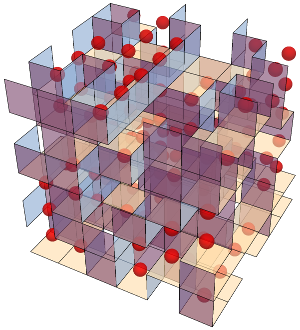
Quiet discomfort: far too many degrees of freedom to describe a local theory (but they are gapped…). Issues of universality.

Longer term directions

Many questions: if we use higher form symmetries (and anomalies), what is the new Landau paradigm? Does it include all phases of matter?

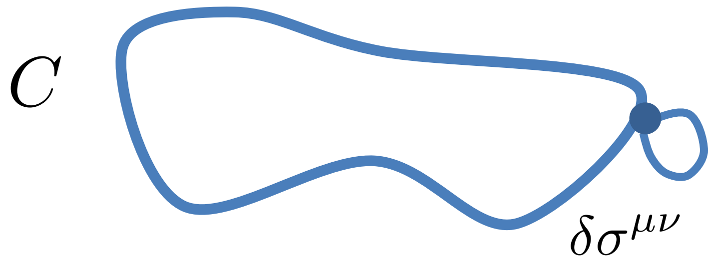


Can string theory – in this putative "mean string field theory" form -- help us understand critical phenomena and statistical physics? Or vice-versa?



Summary

1. Higher form symmetries are a new kind of global symmetry that may let us build a new Landau paradigm.
2. Possible to build a Landau-Ginzburg "mean string field theory" that non-perturbatively describes the condensation of strings.
3. Line operators has an area law in the unbroken phase, and Goldstone modes in the broken phase.
4. Theory is not yet well-understood; much to be explored.



The End.