Stabilization of the extended horizons

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$\Longrightarrow$ Landau-Lifshitz, vol5, *Thermodynamic inequalities*:

$$\frac{c_v}{s} \equiv \frac{T}{s} \left(\frac{\partial s}{\partial T}\right)_v > 0$$

i.e., a condition of the **equilibrium thermal stability**

$\Longrightarrow$ According to gauge/string theory correspondence, black holes/black branes in equilibrium are dual to thermal equilibrium states of the corresponding boundary holographic theory

$\Longrightarrow$ Black holes/branes are **stable** if their QNMs are stable:

$$\mathfrak{w}(\mathfrak{q}) = \frac{w(\mathfrak{q})}{2\pi T}\,, \qquad \mathfrak{q} = \frac{k}{2\pi T}\,, \qquad \text{Im}\,[\mathfrak{w}] > 0$$

$\Longrightarrow$ In early 2000' Gubser and Mitra in a series of paper formulated a **correlated stability conjecture**:

A black brane thermodynamic instability correlates with its dynamical instability

- translational invariance of the horizon
- Schwarzschild black holes are stable
- dynamical instability means the presence of the unstable QNMs

$\Longrightarrow$ In one direction the conjecture is trivial (AB, hep-th/0507275):

$$\textit{thermodynamic instability} \Longrightarrow \textit{dynamical instability}$$

Indeed:

$$\frac{s}{c_v} = c_s^2 \qquad \Longrightarrow \qquad c_v < 0 \Leftrightarrow c_s^2 < 0 \qquad \Longrightarrow \qquad c_s \text{ is purely imaginary}$$

and the sound mode, *i.e.*, the scalar channel black brane QNM,

$$\mathfrak{w}(\mathfrak{q}) = \pm c_s\ \mathfrak{q} - -2\pi i\ \frac{\eta}{s}\left(\frac{2}{3} + \frac{\zeta}{2\eta}\right)\ \mathfrak{q}^2 + \mathcal{O}(\mathfrak{q}^3)$$

$\Longrightarrow$

$$\mathrm{Im}\,[\mathfrak{w}] \qquad \propto \qquad \pm\mathrm{Im}\,[c_s] < 0$$

for some sign and for small enough $\mathfrak{q}$.

$\Longrightarrow$ Have been checked explicitly in a number of examples in holography (also later in this talk)

$\Longrightarrow$ In the other direction the conjecture is simply wrong (a recent very explicit case AB, arXiv:2011.11509)

$$\textit{thermodynamic stability} \not\Longleftarrow\!\!\!\not\Longrightarrow \textit{dynamical stability}$$

$\Longrightarrow$ Example: Holographic conformal order

- It is easy to construct holographic <u>models</u> in asymptotically AdS_{d+2} with, say $\mathbb{Z}_2$ global symmetry, such that

$$\frac{\mathcal{F}}{T^{d+1}} = - \underbrace{\text{const}^2}_{\propto\ c} \times \begin{cases} 1, & \langle\mathcal{O}\rangle = 0 \Longrightarrow \mathbb{Z}_2 \text{ is unbroken;} \\ \kappa, & \langle\mathcal{O}\rangle \neq 0 \Longrightarrow \mathbb{Z}_2 \text{ is broken} \end{cases}$$

  where $0 < \kappa < 1$ is a constant.

- the thermodynamics and the hydrodynamics of both phases is identical, *i.e.*, that of the CFT_{d+1}

- There is a (non-hydrodynamic) branch of the scalar sector QNMs that renders the symmetry broken phase *perturbatively unstable*, *i.e.*, there is QNM with

$$\text{Im}\left[\mathfrak{w}(\mathfrak{q}=0)\right] > 0$$

$\Longrightarrow$ In this talk I will try to 'fix' the instabilities of the extended horizons by compactifying the space of holographic CFT_{d+1}

$$\mathbb{R}^d \longrightarrow L^2 \; S^d \,, \qquad K \equiv \frac{1}{L^2}$$

Motivation:

- fix instabilities in the hydrodynamic sector, models with $c_s^2 < 0$; example: *Klebanov-Strassler black branes/holes*
- fix instabilities in the non-hydrodynamic sector; example: *holographic conformal order on S^d*

Outline

- General formalism for the QNMs
- an application: $\mathcal{N} = 2^*$ model

$\Longrightarrow$ Most of you are familiar with the formalism of Kovtun-Starinets for computing the QNMs of black branes:

- form gauge invariant fluctuations (wrt residual diffeomorphisms preserving the black-brane metric ansatz)
  - tensor sector/scalar channel (helicity $h = 2$) [shear viscosity and universality]
  - vector sector/shear channel (helicity $h = 1$) [diffusion]
  - scalar sector/sound channel (helicity $h = 0$) [bulk viscosity]
- derive EOM, impose the incoming-wave bc, (typically numerically) solve them
- interpret results as in various talks on HoloTube

$\Longrightarrow$ Advantages of Kovtun-Starinets:

- intuitive and straightforward
- hydro regime $\mathfrak{w} \to 0$ and $\mathfrak{q} \to 0$ can often be treated analytically

$\Longrightarrow$ Disadvantages of Kovtun-Starinets:

- not suitable for S^d, as we need
- difficult to prove general stability theorems (in alternative method a theorem can be proven that there can never be instabilities in $h = 2$ and $h = 1$ (no bulk gauge fields) sectors)
- the difficulty (above) is due to the fact that in Kovtun-Starinets QNM eqs both $\mathfrak{w}$ and $\mathfrak{q}$ enter in a highly nonlinear fashion

$\Longrightarrow$ The world before Kovtun-Starinets, **Kodama-Ishibashi**, hep-th/0305147:

$$A\ master\ equation\ for\ gravitational\ perturbations\ of\ maximally$$
$$symmetric\ black\ holes\ in\ higher\ dimensions$$

$\Longrightarrow$ A very relevant generalization to Einstein-Maxwell-single scalar system in space-time with $\Lambda \neq 0$ and $K \neq 0$ by **Jansen-Rostworowski-Rutkowski** (JRR):

$$Master\ equations\ and\ stability\ of\ Einstein - Maxwell - scalar$$
$$black\ holes$$

$\Longrightarrow$ Advantages of JRR:

- what we need!
- as I review in a sec, master equations are nonlinear in $\mathfrak{q}$, but quadratic $\mathfrak{w}$ — this was leveraged to prove the theorem that no instabilities in $h = 2$ and $h = 1$ sectors of the QNMs

$\Longrightarrow$ Disadvantages of JRR/shortcomings:

- need to generalize to systems with many scalars
- (I believe) impossible to do take analytic hydrodynamic limit — the $h = 2$ sector equations are singular in the limit $\mathfrak{q} = 0$.
- all the right words, clear presentation, and extremely many typos, pretty much forced me to rederive everything

$\Longrightarrow$ KI-JRR formalism (as in appendix of my forthcoming paper):

- An effective action:

$$S_5 = \int_{\mathcal{M}_5} d^{3+2}\xi \, \sqrt{-g} \left[R - \sum_{j=1}^{p} \eta_j \left(\partial \phi_j \right)^2 - V \left(\{ \phi_j \} \right) \right]$$

arbitrary number of scalars $j = 1 \cdots p$, constants η_j, arbitrary potential V

- black hole/black brane background

$$ds_5^2 = -c_1^2 \, dt^2 + c_2^2 \, dX_{3,K}^2 + c_3^2 \, dr^2$$

where $c_i = c_i(r)$, $\phi_j = \phi_j(r)$ and

$$dX_{3,K}^2 = \begin{cases} d\boldsymbol{x}^2 \equiv dx_1^2 + dx_2^2 + dx_3^2 \,, & K = 0 \,, \qquad \text{planar} \\ d\Omega_{(3)}^2 \,, & K > 0 \,, \qquad \text{spherical} \\ dH_{(3)}^2 \,, & K < 0 \,, \qquad \text{hyperbolic} \end{cases}$$

explicitly:

$$dX_{3,K}^2 = \frac{dx^2}{(1 - Kx^2)} + (1 - Kx^2) \left[\frac{dy^2}{(1 - Ky^2)} + (1 - Ky^2) \, dz^2 \right]$$

$\Longrightarrow$ We organize all the gauge invariant fluctuations into three sets of master scalars of different helicity h:

- the helicity $h = 2$ set, $\{\Phi_2^{(2)}\}$;
- the helicity $h = 1$ set, $\{\Phi_2^{(1)}\}$;
- the helicity $h = 0$ set, $\{\Phi_2^{(0)}, \Phi_{(0,j)}^{(0)}\}$, $j = 1 \cdots p$.

$\Longrightarrow$ Any master scalar $\Phi_s^{(h)}$ ($s = 2$ or $s = (0, j)$ and $h = \{0, 1, 2\}$) is assumed to have the following dependence:

$$\Phi_s^{(h)}(\xi) = F_s^{(h)}(t, r)\ S(X_{3,K})$$

where

$$\underbrace{\Delta_K}_{Laplacian\ on\ X_{3,K}} \qquad S + k^2\ S = 0$$

$$\text{planar horizon :} \qquad S = e^{i\boldsymbol{k}\boldsymbol{x}}\,, \qquad k = |\boldsymbol{k}| \text{ is any}$$

$$S^3 \text{ horizon :} \qquad k^2 = K\ell(\ell+2)\,, \qquad \ell = 0, 1, 2, \cdots$$

- Each of the master scalars satisfies a coupled master equation

$$\Box\, \Phi_s^{(h)} - W_{s,s'}^{(h)}(r,k)\; \Phi_{s'}^{(h)} = 0$$

  where $\Box$ is the wave operator on the full $D=5$ metric

- Note:
  - $\mathfrak{w}$ dependence comes only from the $\Box$, thus master equations are only quadratic in $\mathfrak{w}$
  - potentials are symmetric (even with multiple scalars!)

$$W_{s,s'}^{(h)} = W_{s',s}^{(h)}$$

  - explicit expressions in my paper, but for now:

$$\mathfrak{h}_{xr} = \frac{c_2^2 c_3^2 c_1}{D} \sum_{j=1}^{p} \left\{ \sqrt{\frac{\eta_j}{2}}\phi_j'\; F_{(0,j)}^{(0)} \right\} + \left(-\frac{3\sqrt{3} c_1 c_2 c_2' c_3^2 K}{k\tilde{k}\;\, D} + \frac{c_2 c_3^2\; k}{2\sqrt{3} c_2' D\;\, \tilde{k}} \left(c_3^2 c_1 k^2 + 3(c_2')^2 c_1 + 3 c_1' c_2' c_2 \right) \right) F_2^{(0)} + \frac{\sqrt{3} c_2^2}{2\; k\tilde{k}}\; \partial_r F_2^{(0)}$$

- where

$$\tilde{k}^2 \equiv K\Big(\ell(\ell+2)-3\Big)$$

$\Longrightarrow$

$$\boxed{k\tilde{k}} = 0 \qquad \Longleftrightarrow \qquad \ell = 0 \text{ or } \ell = 1$$

The reason for the $\ell = \{0,1\}$ singularity is because in this cases the metric fluctuations are pure gauge; must (and can) be treated separately

- Can prove (with math rigor), even with multiple scalars, that **all** QNMs in $h=2$ and $h=1$ sectors have

$$\mathrm{Im}\,[\mathfrak{w}] < 0$$

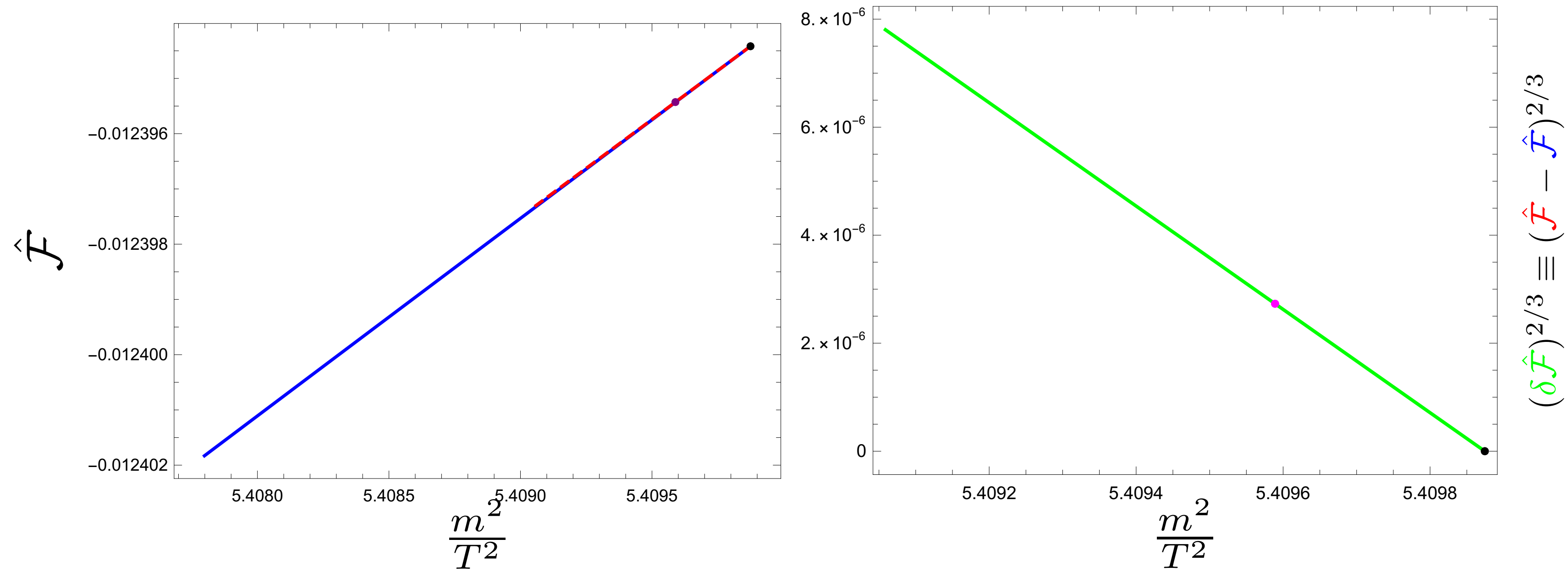
i.e., , are stable

$\Longrightarrow$ Onto application:

$$S_5 = \int_{\mathcal{M}_5} d^{3+2}\xi \, \sqrt{-g} \left[R - \sum_{j=1}^{p} \eta_j \left(\partial\phi_j\right)^2 - V\left(\{\phi_j\}\right) \right]$$

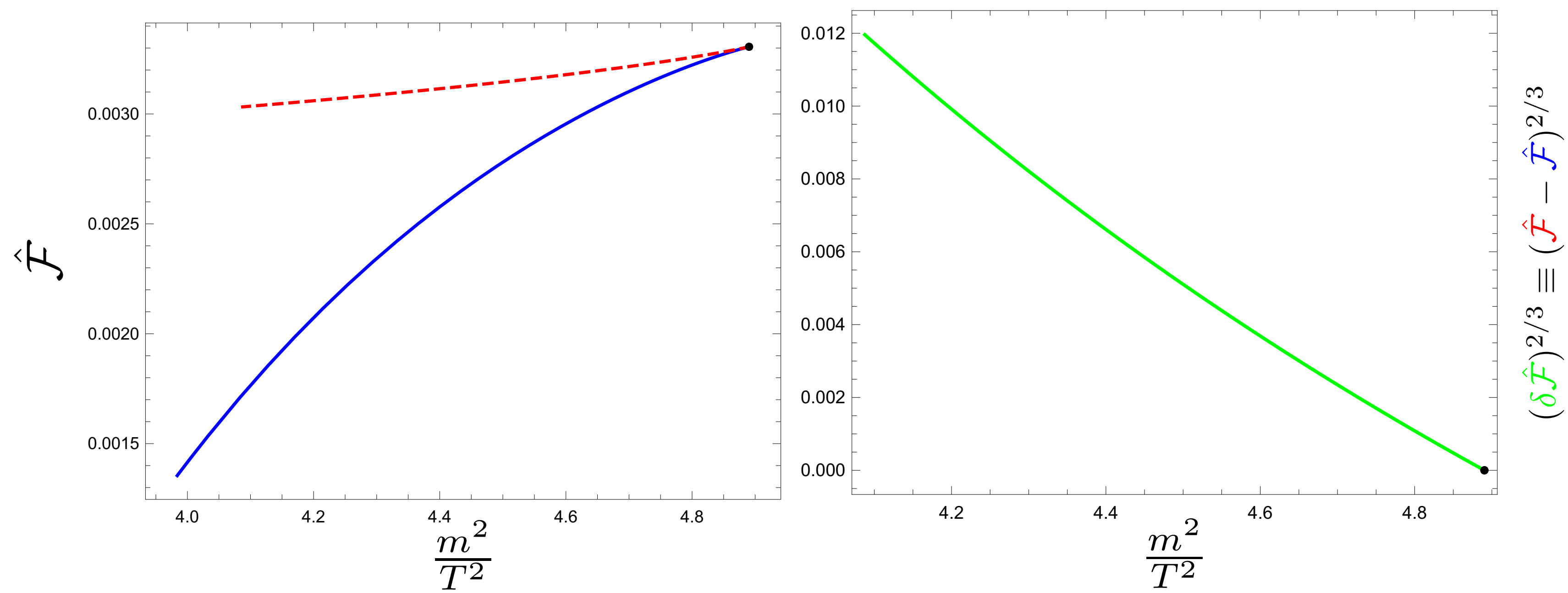
$\Longrightarrow$ A holographic dual to $\mathcal{N} = 4$ SYM with a bosonic mass term, *aka* $\mathcal{N} = 2^*$ model:

- $$p = 1$$
- $$\phi_1 \equiv \alpha \,, \qquad \eta_1 = 12$$
- $$V = -e^{-4\alpha} - 2\, e^{2\alpha}$$
- $$\alpha = \frac{8m^2}{3}\, \rho^2 \ln\rho + \cdots \,, \qquad c_2 = \frac{1}{2\rho} + \cdots$$



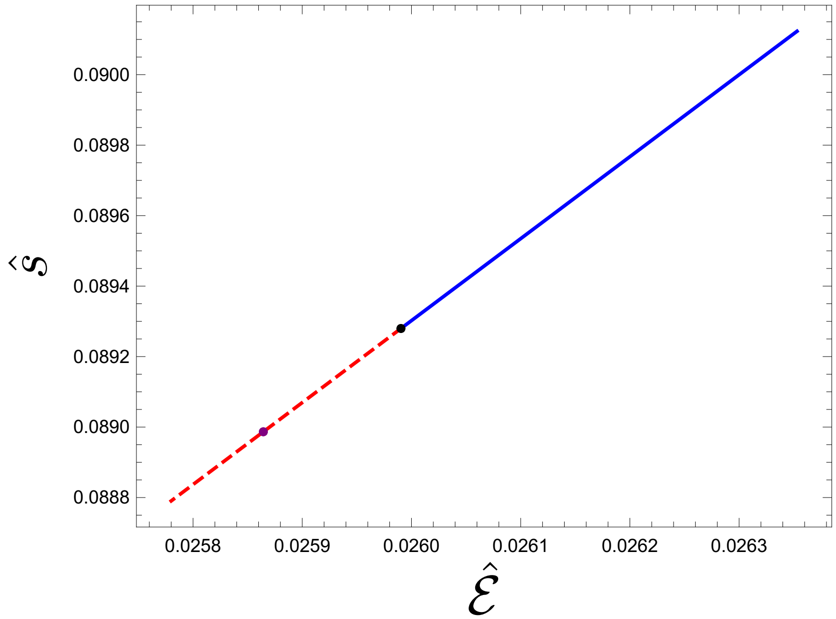
- Macrocanonical ensemble,

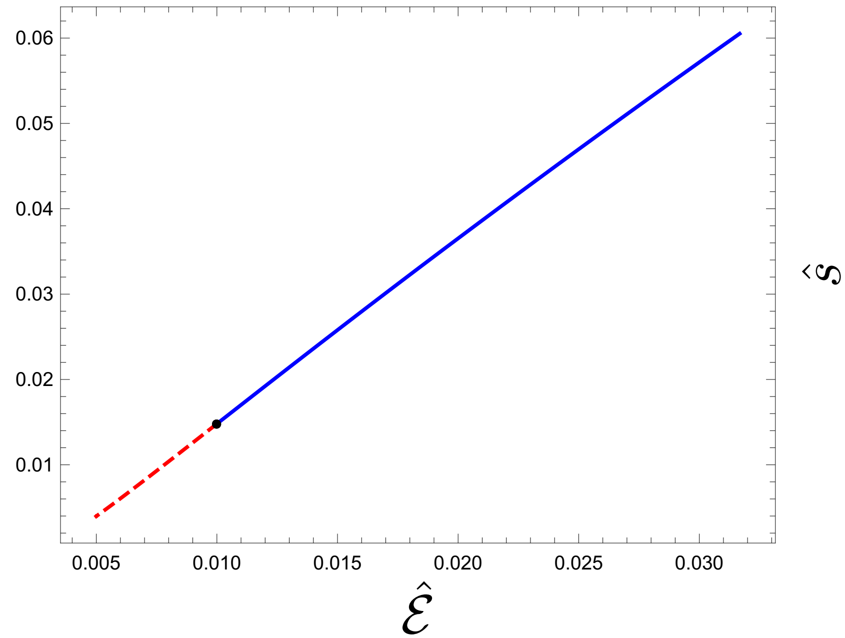
$$\frac{K}{m^2} = 0$$



- Macrocanonical ensemble,

$$\frac{K}{m^2} = 1$$



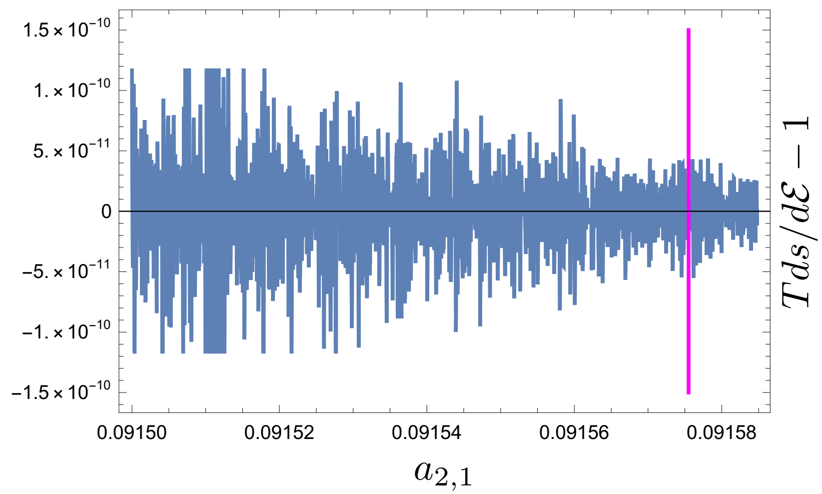


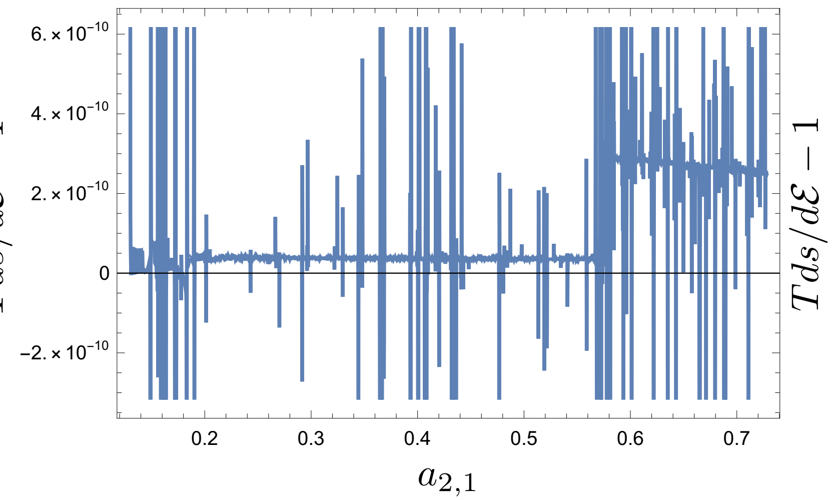
- Microcanonical ensemble,

$$\text{Left :} \qquad \frac{K}{m^2} = 0$$

$$\text{Right :} \qquad \frac{K}{m^2} = 1$$

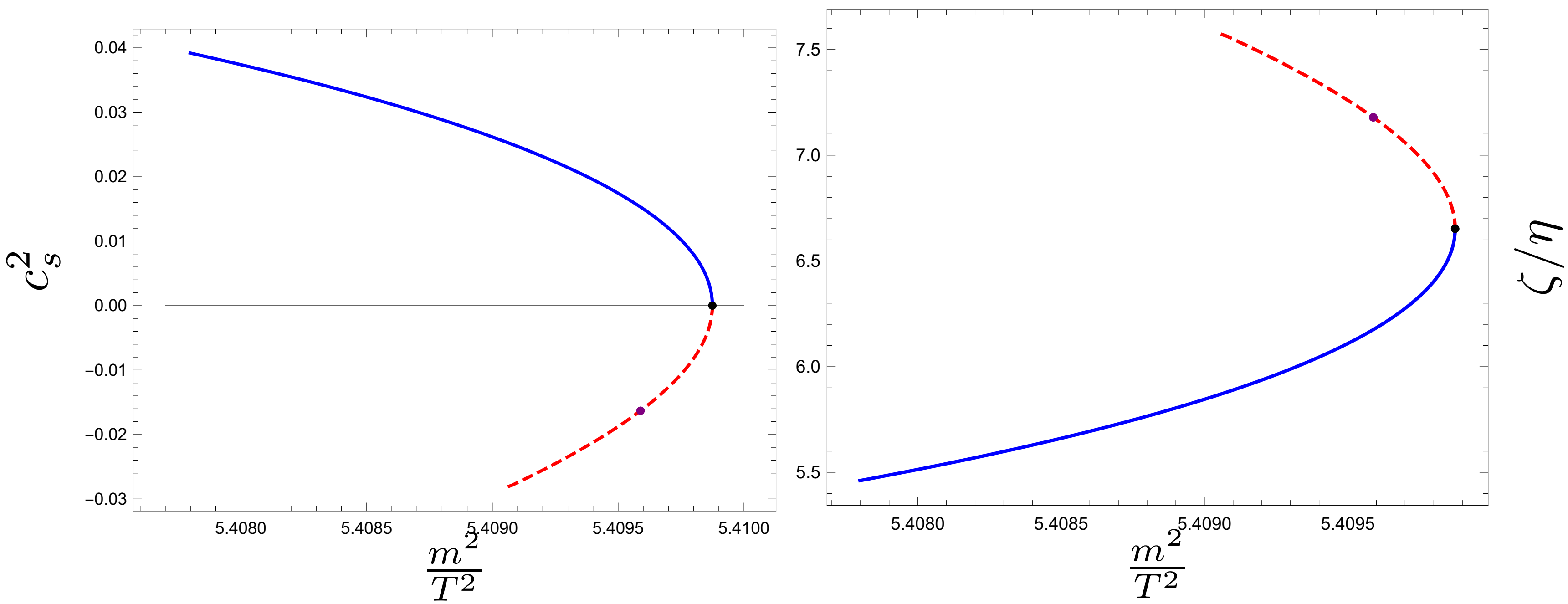
$\Longrightarrow$ Can you trust my numerics?





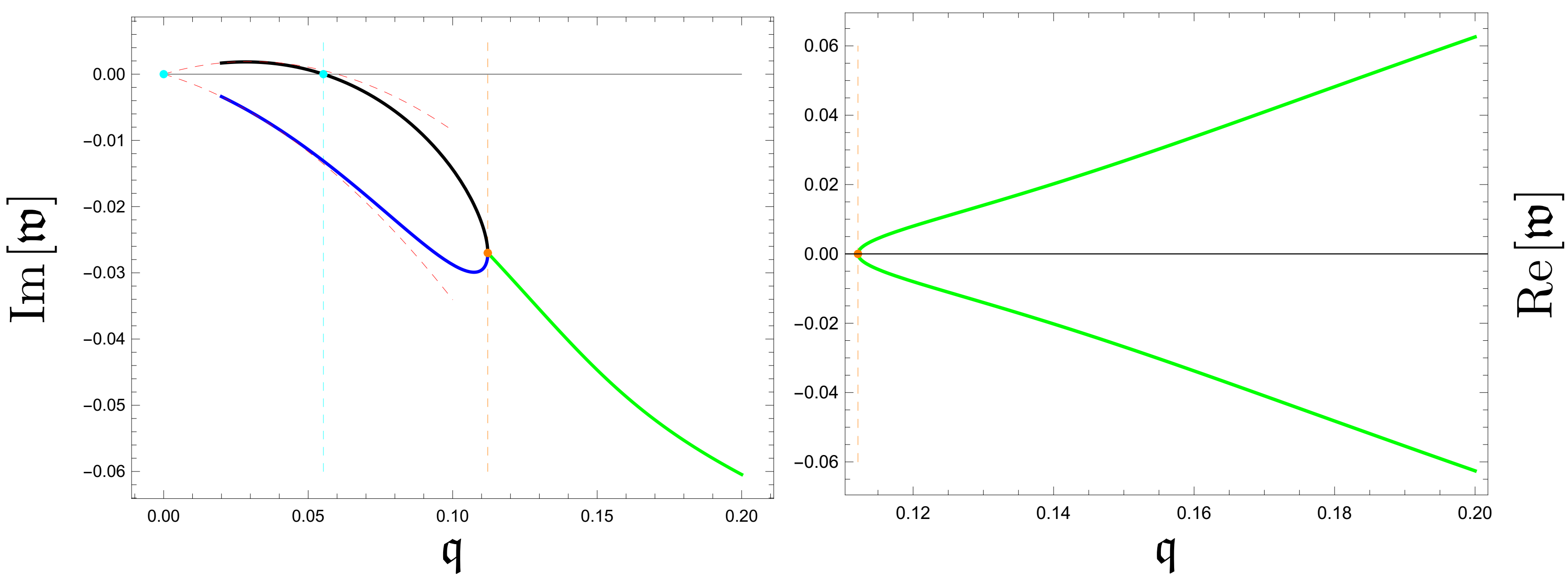
$$\frac{K}{m^2} = 0$$

$$\frac{K}{m^2} = 1$$



$$\mathfrak{w}(\mathfrak{q}) = \pm c_s\ \mathfrak{q} - -2\pi i\ \frac{\eta}{s}\left(\frac{2}{3} + \frac{\zeta}{2\eta}\right)\ \mathfrak{q}^2 + \mathcal{O}(\mathfrak{q}^3)$$

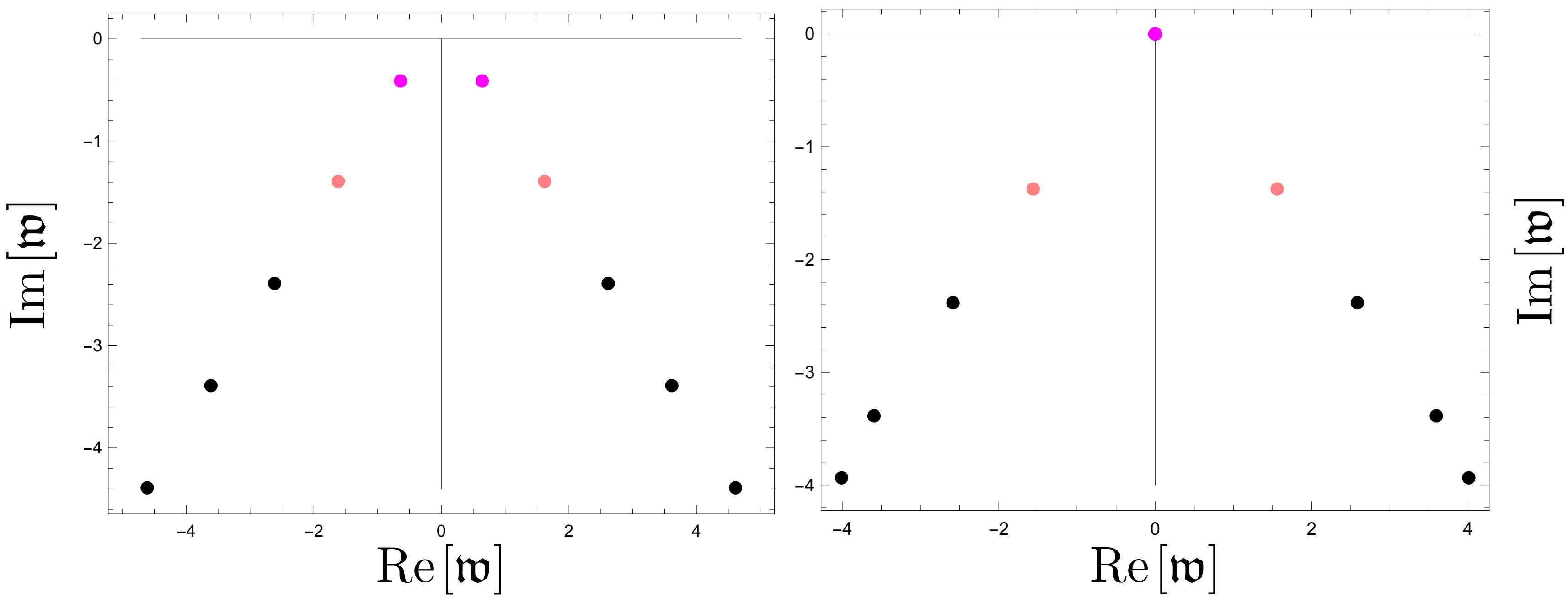
$\Longrightarrow$ Magenta point hydrodynamics:



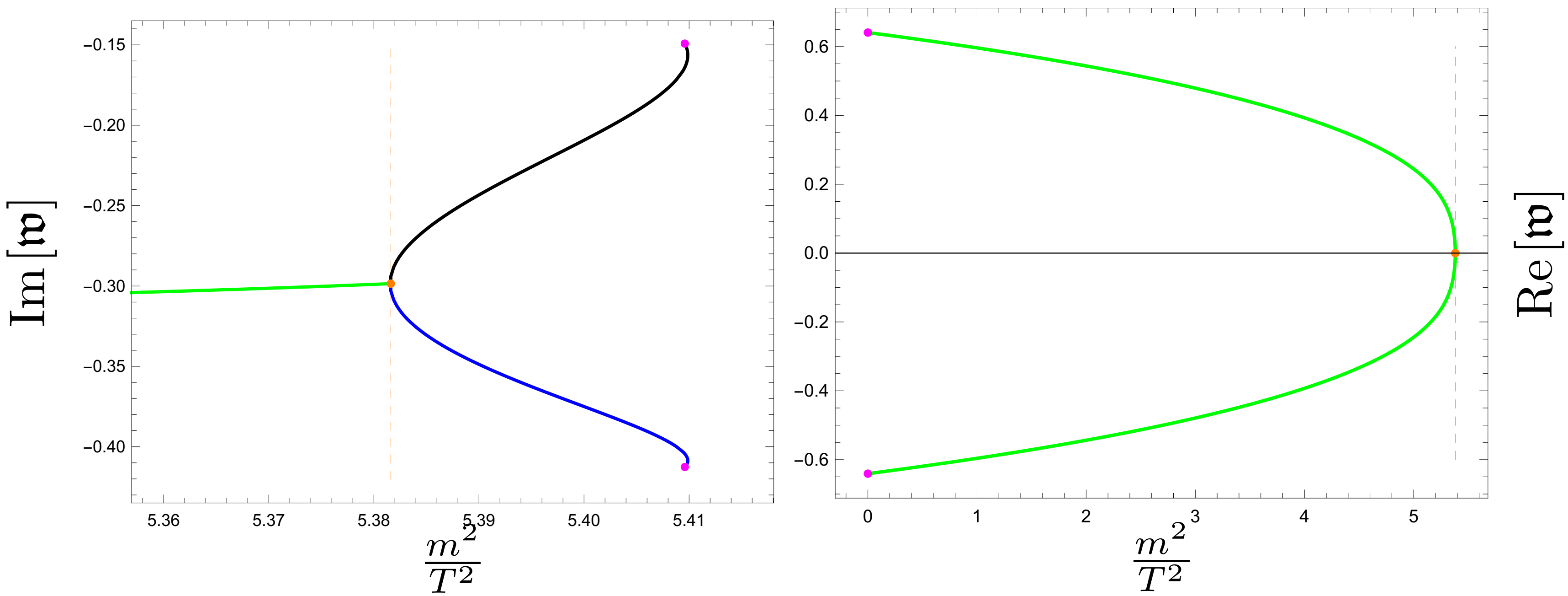
$\Longrightarrow$ We are now going to follow the flow of the QNM from $m = 0$ to our magenta point

- $\frac{m^2}{T^2} = 0$, *i.e.*, AdS_5 QNMs

- $s = 0$ at $\mathfrak{q} = 0$ (left) and $s = 2$ at $\mathfrak{q} = \frac{1}{100}$ (right)

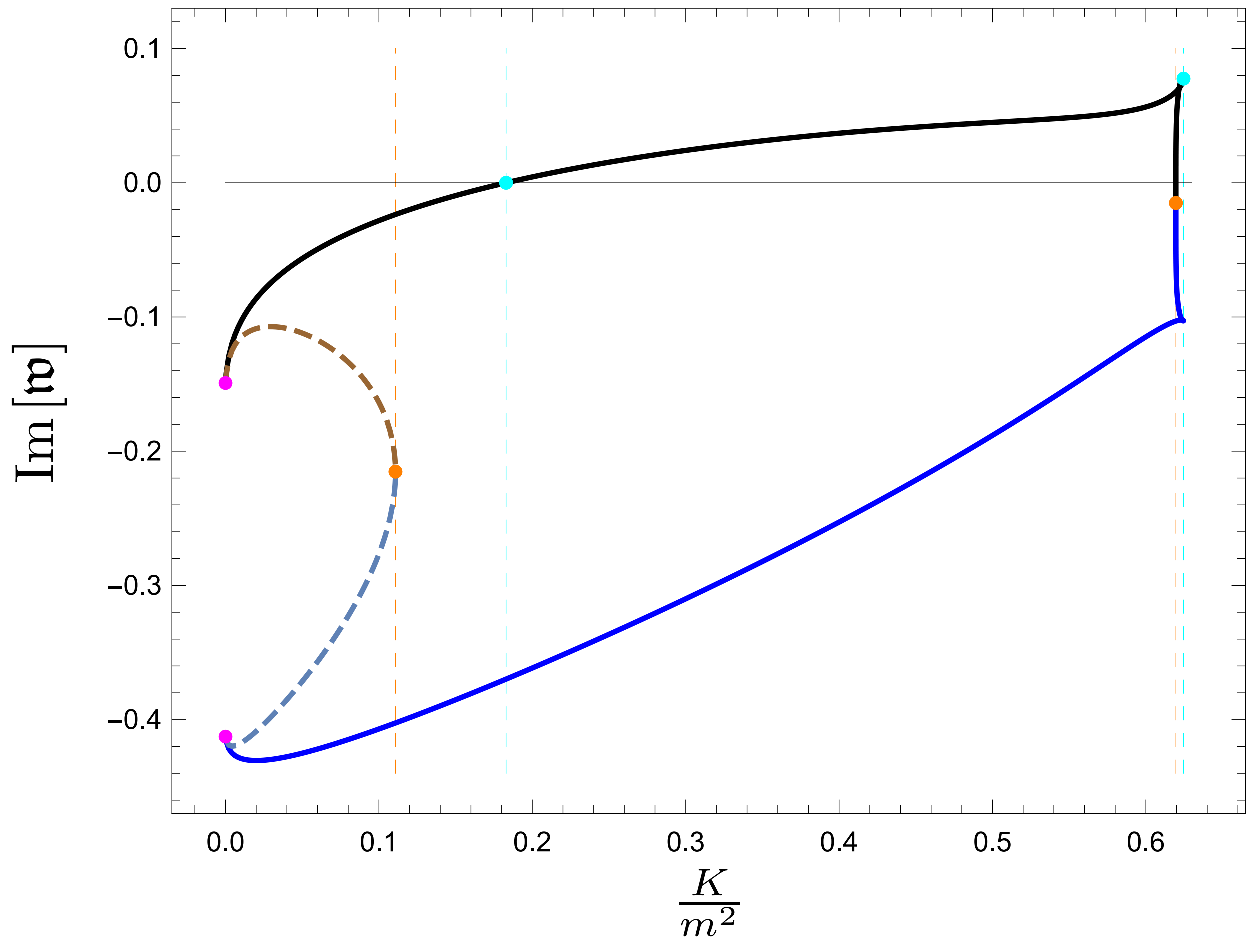


- $s = 0$ at $\mathfrak{q} = 0$



$\Longrightarrow$ We now turn on $K \neq 0$, focusing on magenta points.

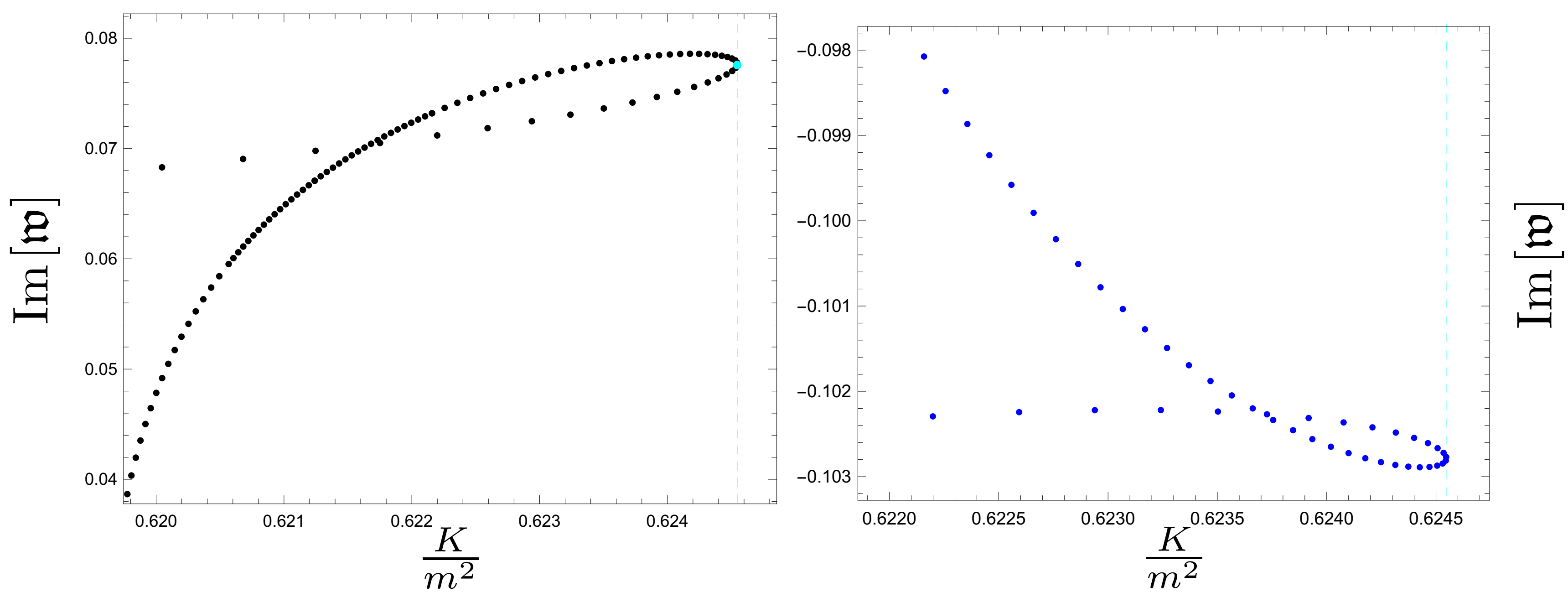
- $\ell = 0$ (solid curves) and $\ell = 1$ (dashed curves), fixed $\frac{m^2}{T^2}$



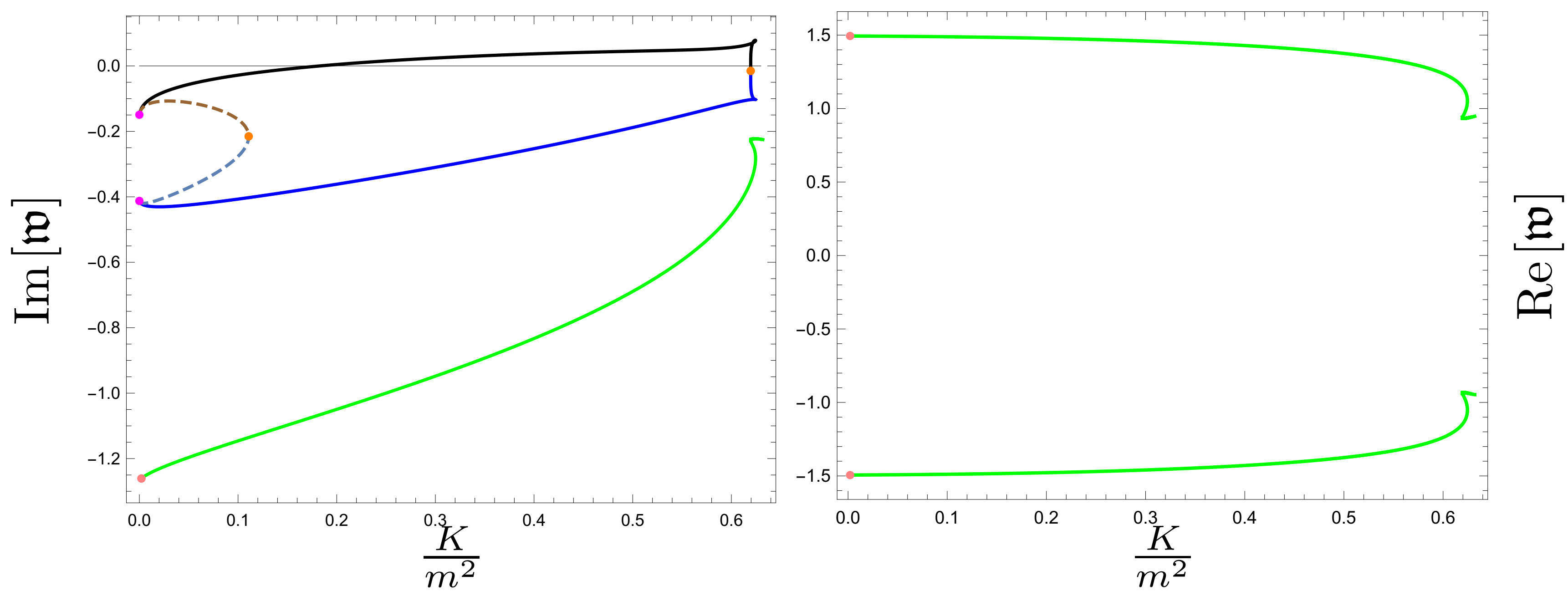
$\Longrightarrow$ unstable

$$0.18 \lesssim \frac{K}{m^2} \lesssim 0.62$$

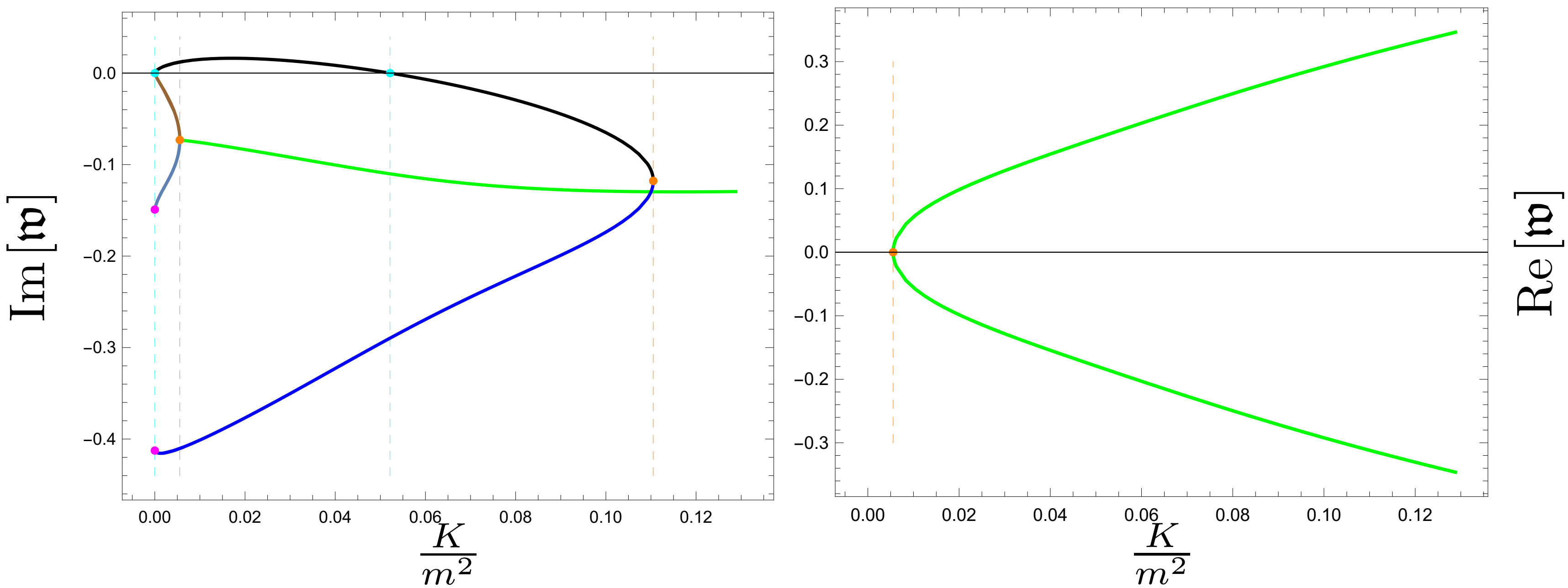
- $\ell = 0$ QNMs, fixed $\frac{m^2}{T^2}$



$\Longrightarrow$ What about $s = 0$ AdS_5 'excited' QNM?



- $\ell = 2$ QNMs, fixed $\frac{m^2}{T^2}$

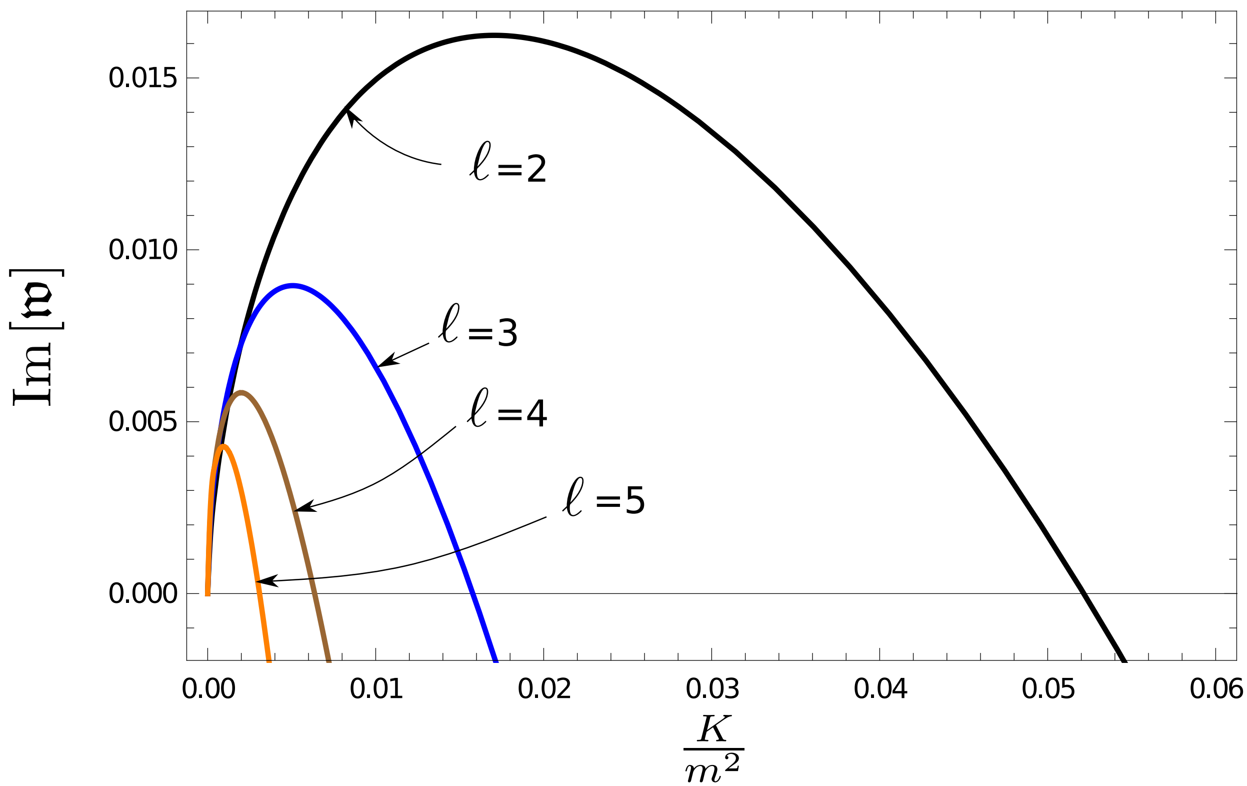


$\Longrightarrow$ unstable

$$0 \;\leq\; \frac{K}{m^2} \;\lesssim\; 0.052$$

$\Longrightarrow$ Note 2 sub-branches of the $\ell = 2$ QNMs: from the magenta points, and from the 'hydrodynamic' point

$\Longrightarrow$ arbitrarily higher ℓ's are also unstable (the instability is driven by the hydrodynamic point), but are stabilized sooner:



Conclusions

- Extended horizons with 'crumpling' instability can be stabilized by positively curving them
- For small K, the instability is driven by 'deformed' hydrodynamic sound mode; implying that *all* higher ℓ modes are unstable

$$0 \leq \frac{K}{m^2} \leq \frac{K^{unstable}_{\ell=2}}{m^2}$$

- Hidden (at $K = 0$) instability appears in $\ell = 0$ sector once K is large enough, and it can be cured again with sufficiently large K:

$$\frac{K^{stable}_{\ell=0}}{m^2} \leq \frac{K}{m^2} \leq \frac{K^{unstable}_{\ell=0}}{m^2}$$

- No instability in $\ell = 1$ sector. (Is this always the case?)
- Stability range (is it always present?):

$$K^{unstable}_{\ell=2} \leq \; K \; \leq K^{stable}_{\ell=0}$$

Future

- Compactified conformal order
- Klebanov-Strassler black holes
- Explore universal(?) aspects of stabilization