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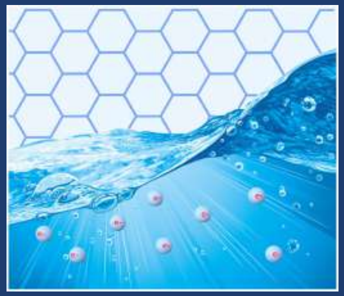
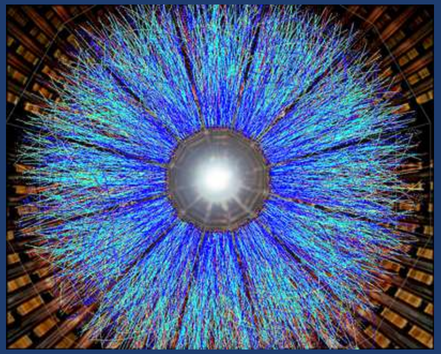
BOUNDS ON TRANSPORT FROM UNIVALENCE

HOLOTUBE, 20.4.2021

GENERAL MOTIVATION

precise analytic understanding of transport



hydrodynamics works unreasonably well

quantum chaotic mess drives collective transport

are hydrodynamic observables, such as diffusion and the speed of sound, bounded?

can we find exact and rigorous bounds on transport, given some precise set of analyticity conditions?

they need *not* be universal

HYDRODYNAMICS

- low-energy limit of QFTs – a Schwinger-Keldysh effective field theory
  [Grozdanov, Polonyi (2013); Crossley, Glorioso, Liu (2015); Haehl, Loganayagam, Rangamani (2015); …]

- conservation laws (equations of motion) of globally conserved operators

$$\nabla_\mu T^{\mu\nu} = 0 \qquad \nabla_\mu J^{\mu} = 0 \qquad \ldots \; \nabla_\mu J^{\mu\nu} = 0$$

higher-form currents in MHD [Grozdanov, Hofman, Iqbal, PRD (2017)]

- tensor structures (symmetries, gradient expansions) and transport coefficients (QFT)

$$T^{\mu\nu} = \sum_{n=0}^{\infty} \left[\sum_{i}^{N} \lambda_i^{(n)} \mathcal{T}^{\mu\nu}_{(n)} \right] \xrightarrow[u^\mu \sim T \sim e^{-i\omega t + iqz}]{\nabla_\mu T^{\mu\nu} = 0} \omega(q) = \sum_{n=1}^{\infty} \alpha_n q^n$$

$$\partial u^\mu \sim \partial T \ll 1 \longrightarrow \omega/T \sim q/T \ll 1$$

equilibrium temperature $q = \sqrt{\mathbf{q}^2}$

- dispersion relations:

shear diffusion	sound
$\omega = -iDq^2$	$\omega = \pm v_s q - i\Gamma q^2$

OUTLINE

- complex spectral curves and convergence
- quantum chaos through *pole-skipping*
- **bounds from univalence**
- summary and future directions

COMPLEX SPECTRAL CURVES AND CONVERGENCE

[Grozdanov, Kovtun, Starinets, Tadić, PRL (2019); …]

HYDRODYNAMICS AS COMPLEX SPECTRAL CURVES

- hydrodynamic modes as complex spectral curves
  [Grozdanov, Kovtun, Starinets, Tadić, PRL (2019) and JHEP (2019)]

hydro: $\det \mathcal{L}(\mathbf{q}^2,\omega) = 0$
QNM: $a(\mathbf{q}^2,\omega) = 0$

$\longrightarrow$ $\boxed{P(\mathbf{q}^2,\omega) = 0} \implies \boxed{\omega_i(\mathbf{q}^2)}$

$$\mathfrak{w} = \frac{\omega}{2\pi T},\ \mathfrak{q} = \frac{|\mathbf{q}|}{2\pi T} \in \mathbb{C}$$

- e.g., first-order hydrodynamics: $P_1(\mathbf{q}^2,\omega) = \left(\omega + iD\mathbf{q}^2\right)^2 \left(\omega^2 + i\Gamma\omega\mathbf{q}^2 - v_s^2\mathbf{q}^2\right) = 0$
- Puiseux theorem: there exists a convergent series around a critical point of any order

$$\boxed{P(\mathbf{q}_*^2,\omega_*) = 0,\ \partial_\omega P(\mathbf{q}_*^2,\omega_*) = 0,\ \ldots,\ \partial_\omega^p P(\mathbf{q}_*^2,\omega_*) \neq 0}$$

- convergence guaranteed up to the nearest (non-trivial) critical point (branch point)
  cf., the Newton polygon or the Darboux theorem [Grozdanov, Starinets, Tadić, *to appear*]

$$\boxed{f(t) \sim r(t)\left(1 - \frac{t}{t_0}\right)^{-\nu}, \quad t \to t_0} \qquad \boxed{\nu = \lim_{n\to\infty}\left(t_0(n+1)\frac{a_{n+1}}{a_n} - n\right)}$$

next critical point

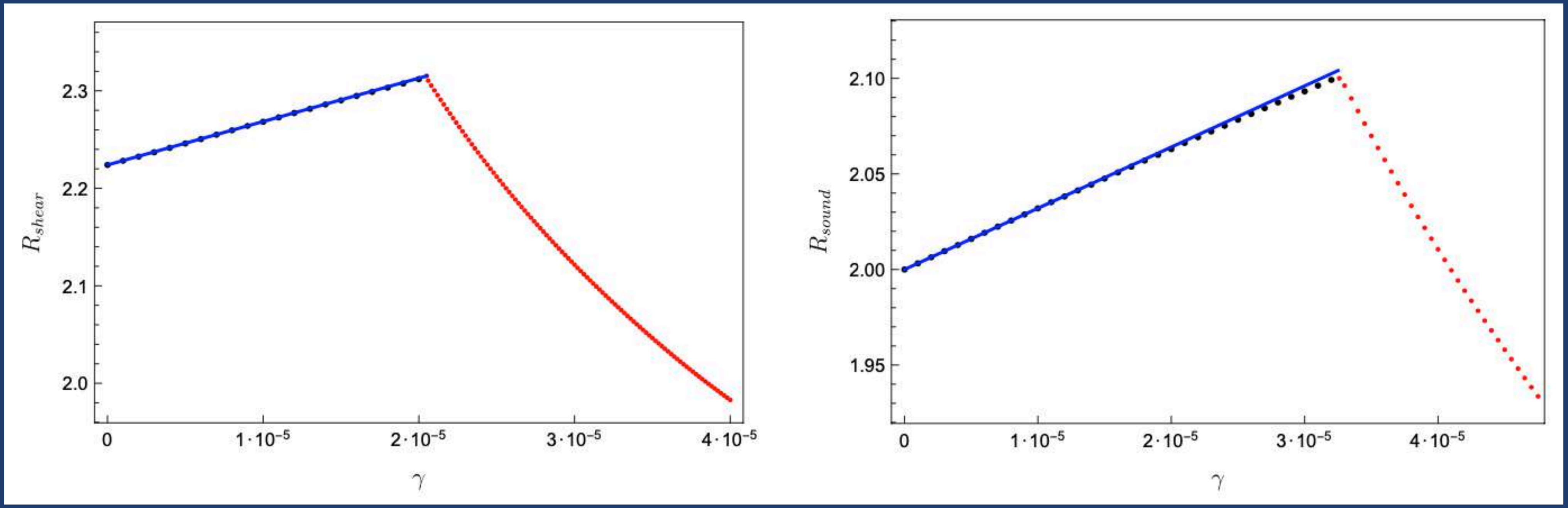
HYDRODYNAMICS AS COMPLEX SPECTRAL CURVES

- hydrodynamic series are convergent Puiseux series (shear *p=1*, sound *p=2*)

$$\mathfrak{w}_{\text{shear}} = -i\sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i\mathfrak{D}\mathfrak{q}^2 + \dots$$

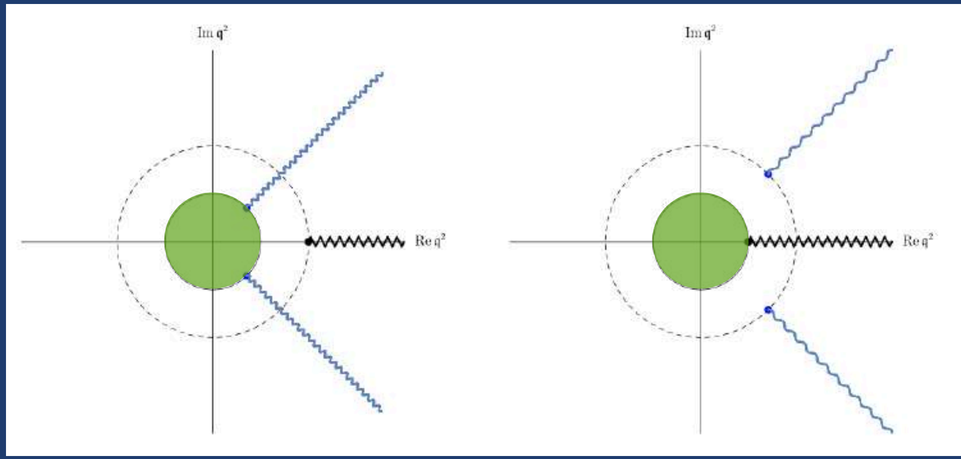
$$\mathfrak{w}_{\text{sound}} = -i\sum_{n=1}^{\infty} a_n e^{\pm\frac{i\pi n}{2}} \left(\mathfrak{q}^2\right)^{n/2} = \pm v_s\mathfrak{q} - \frac{i}{2}\mathfrak{G}\mathfrak{q}^2 + \dots$$

- radius of convergence in $\mathcal{N}=4$ SYM [Grozdanov, Starinets, Tadić, *to appear*]



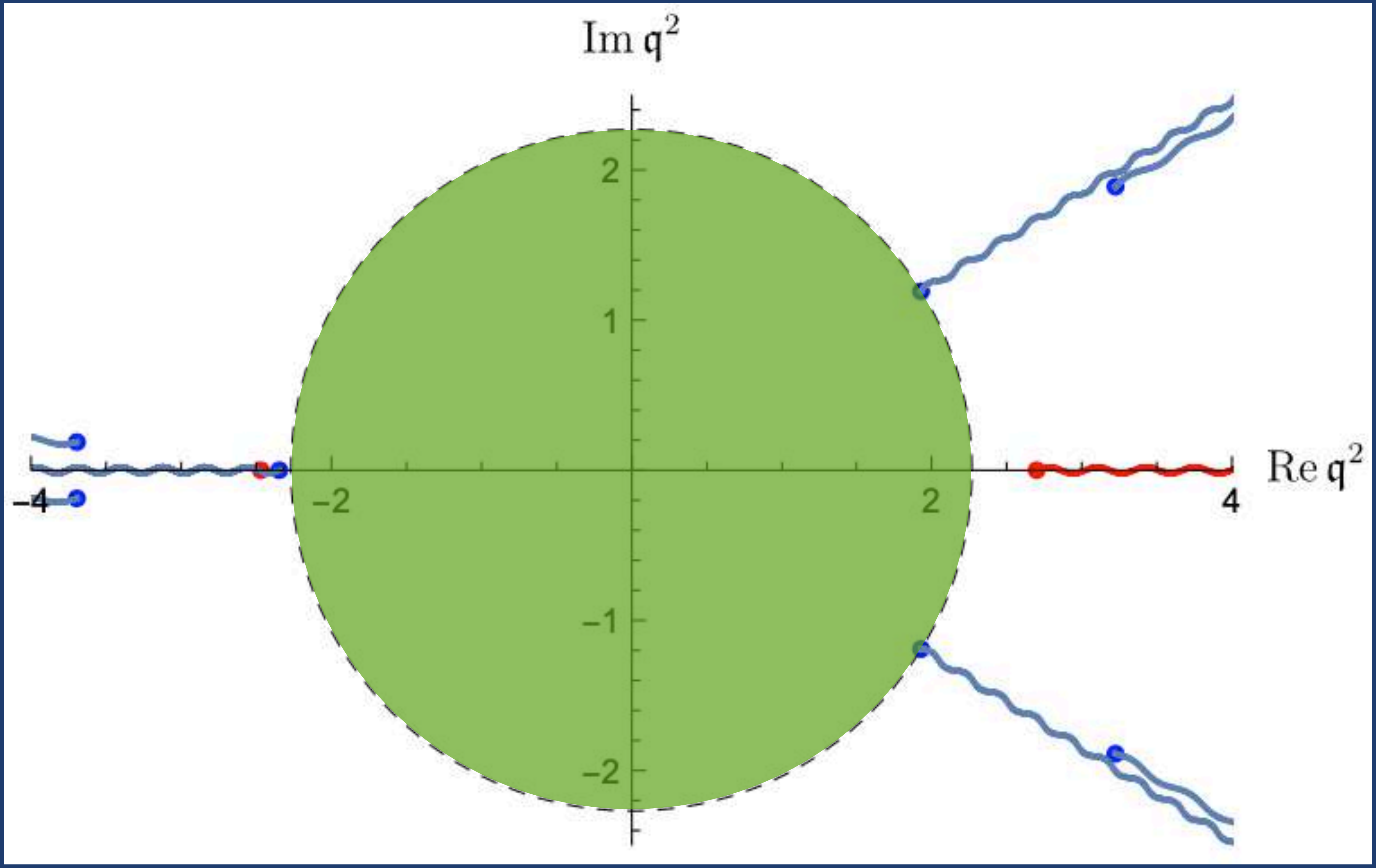
$$R_{\text{shear}}(\lambda) = 2.22\left(1 + 674.15\,\lambda^{-3/2} + \cdots\right)$$
$$R_{\text{sound}}(\lambda) = 2\left(1 + 481.68\,\lambda^{-3/2} + \cdots\right)$$

perturbative to non-perturbative transition

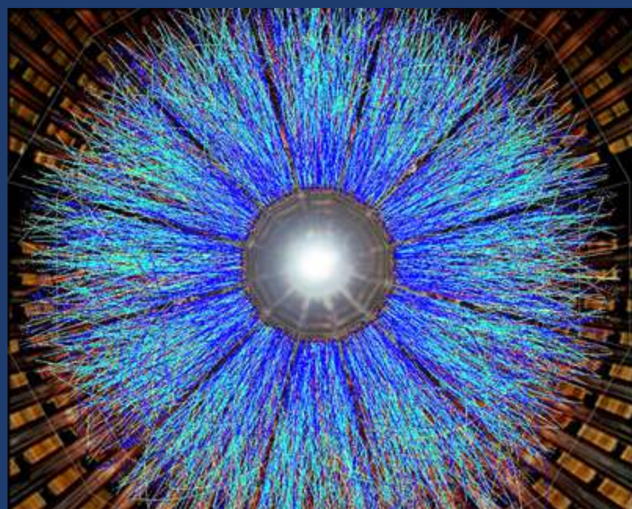


ANALYTIC STRUCTURE

- analytic structure of dispersion relations for $\mathcal{N} = 4$ supersymmetric Yang-Mills theory
- dispersion relations are Riemann surfaces connecting physical modes (analytic continuation)



UNREASONABLE EFFECTIVENESS

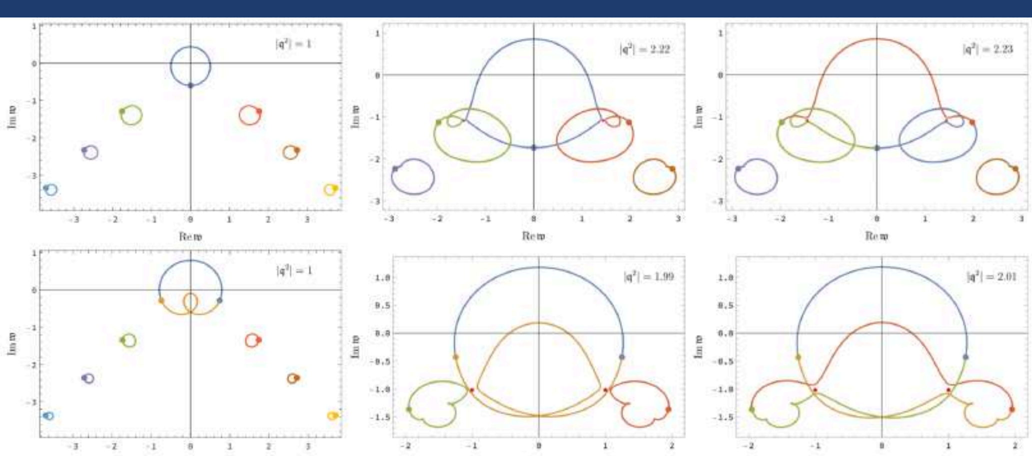


"unreasonable": hydro works for large derivatives;
it has exceptional analytic properties at strong coupling

$$\omega(q) = \sum_{n=1}^{\infty} \alpha_n q^n$$

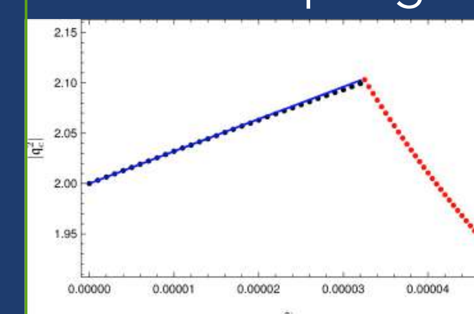
radius of convergence in *N*=4 SYM at strong coupling

$$q/T \sim O(10)$$



microscopic input from holography

finite coupling:



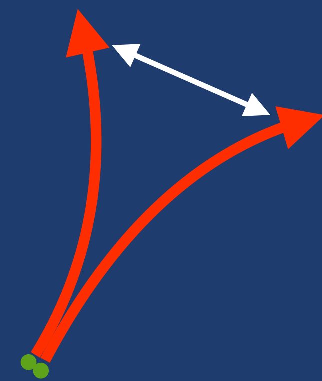
orders of magnitude larger radius of convergence than naive $q/T \ll 1$ –
this is a precise incarnation of the "unreasonable effectiveness of hydrodynamics"

QUANTUM CHAOS AND *POLE-SKIPPING*

[Grozdanov, Schalm, Scopelliti, PRL (2017); …]

CHAOS

- classical chaos means extreme sensitivity to initial conditions



$t = t_i$

Lyapunov exponent → butterfly velocity ↓

$$|\Delta Z(t,\mathbf{x})| \approx |\Delta Z(t_i,\mathbf{x}_i)|\, e^{\lambda_L (t - |\mathbf{x}|/v_B)}$$

- "what is quantum chaos?"
  a measure: "out-of-time-ordered" correlation functions [Larkin, Ovchinnikov; Kitaev]

$$C(t,\mathbf{x}) = \left\langle [W(t,\mathbf{x}), V(0,\mathbf{0})]^\dagger [W(t,\mathbf{x}), V(0,\mathbf{0})] \right\rangle_T \sim \epsilon\, e^{\lambda_L (t - |\mathbf{x}|/v_B)}$$

'quantum' Lyapunov exponent

butterfly velocity

- the Maldacena-Shenker-Stanford bound on exponential Lyapunov chaos

OTOC of $\mathcal{O}(t,x)$

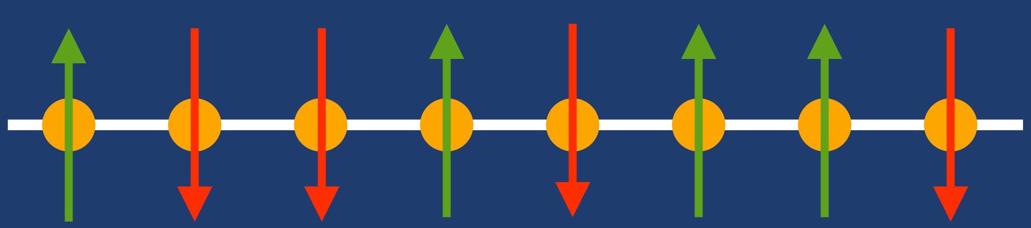
$$C(t,x) \sim \epsilon\, e^{\lambda_L (t - x/v_B)}$$

$$\lambda_L \leq 2\pi T/\hbar$$

- in finite-*N* systems, quantum chaos spreads polynomially with a bounded rate of growth – weak quantum chaos [Kukuljan, Grozdanov, Prosen, PRB (2017)]

OTOC of $\int d^d x\, \mathcal{O}(t,x)$

$$c(t) \leq A t^{3d}$$



POLE-SKIPPING

- precise analytic connection between 'low-energy' hydrodynamics and quantum chaos [Grozdanov, Schalm, Scopelliti, PRL (2017); Blake, Lee, Liu, JHEP (2018); Blake, Davison, Grozdanov, Liu, JHEP (2018); Grozdanov, JHEP (2019)]

- resumed all-order hydrodynamic series (e.g. sound)

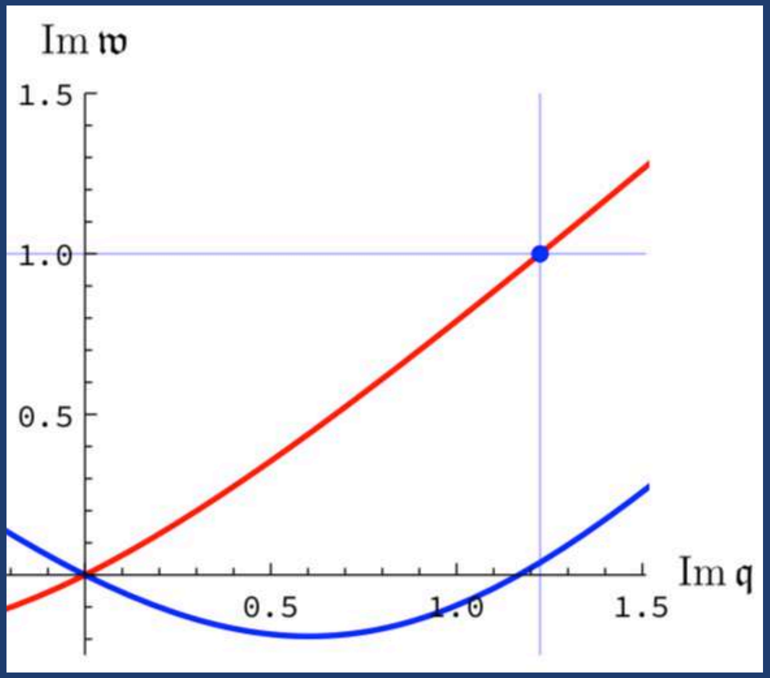
$$\omega(q) = \sum_{n=1}^{\infty} \alpha_n \left(T, \mu_i, \langle \mathcal{O}_j \rangle, \lambda\right) q^n$$

passes through a "chaos point" at imaginary momentum

$$\omega(q = i\lambda_L / v_B) = i\lambda_L = 2\pi T i$$

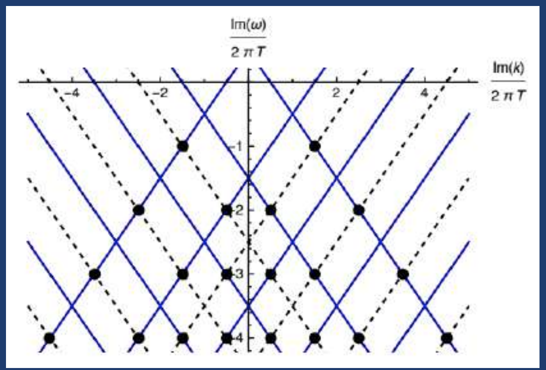
where the associated *2*-pt function is "0/0":

$$\operatorname{Res} G_R^{\varepsilon\varepsilon}\left(\omega = i\lambda_L, q = i\lambda_L / v_B\right) = 0$$



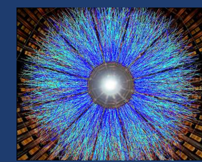
- triviality of Einstein's equations at the horizon [Blake, Davison, Grozdanov, Liu, JHEP (2018)]

- infinite constraints on correlators at multiples of Matsubara frequencies [Grozdanov, Kovtun, Starinets, Tadić, JHEP (2019); Blake, Davison, Vegh, JHEP (2019)]



$$\mathcal{C}\hbar/(k_B T) \lesssim \tau_R$$

relaxation time
[Sachdev; …]

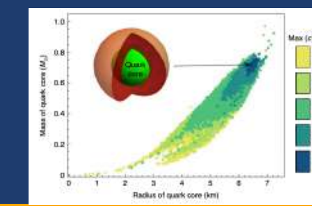


$$\hbar/(4\pi k_B) \leq \eta/s$$

shear viscosity
[Kovtun, Son, Starinets; …]

$$\mathcal{C} v_B^2/\lambda_L \lesssim D$$

diffusion vs. chaos
[Hartnoll; Blake, …]



$$v_s \leq c/\sqrt{3}$$

sound
[Hohler, Stephanov; Cherman, Cohen, Nellore; …]

BOUNDS FROM UNIVALENCE

[Grozdanov, PRL (2021)]

$$\lambda_L \leq 2\pi T/\hbar$$

exponential Lyapunov chaos

$$c(t) \leq A t^{3d}$$

polynomial weak chaos

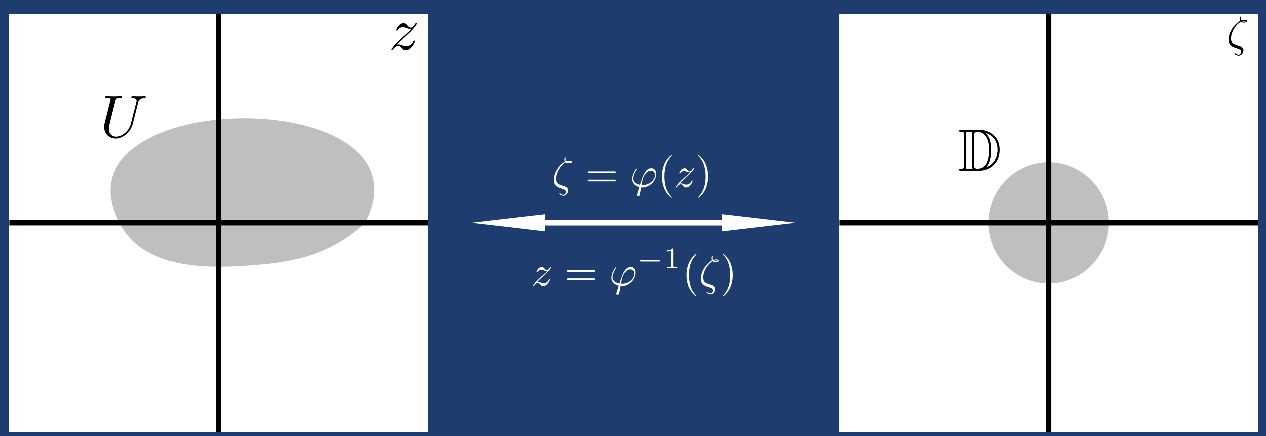
can we find exact and rigorous bounds on transport, given some precise set of analyticity conditions?

they need *not* be universal

… closer in the spirit of exactness and non-universality, but not really, to [Grozdanov, Lucas, Sachdev Schalm, PRL (2015)]

UNIVALENT FUNCTIONS

- univalent function $f(z),\ z \in \mathbb{C}$ is a complex, holomorphic and injective function
- injectivity: $f(z_1) \neq f(z_2)$ for all $z_1 \neq z_2$
- for $f(z)$ univalent in some region U, by the Riemann mapping theorem:



$\mathbb{D} = \{\zeta \,|\, |\zeta| < 1\}$

$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n$$

is univalent in $\mathbb{D}$

- holomorphic functions are "stiff", univalent function even more so…
- the growth theorem:

$$\frac{|\zeta|}{(1+|\zeta|)^2} \leq |f(\zeta)| \leq \frac{|\zeta|}{(1-|\zeta|)^2}$$

- the famous Bieberbach conjecture (1916), now de Branges's theorem (1985):

$$|b_n| \leq n, \quad \text{for all } n \geq 2$$

EXTREME UNIVALENCE: THE KOEBE FUNCTION

- the Koebe function defined on $\mathbb{D} = \{\zeta \,|\, |\zeta| < 1\}$:

$$f_K(\zeta) = \frac{\zeta}{(1-\zeta)^2} = \zeta + 2\zeta^2 + 3\zeta^3 + \ldots = \sum_{n=1}^{\infty} n\zeta^n$$

- it saturates all univalence inequalities
- it is a conformal map to the complex plane without a semi-infinite line:

$$f_K : \mathbb{D} \to \mathbb{C} \setminus (-\infty, -1/4]$$

- the Koebe *1/4* theorem (proven by Bieberbach):

any univalent function $f(\zeta)$ *mapping* $\mathbb{D} \to \mathbb{C}$

contains a disk of radius $|f(\zeta)| = 1/4$

WHEN IS A FUNCTION UNIVALENT?

- when is $f(z)$ univalent and what is U?
- local univalence $f'(z) \neq 0$ everywhere is insufficient
- global univalence is tricky!
- sufficient vs. necessary conditions; e.g., in terms of $\{f(z), z\} = f'''(z)/f'(z) - (3/2)\left(f''(z)/f'(z)\right)^2$

$f(z)$ is univalent if $|\{f(z), z\}| \leq \frac{2}{(1-|z|^2)^2}, \quad |z| < 1$

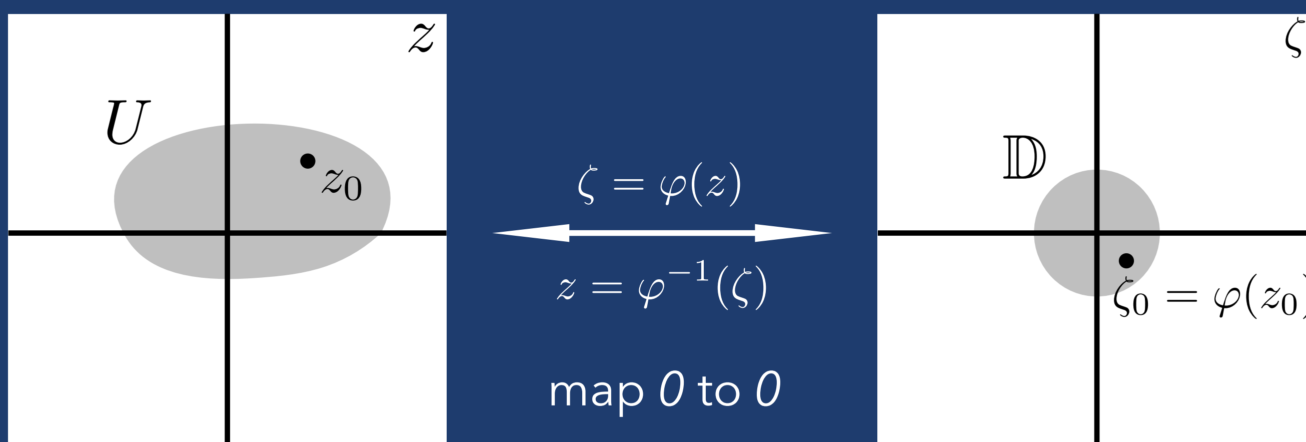
if $f(z)$ is univalent, then $|\{f(z), z\}| \leq \frac{6}{(1-|z|^2)^2}, \quad |z| < 1$

- we will choose $\mathrm{Re}\, f'(z) > 0$ in any convex $z \in U \subset \mathbb{C}$
- if, moreover, $\mathrm{Re}\, f'(\zeta) > 0, \ \zeta \in \mathbb{D}$, then we get even stronger bounds (MacGregor):

$$-|\zeta| + 2\ln(1+|\zeta|) \leq |f(\zeta)| \leq -|\zeta| - 2\ln(1-|\zeta|)$$
$$|b_n| \leq 2/n, \quad \text{for all } n \geq 2$$

BOUNDS ON HYDRODYNAMICS

- series: $\omega_{\text{diff}}(z \equiv \mathbf{q}^2) = -i \sum_{n=1}^{\infty} c_n z^n, \quad c_1 = D;$ $\quad \omega_{\text{sound}}^{\pm}(z \equiv \sqrt{\mathbf{q}^2}) = -i \sum_{n=1}^{\infty} a_n e^{\pm \frac{i\pi n}{2}} z^n$
- dispersion relations have a finite region of univalence U
- def: $f_{\text{diff}}(z) = i\omega_{\text{diff}}(z) \quad \longleftrightarrow \quad f_{\text{diff}}(\zeta) \equiv \dfrac{i\omega_{\text{diff}}(\varphi^{-1}(\zeta))}{D\partial_\zeta \varphi^{-1}(0)} = \zeta + \sum_{n=2}^{\infty} b_n^{\text{diff}} \zeta^n$



need to know the value of the dispersion relation at one point $\omega_0 \equiv \omega(z_0)$

- exact and rigorous bounds on *all coefficients* immediately follow (similarly for sound):

$$\frac{|\omega_0|\,(1-|\zeta_0|)^2}{|\zeta_0|\,|\partial_\zeta \varphi^{-1}(0)|} \le D \le \frac{|\omega_0|\,(1+|\zeta_0|)^2}{|\zeta_0|\,|\partial_\zeta \varphi^{-1}(0)|}$$

$$|b_n| \le n \;\Rightarrow\; \left| c_2 + \frac{D}{2} \frac{\partial_\zeta^2 \varphi^{-1}(0)}{[\partial_\zeta \varphi^{-1}(0)]^2} \right| \le \frac{2D}{|\partial_\zeta \varphi^{-1}(0)|}, \;\ldots$$

- stronger bounds with *logs* follow if $\text{Re}\, f'(\zeta) > 0, \; |\zeta| < 1$

BOUNDS ON HYDRODYNAMICS

- are hydrodynamic dispersion relations really univalent?
- they are holomorphic and invertible at $z = 0$ (Puiseux), hence locally univalent; a finite $U = \{z \,|\, |z| < \min[|z_g|, R]\}$ with group velocity $v_g = \partial\omega/\partial q$ and

$$\begin{aligned} \text{diffusion}: \quad & \operatorname{Re} f'(z) > 0 \quad \Longrightarrow \quad z_g = q_g^2 \equiv \min q^2 \,|\, \operatorname{Re} v_g \operatorname{Im} q = \operatorname{Im} v_g \operatorname{Re} q \\ \text{sound}: \quad & \operatorname{Re} f'(z) > 0 \quad \Longrightarrow \quad z_g = q_g \equiv \min q \,|\, \operatorname{Re} v_g = 0 \end{aligned}$$

- sound seems to lose univalence via the local condition (is this generic?):

$$f'(z_g) = 0 \quad \Longrightarrow \quad q_g \equiv \min q \,|\, v_g = 0$$

- with no additional input, we can get

$$\begin{aligned} \text{diffusion}: \quad & 0 \le D \le 4\,|v_{ph}(\bar{q}^2)/\bar{q}| \\ \text{sound}: \quad & 0 \le v_s \le 4\,|v_{ph}(\bar{q})| \end{aligned}$$

or $1/(2\ln 2 - 1) \approx 2.59$

$|\bar{q}| = \min[|q_g|, |q_*|]$

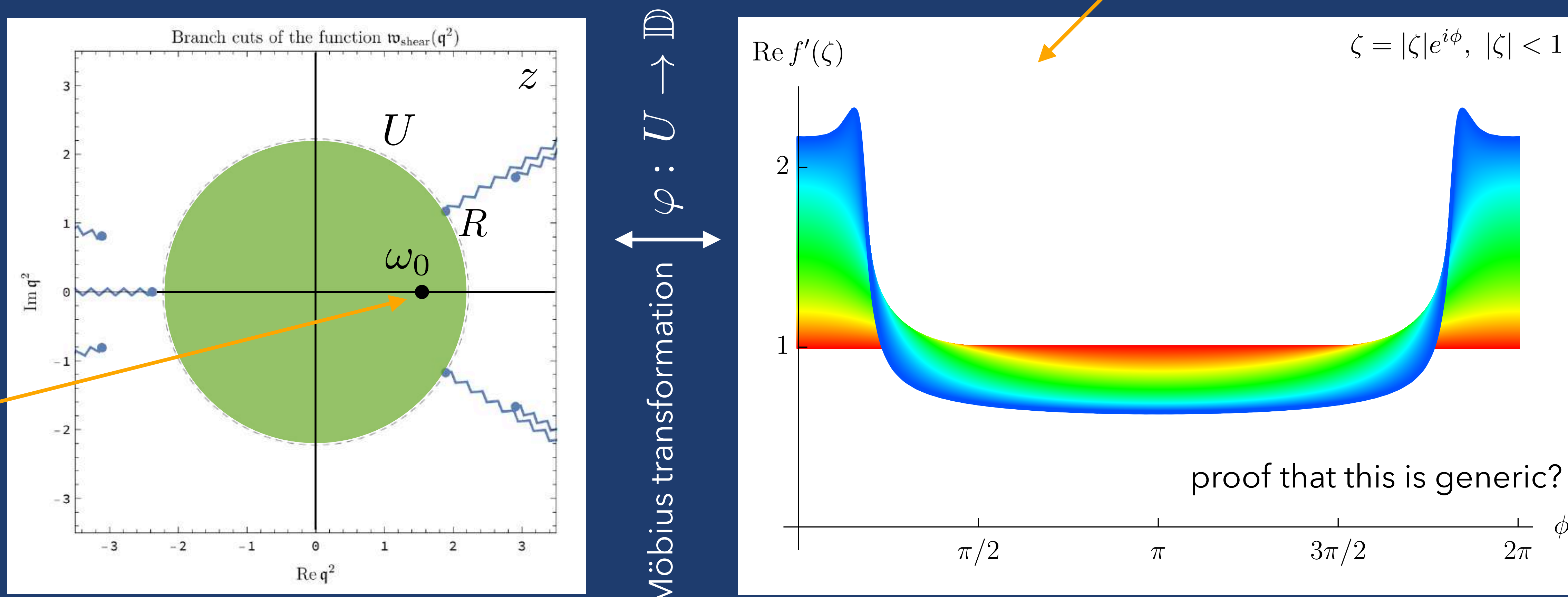
critical q setting R

phase velocity $v_{ph} = \omega/q$

COMPLICATED DIFFUSION

- example: momentum diffusion in $\mathcal{N} = 4$ SYM theory

$\mathrm{Re}\, f'(\zeta) > 0$
so more stringent bounds with *logs*



pole-skipping

$$0.046/T \leq D = 1/4\pi T \approx 0.080/T \leq 0.201/T$$

$$-\frac{2D}{nR^{n-1}} \leq c_{n\geq 2} \leq \frac{2D}{nR^{n-1}}$$

better convergence squeezes the series

input: pole-skipping + radius of convergence *R*

SIMPLE DIFFUSION, BUT EXTREME UNIVALENCE

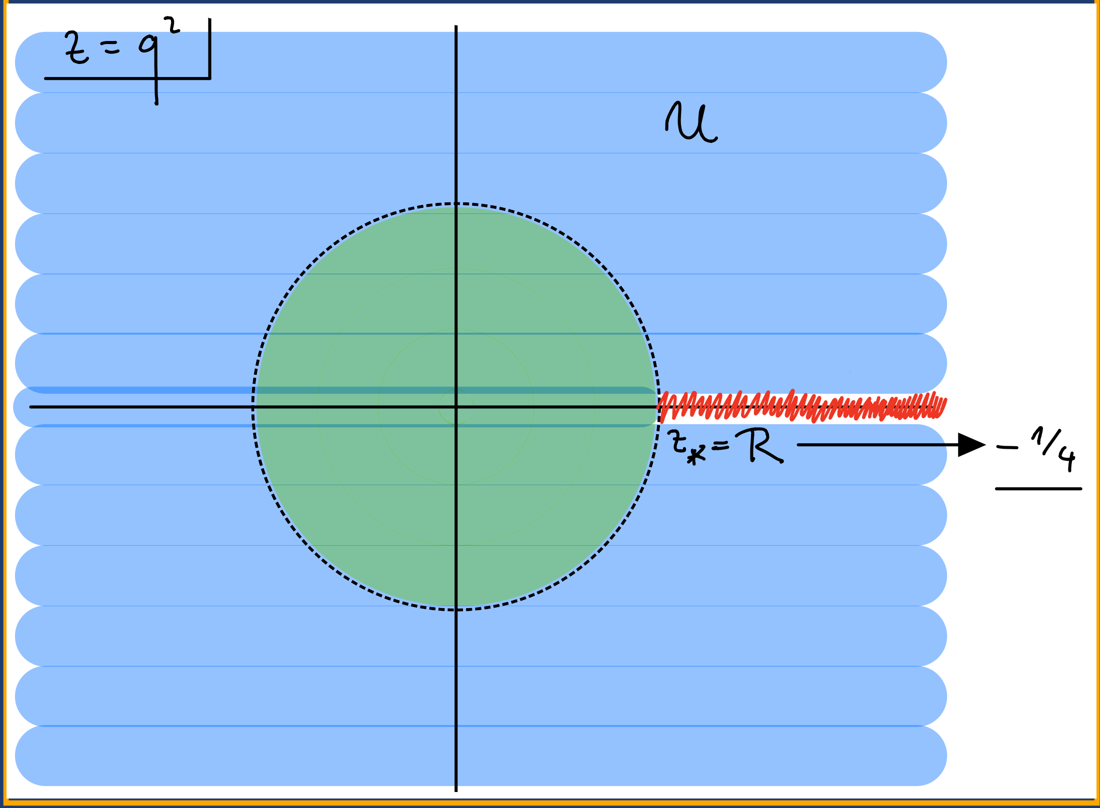
- assume a dispersion relation is univalent everywhere except at a single branch cut
  e.g.: self-dual axion model
  [Davison, Gouteraux (2014)]

$$\omega(z=\mathbf{q}^2) = -i\pi T\left(1-\sqrt{1-\frac{z}{\pi^2T^2}}\right)$$

$$\zeta = \varphi(z) = \frac{z - 2z_* + 2\sqrt{z_*^2 - zz_*}}{z},$$

$$z = \varphi^{-1}(\zeta) = -4z_* f_K(\zeta) = -\frac{4z_*\zeta}{(1-\zeta)^2}$$

$$\partial_\zeta^n \varphi^{-1}(0) = -4n^2(n-1)!z_*$$



- optimal bounds
  (explicit Blake-type proposal $D \gtrsim v_B^2/\lambda_L$):

energy diff: $z_0 = -\frac{\lambda_L^2}{v_B^2} < 0: \quad \frac{v_B^2}{\lambda_L} \le D \le \frac{v_B^2}{\lambda_L} + \frac{\lambda_L}{R}$

momentum diff: $0 < z_0 = \frac{\lambda_L^2}{v_B^2} < R: \quad \frac{v_B^2}{\lambda_L} - \frac{\lambda_L}{R} \le D \le \frac{v_B^2}{\lambda_L}$

higher orders: $0 \le c_2 \le \frac{D}{R}, \quad \cdots$

- infinite radius of convergence (*squeeze*):

$$\omega_{\text{diff}}(\mathbf{q}^2) = -iD\mathbf{q}^2 = -i\frac{v_B^2}{\lambda_L}\mathbf{q}^2$$

positivity of c_2 and connection to Wu, Baggioli, Li, arXiv:2102.05810

COMPLICATED SOUND

- univalence breakdown is set by the local condition

$$f'(z_g) = 0 \quad \Longrightarrow \quad q_g \equiv \min q \,|\, v_g = 0$$

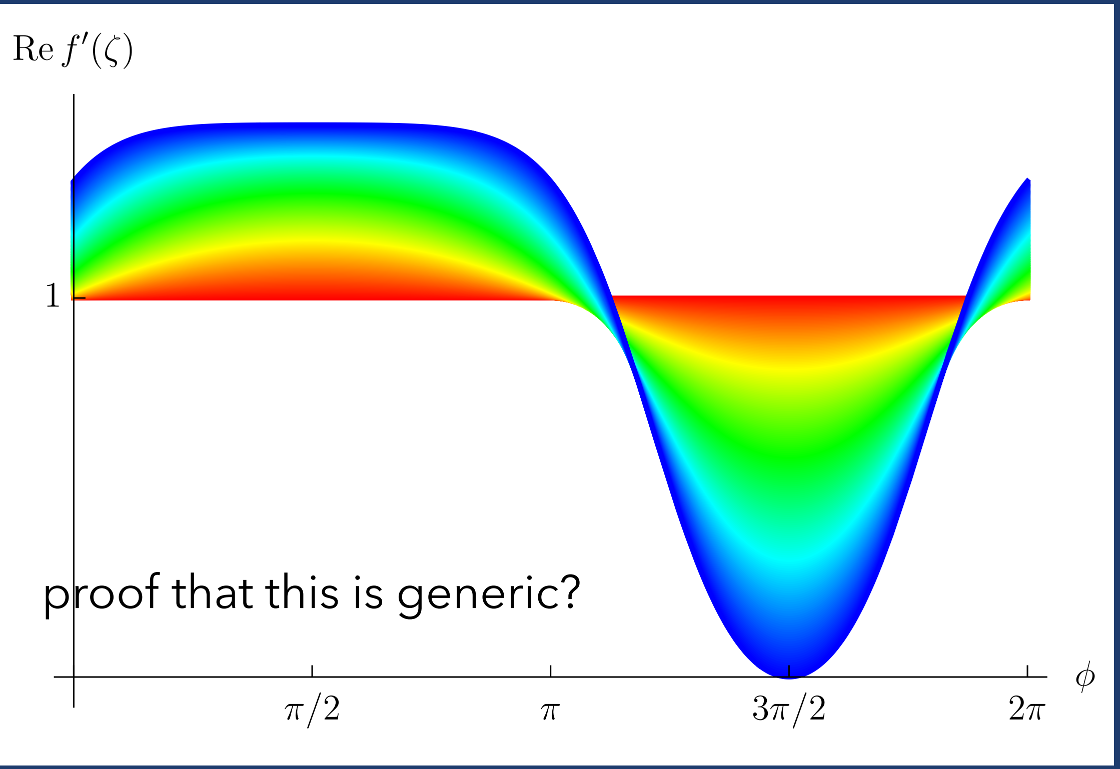
- example: $\mathcal{N} = 4$ SYM theory

$$z_g = \sqrt{\mathbf{q}_g^2} \approx -3.791\, iT$$

$$z_g = \sqrt{\mathbf{q}_g^2} \approx -3iv_s/4D = -5.441\, iT$$

exact (numerics)

approximate (hydro)



- construct a sufficient analyticity condition for the conformal bound on the speed of sound [Cherman, Cohen, Nellore (2009); Hohler, Stephanov (2009)], but stay tuned…

$$|\partial_\zeta \varphi^{-1}(0)| = 4\sqrt{3}|\omega_0(z_0)| \;\wedge\; |\zeta_0| = |\varphi(z_0)| \to 1 \quad \Longrightarrow \quad 0 \le v_s \le \sqrt{\frac{1}{3}}$$

SUMMARY AND
FUTURE DIRECTIONS

SUMMARY AND FUTURE DIRECTIONS

- complex analytic structures of transport reveal new physical properties
- holographic duality is an invaluable tool for exploring QFTs

- hydrodynamics has exceptional analytic (convergence) properties
- pole-skipping is a precise relation between hydrodynamics and quantum chaos
- new methods for rigorous lower and upper bounds on *all* coefficients of hydrodynamic dispersion relations

- a lot to be done: generic results (stay tuned), applications, …
- uninsightful comment: application of these techniques beyond large-N
- application to other problems like scattering amplitudes
  [Khuri, Kinoshita, Phys. Rev. (1965);
  Haldar, Sinha, Zahed, arXiv:2103.12108]

Quantum field theory and the Bieberbach conjecture

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THANK YOU!