Hydrodynamics

panel discussion

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Outline:

1) Divergence of non-linear hydro derivative expansion in position space → 1.1, 1.2, 1.3

2) Convergence of linear hydro derivative expansion → 2.1, 2.2

3) Effective field theory of hydro → 3.1, 3.2, 3.3

4) Hydrodynamic frames: Causality and Stability → 4.1, 4.2

- Each slide includes "**A very short review**" + "**A number of questions**" ⟶ 2min
- We do not intend to introduce any paper, just refer to (1-2) well-known ones

1.1 Divergence of the derivative expansion in real space

[Heller, Janik, Witaszczyk, 1302.0697]

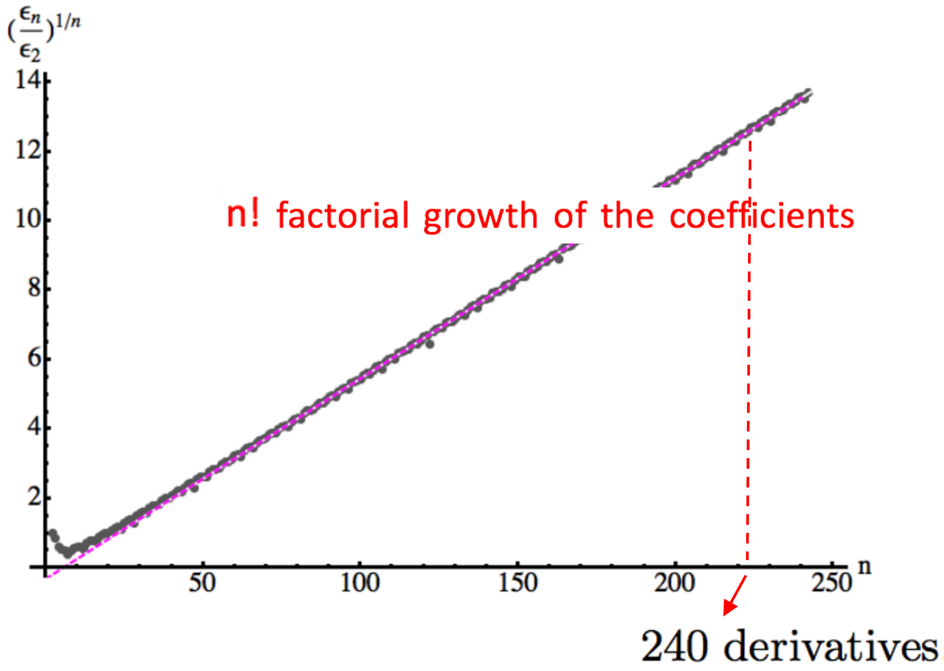
Search for **1-dim boost invariant flow from holography**

$$ds^2 = 2d\tau dr - Ad\tau^2 + \Sigma^2 e^{-2B} dy^2 + \Sigma^2 e^B (dx_1^2 + dx_2^2)$$

- Hydro in Large τ limit: $\frac{\dot{T}}{T^2} \sim \frac{1}{\tau^{2/3}}$
- The energy density (**sensitive to nonlinearities**

  given in a Derivative expansion:

$$\epsilon = \frac{3}{8} N_c^2 \pi^2 \underbrace{\frac{1}{\tau^{4/3}}}_{\sim T^4} \left(\epsilon_2 + \epsilon_3 \underbrace{\frac{1}{\tau^{2/3}}}_{\sim \frac{\dot{T}}{T^2}} + \epsilon_4 \frac{1}{\tau^{4/3}} + \dots \right)$$



1. **Is it sufficient to say that the derivative expansion is divergent?**
2. **Is this factorial growth specific to the particular choice of a highly symmetric flow (Bjorken)?**
3. **The origin of such N! growth?**
4. **Any analogue in QFT?**
5. **Any resolution? Resummation?!!!!** (→ Next slide)

1.2 Resumming the derivative expansion.

[Heller, Janik, Witaszczyk, 1302.0697]

Borel-transformed series:

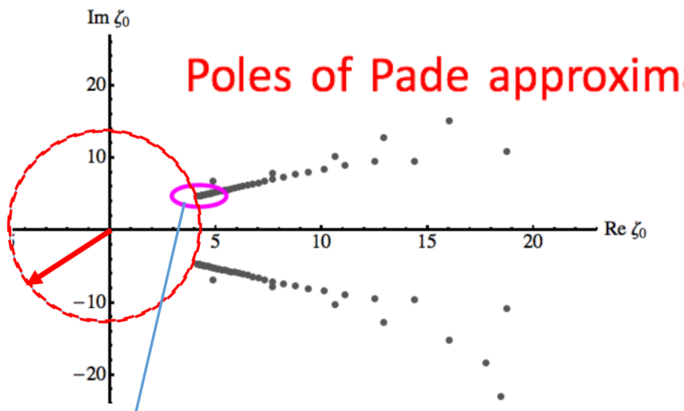
$$\sum_{n=0} \epsilon_{n+2} \left(\frac{1}{\tau^{2/3}}\right)^n \xrightarrow{\tilde{\epsilon}_n = \epsilon_n / n!} \sum_{n=0} \tilde{\epsilon}_{n+2} \left(\frac{1}{\tau^{2/3}}\right)^n$$

$$\epsilon_{resum}(\tau) = \int_0^\infty \tilde{\epsilon}(\zeta\tau)\, e^{-\zeta}\, d\zeta$$

- Analytic continuation is needed for Borel transform:

Pade (rational function instead of $\tilde{\epsilon}(\zeta\tau)$)

- The radius of convergence of Pade is set by the location of branch point $|\omega_{\text{Borel}}| = 4.1565$



$$\delta\epsilon \sim \tau^{\alpha_{\text{Borel}}} \exp\left(-i\,\frac{3}{2}\,\omega_{\text{Borel}}\,\tau^{2/3}\right)$$

Consistent with the lowest QNM: $\delta\epsilon \sim \tau^{\alpha_{qnm}} \exp\left(-i\,\frac{3}{2}\,\omega_{qnm}\,\tau^{2/3}\right)$

[Janik, Peschanski, 0606149]

1. **Ambiguity in resummation?**
2. **Which contour?**
3. **Resumming 240 terms gives the first QNM, is it possible to get more?**
4. **The whole micro from the derivative expansion?**

1.3 Beyond the derivative expansion: Attractors.

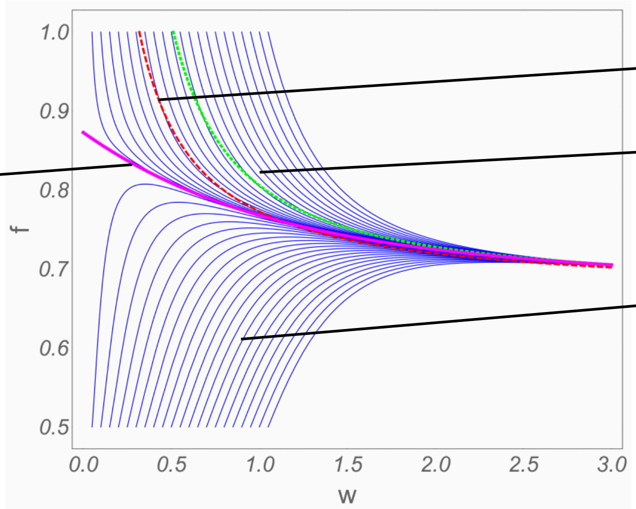
[Heller, Spalinski, 1503.07514]

MIS + Bjorken flow

$$C_{\tau\Pi} w f f' + 4C_{\tau\Pi} f^2 + \left(w - \frac{16C_{\tau\Pi}}{3}\right) f - \frac{4C_\eta}{9} + \frac{16C_{\tau\Pi}}{9} - \frac{2w}{3} = 0,$$

$$w = \tau T, \qquad f = \tau \frac{\dot{w}}{w}$$

attractor

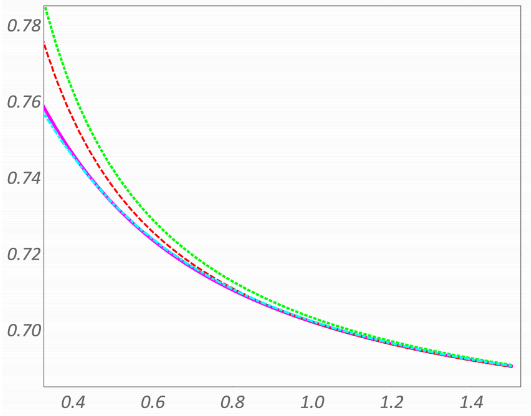


1^{st} order hydro

2^{nd} order hydro

$$f(w) = \frac{2}{3} + \frac{4C_\eta}{9w} + \frac{8C_\eta C_{\tau\Pi}}{27w^2} + O(\frac{1}{w^3}).$$

Numerical solution with different initial conditions



Just like in N=4 SYM, the gradient series is divergent: $f(w) = \sum_{n=0}^{\infty} f_n w^{-n}$

"Borel transformation + Resurgence" indeed follows the attractor

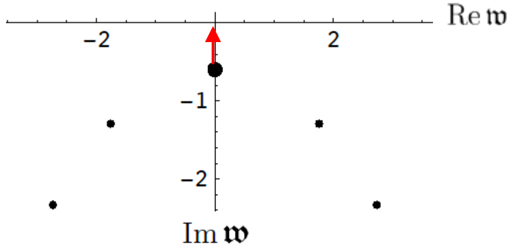
1. **How about the systems other than MIS?** [Romatschke, 1704.08699]
2. **Is it the reason for the fact that low order hydro accurately describes out of equ. systems with gradients of order unity?**
3. **The relation with the fluid/gravity geometry?**
4. **What are actually the non-hydro modes?**

2.1 Linearized hydro: Derivative expansion in momentum space:

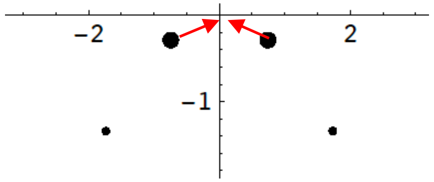
[Withers, 1803.08058]
[Grozdanov, Kovtun, Starinets, Tadic, 1904.01018]

Linearized hydro near equilibrium $P(\mathbf{q}^2,\omega) = F^2_{\text{shear}} F_{\text{sound}} = 0$

$\mathfrak{q} = 1$



$\mathfrak{q} \in \mathbf{R}$

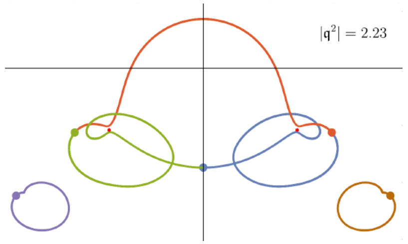


$\mathfrak{q} = 1$

[Kovtun, Starinets, 0506184]

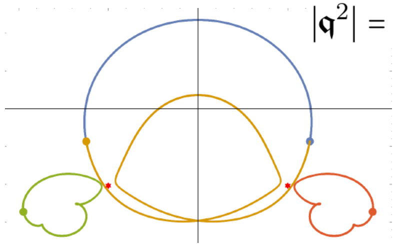
$$\mathfrak{w}_{\text{shear}} = -i\sum_{n=1}^{\infty} c_n \mathfrak{q}^{2n}$$ ← Derivative expansion in momentum space → $$\mathfrak{w}^{\pm}_{\text{sound}} = -i\sum_{n=1}^{\infty} a_n e^{\pm\frac{i\pi n}{2}} \mathfrak{q}^n$$

$|\mathfrak{q}^2| = 2.23$



$\mathfrak{q}^2 = |\mathfrak{q}^2|\, e^{i\theta}$

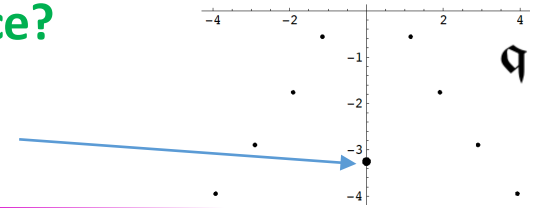
$|\mathfrak{q}^2| = 1.99$



Radius of convergence is set by $|\mathfrak{q}^2|$ of the point at which **level crossing** occurs.

1. **Why linearized hydro is able to catch the non-hydro modes, too? Non-linearities are unimportant?**
2. **How to interpret a hydro mode being outside its domain of convergence?**

   For the charge diffusion: $|\mathfrak{q}_c| \approx 0.72$ **Is it actually a hydro mode?**



$\mathfrak{q} = 1$

2.2 Real space vs. momentum space

[Heller, Serantes, Spalinski, Swensson,Withers,2007.05524]

[Withers, Holotube talk]

Reformulate **linear hydrodynamics** in terms of the fundamental blocks

$$\sigma_{jl} = \left(\partial_j u_l + \partial_l u_j - \frac{2}{d-1}\delta_{jl}\partial_r u^r\right),$$

$$\pi^{\epsilon}_{jl} = \left(\partial_j\partial_l - \frac{1}{d-1}\delta_{jl}\partial^2\right)\epsilon,$$

$$\pi^{u}_{jl} = \left(\partial_j\partial_l - \frac{1}{d-1}\delta_{jl}\partial^2\right)\partial_r u^r.$$

$$\Pi_{jl} = -A(\partial^2)\,\sigma_{jl} - B(\partial^2)\,\pi^u_{jl} - C(\partial^2)\,\pi^{\epsilon}_{jl}$$

Derivative expansion in real space: $A = \sum_{n=0}^{\infty} a_n \left(-\partial^2\right)^n$

Is convergent if: $|\mathfrak{q}_{max}|^2 < \min\left\{|\mathfrak{q}_c^{(A)}|^2, |\mathfrak{q}_c^{(B)}|^2, |\mathfrak{q}_c^{(C)}|^2\right\}$

Support of initial data in momentum space

1. **Does it show that speaking about the convergence of derivative expansion without referring to a specific solution is meaningless?**
2. **How to reconcile it with the divergence of the derivative expansion in the full non-linear hydro?**
3. **Mechanism for the compact support?** [Withers, Holotube talk]
4. **Is it possible to extract the hydrodynamic attractor from linearized hydro?**

3.1 Hydrodynamics from Effective action: **Top-Down**

[Glorioso, Liu 1805.09331]

The first question is: “**How to construct an action with its Euler-Lagrange’s equations being the hydro equations?”**

Diffusion model:

$$e^{W[A_{1\mu},A_{2\mu}]} = \mathrm{Tr}\left(\rho_0 \mathcal{P} e^{i\int d^dx\, A_{1\mu}J_1^\mu - i\int d^dx\, A_{2\mu}J_2^\mu}\right)$$

- Integrating out UV dof:

$$e^{W[A_{1\mu},A_{2\mu}]} = \int D\varphi_1 D\varphi_2\, e^{iI_{\mathrm{EFT}}[\varphi_1,\varphi_2;A_{1\mu},A_{2\mu}]}$$

- Demanding “gauge inv. + (EoM=Conservation equation)”:

$$e^{W[A_{1\mu},A_{2\mu}]} = \int D\varphi_1 D\varphi_2\, e^{iI_{\mathrm{EFT}}[B_{1\mu},B_{2\mu}]} \qquad B_{1\mu} \equiv A_{1\mu} + \partial_\mu \varphi_1$$

- Symmetries:

$I^*_{\mathrm{EFT}}(\varphi_1,\varphi_2;A_{1\mu},A_{2\mu}) = -I_{\mathrm{EFT}}(\varphi_2,\varphi_1;A_{2\mu},A_{1\mu})$

$\mathrm{Im}\, I_{\mathrm{EFT}} \geq 0$

$I_{\mathrm{EFT}}(\varphi_r, A_{r\mu}; \varphi_a = A_{a\mu} = 0) = 0$

dynamical KMS symmetry

Diagonal Shift symmetry

$$\mathcal{L}_{\mathrm{EFT}} = i\frac{\sigma}{\beta_0}B_{ai}^2 + \chi B_{a0}B_{r0} - \sigma B_{ai}\partial_0 B_{ri}$$

$$\partial_0 n - D\partial_i^2 n = -\sigma\partial_i E_i, \qquad n = \hat{J}_r^0$$

FD: $D = \dfrac{\sigma}{\chi}$ 2nd law: $\partial_\mu s^\mu = \beta_0 \sigma(\partial_i \mu)^2 \geq 0$ Onsager

1. **FD, 2nd law and Onsager relations are being obtained from the first principles.**
   Is it to say that there is no anymore constraint on the transport coefficients?

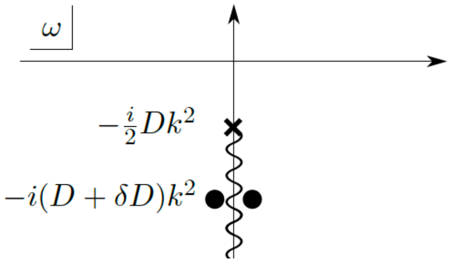
3.2 Applications

1) Long time tail in diffusion

[Chen-Lin, Delacretaz, Hartnoll 1811.12540]

$$\mathcal{L}_1^{\text{free}} = -\varphi_a\left(\partial_t\delta n - D\partial^2\delta n\right) + iT_0\sigma\partial^i\varphi_a\partial_i\varphi_a,$$
$$\mathcal{L}_1^{3\text{pt}} = \frac{1}{2}\lambda\delta n^2\partial^2\varphi_a + i\chi T_0\tilde{\lambda}\delta n\partial^i\varphi_a\partial_i\varphi_a,$$
$$\mathcal{L}_1^{4\text{pt}} = \frac{1}{3}\lambda_4\delta n^3\partial^2\varphi_a + i\chi T_0\tilde{\lambda}_4\delta n^2\partial^i\varphi_a\partial_i\varphi_a.$$

$$\Rightarrow \quad \langle\delta n(p)\varphi_a(-p)\rangle \equiv \frac{1}{F(p)+\Sigma(p)}, \qquad \Sigma(p) \sim \sqrt{k^2 - \frac{2i\omega}{D}}.$$



Long time tail: $\left\langle \frac{1}{2}\{J_a^i(t), J_b^j(0)\}\right\rangle = \frac{T}{\bar{w}}\frac{\delta^{ij}}{12}\left\{\frac{1}{[(D+\gamma_\eta)\,\pi|t|]^{3/2}}\chi\right\}_{ab} + (\text{exponential decay})$

2) Non-universality of hydro

[Jain, Kovtun 2009.01356]

$$\mathcal{L}_2^{4\text{pt}} = i\frac{\vartheta_1}{\chi^2}(\partial^i n\partial_i\varphi_a)^2 - i\frac{\vartheta_1+\vartheta_2}{\chi^2}(\partial^i n\partial_i n)(\partial^j\varphi_a\partial_j\varphi_a) + \cdots$$

Stochastic coefficients

$$\Rightarrow \quad \frac{\omega}{k^2}\,\text{Im}\,G_{ra} = \chi D + \frac{\chi^2\lambda^2 T_0}{32\pi D^{3/2}}\omega^{1/2}k^2 + \ldots - \frac{\lambda^2 T_0(\frac{2}{3}\vartheta_1+\vartheta_2)}{1024\pi^2 D^4}\omega^2 k^4 + \ldots.$$

Fluctuations break the derivative expansion

Not fixed by the constitutive relations

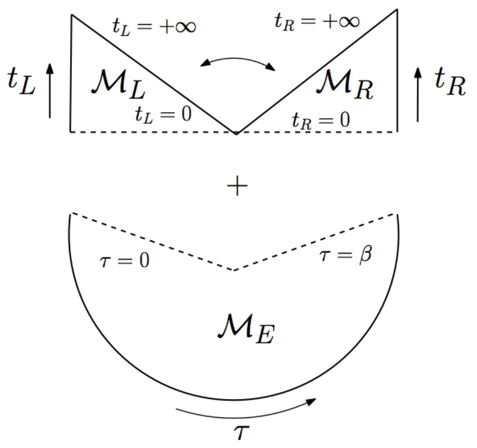
Classical transport coefficients are not enough!

1. Is the above long time tail just a confirmation of "long time tail" computed in [Kovtun, Yaffe 0303010] ?
2. Which of the following problems is more problematic for classical hydro?
   "Breakdown of derivative expansion" or "appearance of stochastic coefficients"
3. How to measure stochastic coefficients in experiment?

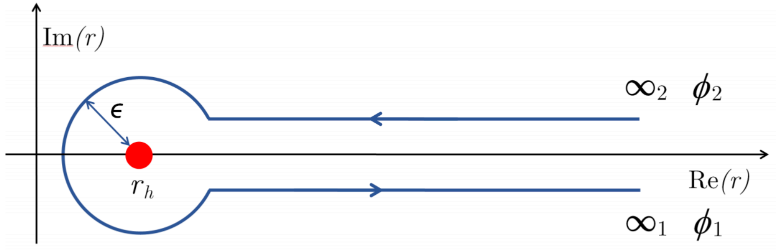
3.3 Holographic Schwinger-Keldysh contour

Limited to the diffusion model

[de Boer, Heller, Pinzani-Fokeeva 1812.06093]



[Glorioso, Crossley, Liu 1812.08785]



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Based on analytic continuation across the horizon

$$I_{\rm EFT}[B_{1\mu}, B_{2\mu}] = \tilde{S}_{\rm on-sell}|_{C_0(r_c)=0}$$

No analytic continuation across the horizon [Skenderis, van Rees 0812.2909]

$$i\,S_{eff} = i\,S_{\rm onshell}\Big|_{u_R=0} - i\,S_{\rm onshell}\Big|_{u_L=0} - S_{\rm onshell}\Big|_{u_E=0}$$

1. **Only quadratic action? How far is it from a complete picture in the bulk? What are the obstacles?**

2. **Do we need one-loop computation in Einstein gravity in the bulk to study the hydro fluctuation on the boundary? Finding stochastic coefficients from gravity?** [Caron-Huot, Saremi 0909.4525]

3. **Any relation to large order behavior?**

4.1 Instability in an uncharged fluid

[Kovtun 1907.08191]

To first order in derivatives: $T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)}$

$T^{\mu\nu}_{(1)}$ is ambiguous;

$$T^{\mu\nu} = \mathcal{E}u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (\mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu) + \mathcal{T}^{\mu\nu}$$

- This is due to the fact that out of equ. T and u^μ are not well-defined.
- Fixing the ambiguity = frame choosing
- Landau-Lifshitz frame: $u_\mu T^{\mu\nu}_{(1)} = 0$ → $T^{\mu\nu} = \epsilon u^\mu u^\nu + \left(p - \zeta\,\partial u\right)\Delta^{\mu\nu} - \eta\,\sigma^{\mu\nu}$

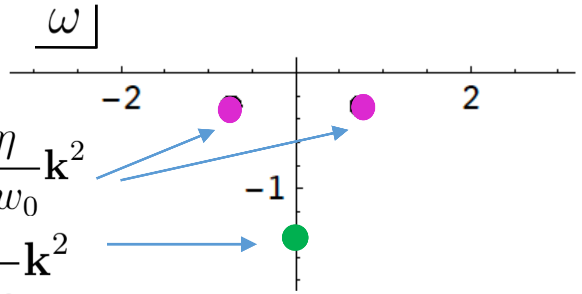
Perturbations about equilibrium: (conformal fluid)

- Fluid at rest $\mathbf{v} = \mathbf{0} + \delta\mathbf{v}$

  $T = T_0 + \delta T$

$$\omega = \pm\frac{1}{\sqrt{3}}|\mathbf{k}| - i\frac{\eta}{3w_0}\mathbf{k}^2$$

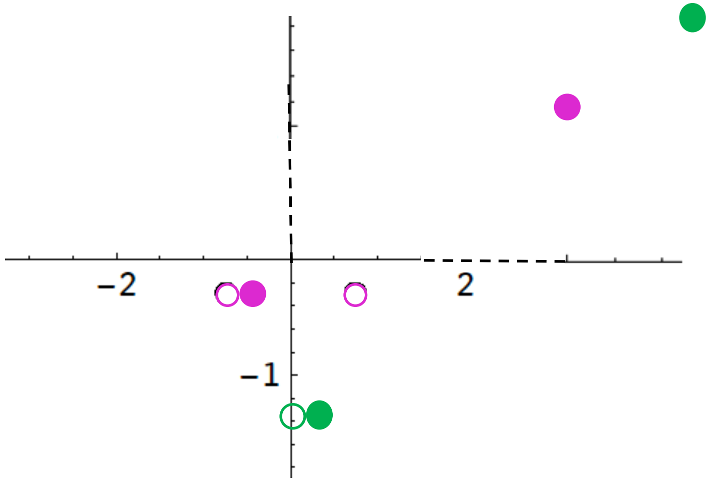
$$\omega = -i\frac{\eta}{2w_0}\mathbf{k}^2$$



All modes in the lower half plane: **stable**

- Moving Fluid $\mathbf{v} = \mathbf{v}_0 + \delta\mathbf{v}$

  $T = T_0 + \delta T$



Two of modes move to the upper half plane: **Unstable** $\mathrm{Im}\,\omega(\mathbf{k}) > 0$

[Hiscock, Lindblom 1985]

4.2 Stability of an uncharged fluid

[Bemfica, Disconzi, Noronha, 1708.06255]
[Kovtun 1907.08191]

In previous case

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + (p - \zeta\,\partial u)\Delta^{\mu\nu} - \eta\,\sigma^{\mu\nu}$$

$$F_{\text{shear}} = i w_0\,\omega - \mathbf{k}^2\,\eta$$

$$F_{\text{sound}} = -w_0\,\omega^2 - i\mathbf{k}^2 w_0\,\omega + \mathbf{k}^2 w_0^2 v_s^2$$

In general frame:

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (\mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu) + \mathcal{T}^{\mu\nu}$$

$$\left.\begin{aligned} \mathcal{E} &= \epsilon + \varepsilon_1 \dot{T}/T + \varepsilon_2 \partial_\lambda u^\lambda \\ \mathcal{P} &= p + \pi_1 \dot{T}/T + \pi_2 \partial_\lambda u^\lambda \\ \mathcal{Q}^\mu &= \theta_1 \dot{u}^\mu + \theta_2/T\,\Delta^{\mu\lambda}\partial_\lambda T \end{aligned}\right\} \Rightarrow$$

$$F_{\text{shear}} = \#\omega^2 + \cdots \Rightarrow \omega(\mathbf{k}) = \frac{i w_0 \sqrt{1-\mathbf{v}_0^2}}{\eta \mathbf{v}_0^2 - \theta}$$

$\theta > \eta > 0$

$$F_{\text{sound}} = \#\omega^4 + \cdots \Rightarrow \begin{cases} \omega(\mathbf{k}) = -i\dfrac{\epsilon_0+p_0}{v_s^2\varepsilon_1} + O(k^2) \\ \omega(\mathbf{k}) = -i\dfrac{\epsilon_0+p_0}{\theta} + O(k^2) \end{cases}$$

$\varepsilon_1 > 0\,, \quad \theta > 0$

Thus there exist more general frames in which, the first order hydro is stable.

1. Why do we trust the modes outside of regime of hydro?
2. Off-shell or on-shell? For example in a conformal fluid: $\varepsilon_1 = 3\pi_1\,, \quad \varepsilon_2 = 3\pi_2\,, \quad \pi_1 = 3\pi_2$

   If on-shell, $\dfrac{\dot{T}}{T} = -v_s^2\,\partial_\lambda u^\lambda$, then we are left only with η !
3. Does it mean that conformal fluid with on-shell constitutive relations is always unstable?
4. What about the fluid/gravity?