Does the hydrodynamic series converge?

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Based on:
1803.08058 - BW
2007.05524 - M. Heller, A. Serantes, M. Spaliński, V. Svensson, BW
2011.nnnnn - M. Heller, A. Serantes, M. Spaliński, V. Svensson, BW

THE ROYAL SOCIETY

STAG RESEARCH CENTRE

HoloTube
17 November 2020

Hydrodynamics is ubiquitous

$$\nabla_\mu \langle T^{\mu\nu} \rangle = 0$$

$$\langle T^{\mu\nu} \rangle = T^{\mu\nu}_{\text{ideal}} + \Pi^{\mu\nu}$$

$$\Pi^{\mu\nu} = \sum_{n=1}^{\infty} \Pi^{\mu\nu}_{(n)} \lambda^n$$

Perturbative expansion in derivatives

Each $\Pi^{\mu\nu}_{(n)}$ given by a sum over allowed symmetric structures

Transport coefficients encode microscopic data

Captures non-equilibrium processes of QFTs, black holes, etc.

$$\nabla_\mu \langle T^{\mu\nu} \rangle = 0$$

$$\langle T^{\mu\nu} \rangle = T^{\mu\nu}_{\rm ideal} + \Pi^{\mu\nu}$$

$$\Pi^{\mu\nu} = \sum_{n=1}^{\infty} \Pi^{\mu\nu}_{(n)} \lambda^n$$

Question 1: does this series converge?

Question 2: can the microscopic theory be recovered by suitable (re)summation?

“To converge or not to converge, that is the question.”

- Holotube organisers

Conformal hydrodynamics

e.o.s.

$$T^{\mu\nu}_{\text{ideal}} = (\epsilon + P)\,U^\mu U^\nu + P g^{\mu\nu}$$

$$\Pi^{\mu\nu} = -\eta\,\sigma^{\mu\nu} + \tau_\pi \eta\,\mathcal{D}\sigma^{\mu\nu} - \frac{1}{2}\theta_1\,\mathcal{D}_\alpha\mathcal{D}^\alpha\sigma^{\mu\nu} - \theta_2\,\mathcal{D}^{\langle\mu}\mathcal{D}^{\nu\rangle}\mathcal{D}_\alpha U^\alpha + \ldots\,,$$

(some terms suppressed)

shear tensor, one derivative of U

Transport coefficients $\eta, \tau_\pi, \theta_1, \theta_2, \ldots$ fixed by microscopic details

e.g. famously in any QFT with an Einstein dual $\eta = \frac{1}{4\pi}s$

To answer our questions we want to compute as many coefficients as we can

- holography to the rescue

Outline of the talk

1. **Bjorken flow (2013)**
2. **Dispersion relations (2018)**
3. **New real space results (2020)**

Spoiler / take-home message:
There is no intrinsic microscopic answer
It is *conditional* on the momentum-space support of a solution
The condition itself is intrinsic to the microscopic theory

(I will use q and k interchangeably for spatial momentum… sorry)

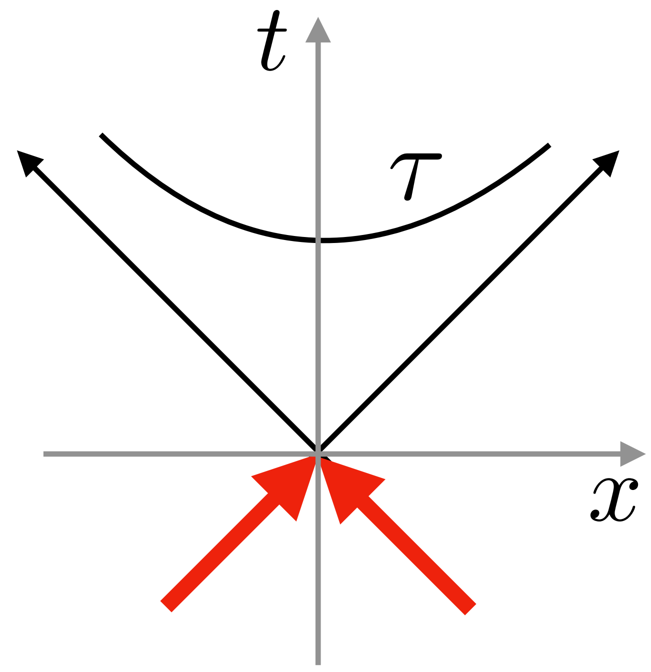
1. Bjorken flow

[Heller, Janik, Witaszczyk (2013)]

(& subsequently many other works)

Boost-invariant flow,
of interest in heavy-ion collisions

Depends only on $\tau = \sqrt{t^2 - x^2}$



Large τ expansion ~ hydrodynamic gradient expansion

$$\epsilon = \frac{1}{\tau^{4/3}} \left(\epsilon_2 + \frac{\epsilon_3}{\tau^{2/3}} + \frac{\epsilon_4}{\tau^{4/3}} + \ldots \right)$$

ϵ_n are transport coefficients, and in holography were generated to order 240

Found to be a **divergent series** $\epsilon_n \sim n!$

Resummation via Borel-Padé finds non-perturbative contributions

$$\delta\epsilon \sim \tau^{\alpha} \exp\left(-i\tfrac{3}{2}\omega_1(0)\tau^{2/3}\right)$$

expected since QNMs are non-perturbative in $1/\tau$

$$e^{-\gamma T(\tau)\tau} \qquad T(\tau) \sim \tau^{-1/3}$$

2. Dispersion relations

[BW 1803.08058]

Hydrodynamic equations of motion $\nabla_\mu \langle T^{\mu\nu} \rangle = 0$

admit plane wave solutions $\delta U \sim e^{iqx - i\omega(q)t}$

In the shear channel, $\omega(q) = -iDq^2 + O(q^4)$

part of a the hydro series $\omega(q) = \sum_{n=1}^{\infty} \omega_n q^{2n}$

For a black hole, this is a long-lived quasinormal mode

Quasinormal modes of RN-AdS4

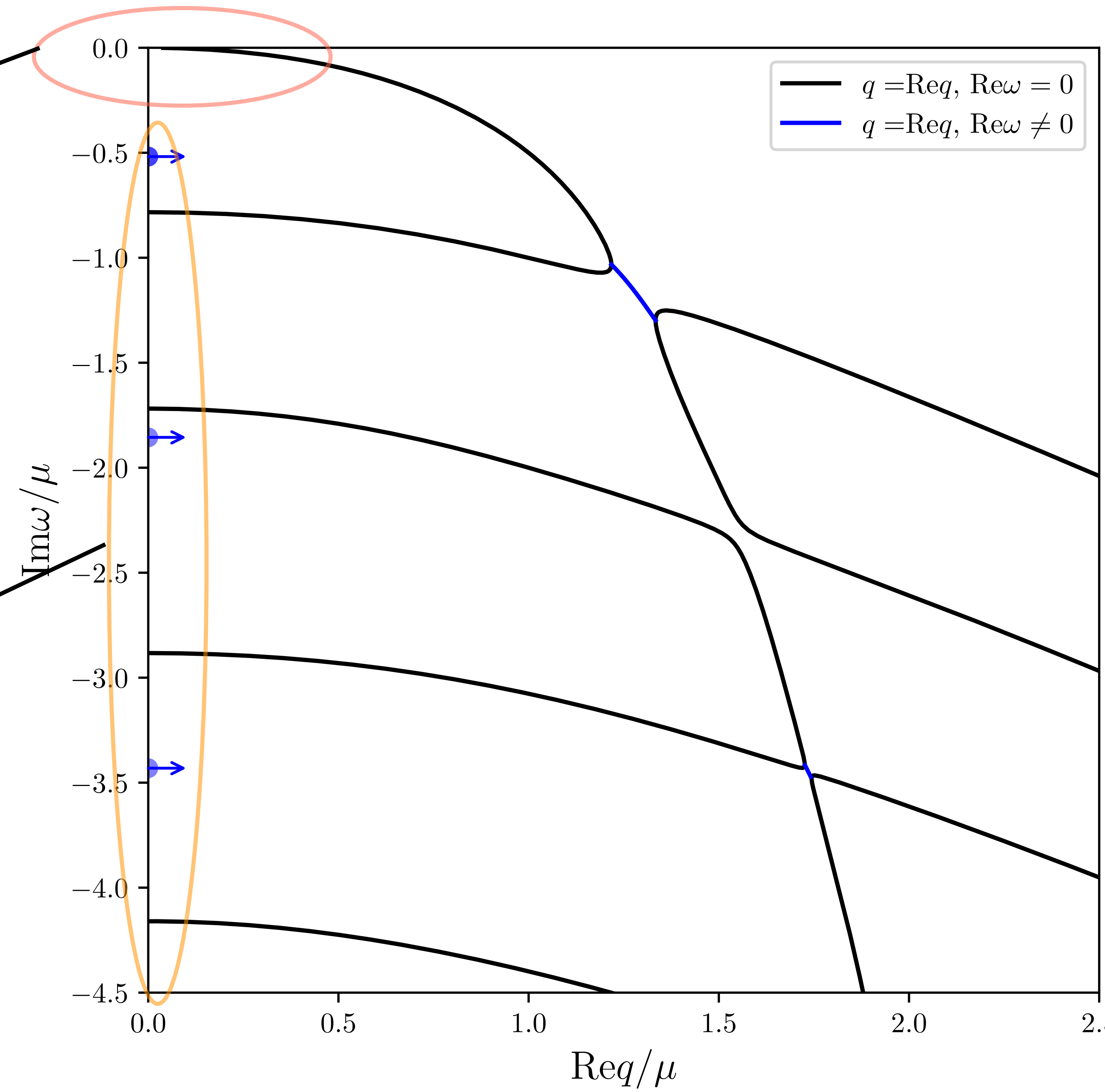
Model choice inspired by **[Brattan & Gentle 2010]** - movies depicting intricate trajectories of shear-mode poles

hydro

$$\omega(q) = \sum_{n=1}^{\infty} \omega_n q^{2n}$$

transients

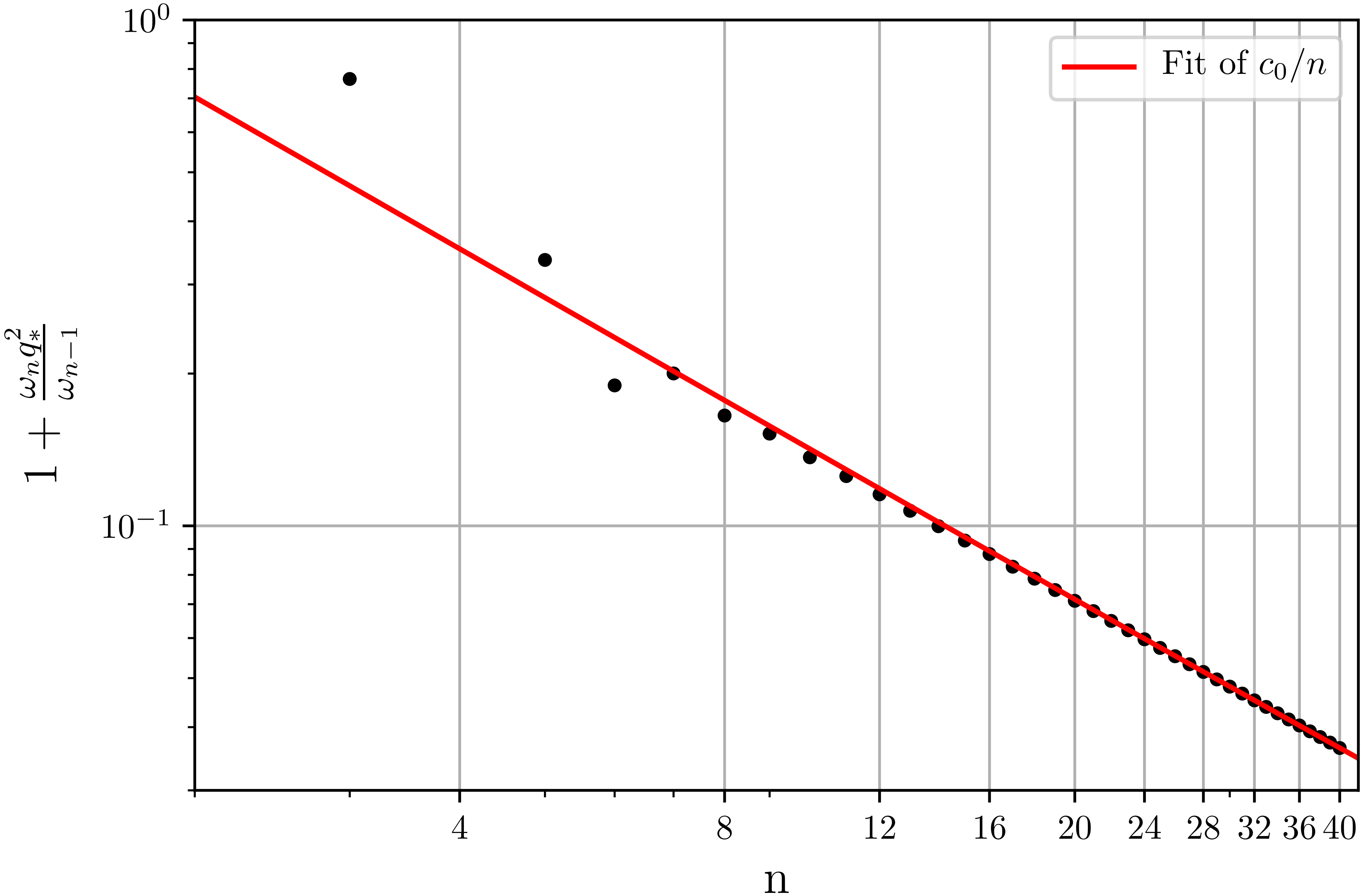
$$\omega_j(q) = -\frac{i}{\tau_j} + O(q)^2$$



Computed ω_n to hydrodynamic order ~80

Radius of convergence

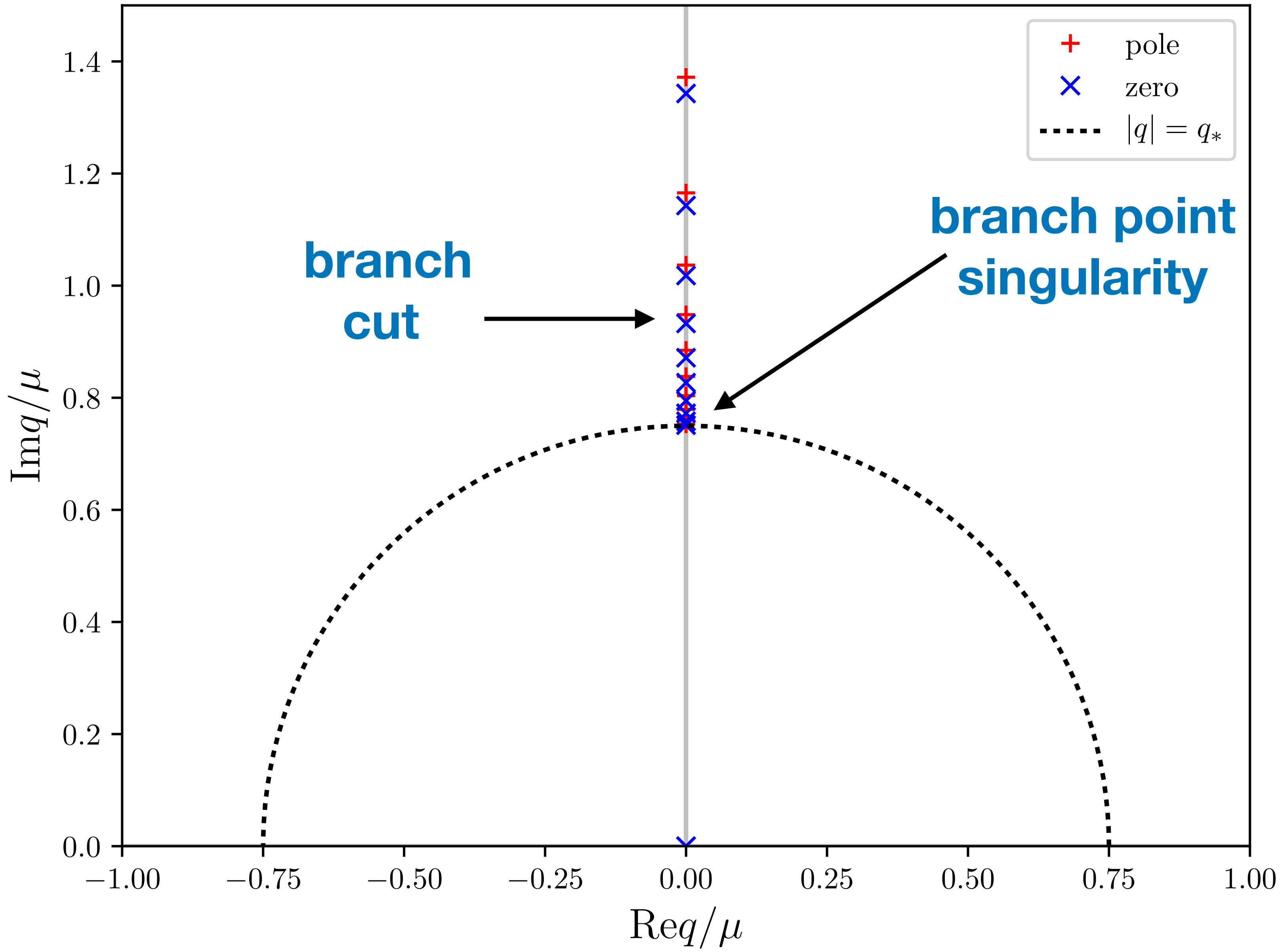
$$r_n \equiv \frac{\omega_n q_*^2}{\omega_{n-1}} \qquad q_* \equiv \frac{\epsilon+p}{2\mu\sqrt{\eta}}$$



- r_n converges to -1 at a rate $1/n$
- radius of convergence is q_* set by a singularity at $q = \pm i q_*$

Padé approximant reveals its nature,

$$\omega(q) = \sum_{n=1}^{\infty} \omega_n q^{2n} \longrightarrow \mathcal{P}_q(q) = \frac{\sum_{i=0}^{N} a_i q^i}{1 + \sum_{j=1}^{N} b_j q^j}$$



Branch-point singularity in the complex-q plane

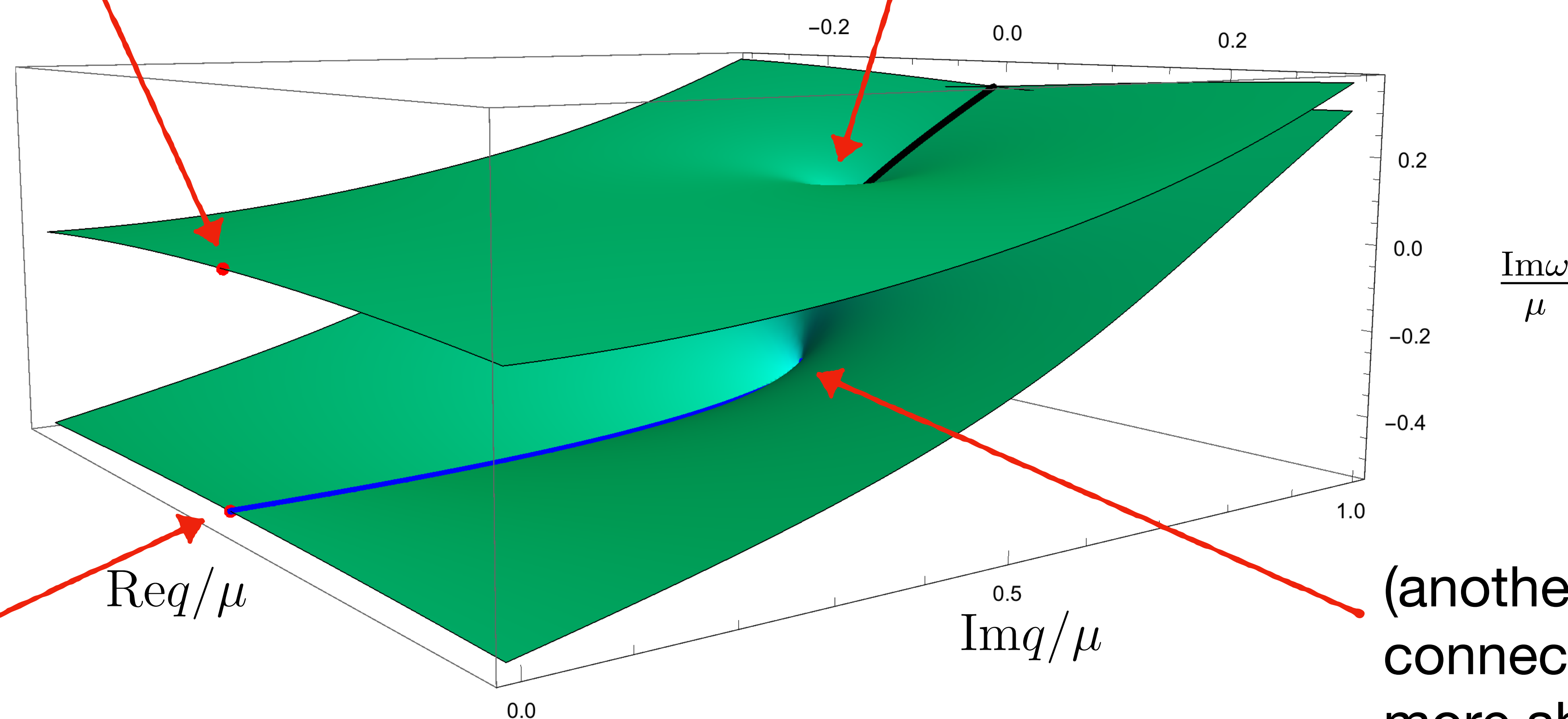
Suggests we should view $\omega(q)$ as a multi-sheeted Riemann surface

Hydrodynamics:
neighbourhood of
origin on sheet 1

$$\omega(q) = \sum_{n=1}^{\infty} \omega_n q^{2n}$$

radius of convergence set by

closest singularity to q=0
on hydrodynamic sheet of w(q)

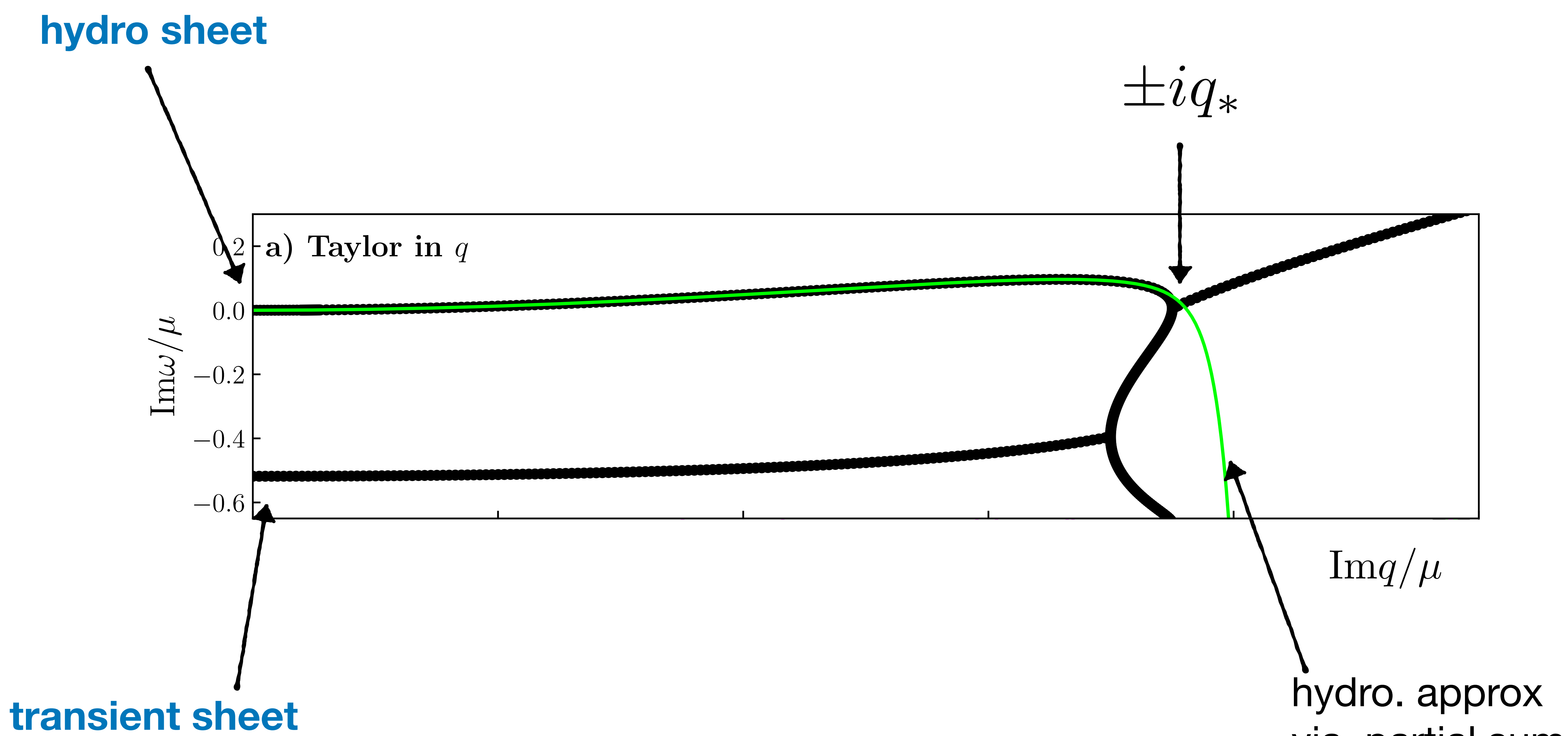


transient/
nonhydro
QNM

(another b.p.
connecting to
more sheets)

there are infinitely many sheets, all seem to be connected for RN-AdS4

A demonstrative slice:



Branch point singularities = well-known mode-collision phenomenon

But here at non-real q

Was not appreciated before that they set the radius of convergence

A note of caution:

$$\pm i q_* \qquad\qquad q_* \equiv \frac{\epsilon+p}{2\mu\sqrt{\eta}}$$

Analytically determined because this branch point appears in the e.o.m.

We have $q_* \to \infty$ as $\mu \to 0$

However, $\not\Longrightarrow$ $\text{radius} \to \infty$

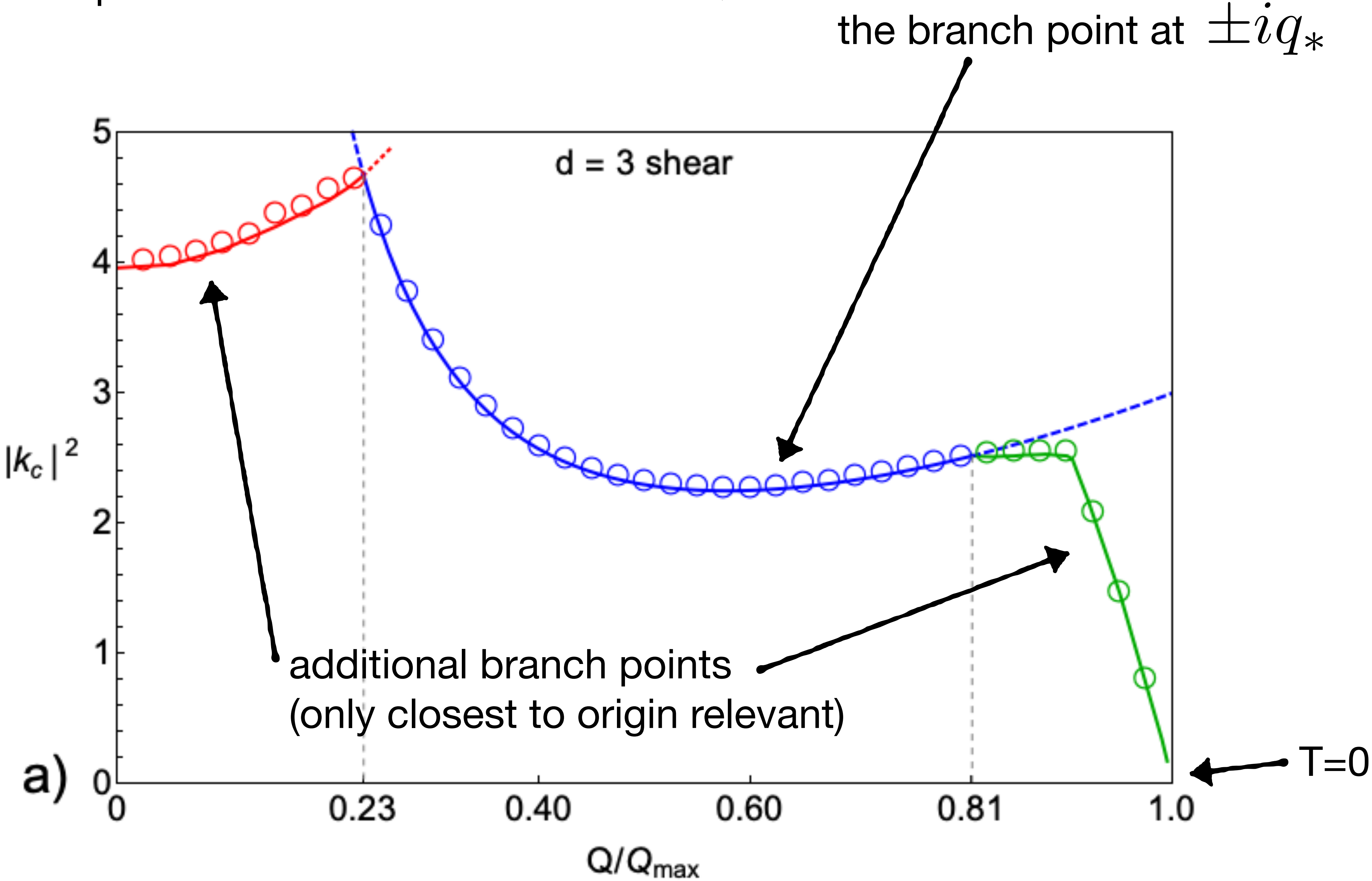
As we have seen, there can be many other branch points = obstructions to convergence.

However, several subsequent papers incorrectly made this step!

Nevertheless, at the value of Q studied, this branch point *does* set the radius. Full picture?

[A. Jansen, C. Pantelidou 2020]

Computed radius for RN-AdS4 for all Q



Curiously, analytic expression for radius in some interval, interval known numerically

see also [N. Abbasi, S. Tahery 2020]

Question 1: radius of convergence

treat $\omega(q)$ at $q \in \mathbb{C}$ describing Riemann surface

radius = $\lvert q \rvert$ of closest singularity to origin on hydrodynamic sheet

We saw mode collisions (*i.e. a branch point singularity*) for RN AdS4 at fixed Q

Collisions observed for several other examples, e.g.

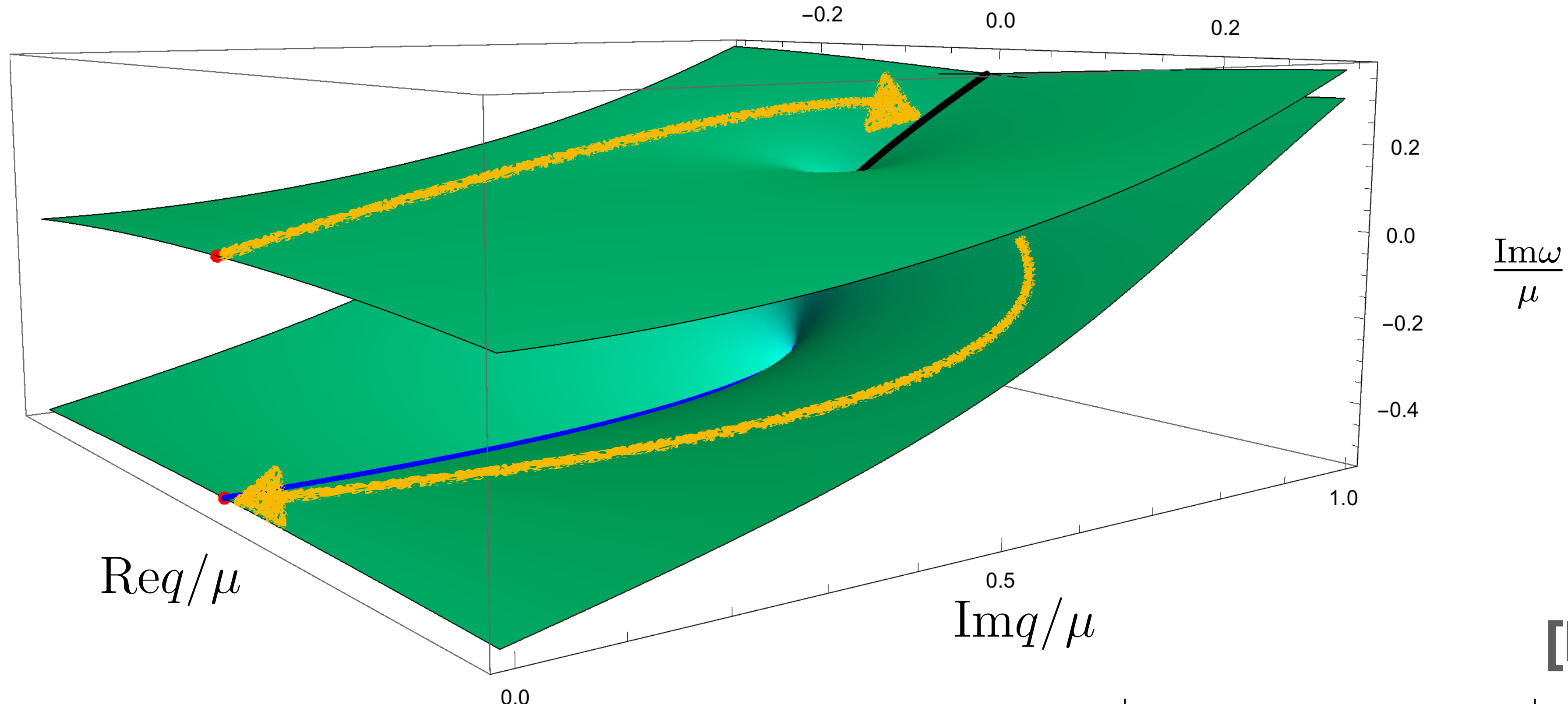
- Already-known dispersion relations of square-root-type (e.g. in MIS/BRSSS)
- Schwarzschild AdS5 **[S. Grozdanov, P. Kovtun, A. Starinets, P. Tadić (2019)]**
- RN AdS4 & RN AdS5 all Q **[A. Jansen, C. Pantelidou (2020)]**

Reasonable to propose that branch-points/mode-collisions are the generic case (but no proof)

a mathematically distinct proposal:
radius set by 'critical point of spectral curve'
A. Starinets on Holotube, 06 Oct. 2020

Question 2: can the microscopic theory be recovered by (re)summation?

Analytically continue the hydrodynamic data itself:



[BW 1803.08058]

hydro. order	$\omega(0)/\mu$ (2nd sheet)	$\omega(0)/\mu$ (3rd sheet)
3	$0.0276 + 0.2475i$	$-0.0276 + 0.2475i$
5	$-0.0312 + 0.0846i$	$0.0312 + 0.0846i$
7	$-1.3495 + 0.4494i$	$1.3495 + 0.4494i$
9	$-0.1133 - 0.3505i$	$0.1133 - 0.3505i$
11	$-0.8291 - 0.3872i$	$0.8291 - 0.3872i$
13	$-0.7264 - 0.4930i$	$0.7264 - 0.4930i$
⋮	⋮	⋮
77	$-0.7494 - 0.5182i$	$0.7494 - 0.5182i$
79	$-0.7493 - 0.5182i$	$0.7493 - 0.5182i$
exact (numerics)	$-0.7493 - 0.5182i$	$0.7493 - 0.5182i$

Recover a pair of transient black hole QNMs

3. New real space results

[M. Heller, A. Serantes, M. Spaliński, V. Svensson, BW (2007.05524)]


Speaker View
Exit Full Screen
Michal P. Heller
Ben Withers
Alex Serantes
Viktor Svensson
Michał Spaliński
Mute
Stop Video
5
Participants
Chat
Share Screen
Record
Reactions
Leave

What does real space divergence (e.g. Bjorken flows) have to do with w(k)?

Consider conformal hydrodynamics, *linearised*.

Key step: a novel reorganisation of $\Pi^{\mu\nu}$

Use eom to replace each ∂_t by a series in ∂_x (it's a redundancy)

In Landau frame, all tensor structures:

$$\sigma_{jl} = \left(\partial_j u_l + \partial_l u_j - \frac{2}{d-1} \delta_{jl} \partial_r u^r \right),$$

$$\pi^{\epsilon}_{jl} = \left(\partial_j \partial_l - \frac{1}{d-1} \delta_{jl} \partial^2 \right) \epsilon,$$

$$\pi^{u}_{jl} = \left(\partial_j \partial_l - \frac{1}{d-1} \delta_{jl} \partial^2 \right) \partial_r u^r.$$

Then $\Pi_{jl} = -A(\partial^2)\, \sigma_{jl} - B(\partial^2)\, \pi^{u}_{jl} - C(\partial^2)\, \pi^{\epsilon}_{jl},$

with $A = \sum_{n=0}^{\infty} a_n \left(-\partial^2\right)^n$

Counting transport coefficients & matching

odd odd even

$$\Pi_{jl} = -A(\partial^2)\,\sigma_{jl} - B(\partial^2)\,\pi^{u}_{jl} - C(\partial^2)\,\pi^{\epsilon}_{jl},$$

first order: 1
each subsequent odd order: 2
each even order: 1

This matches the counting of coefficients in $\omega_{\rm shear}(k), \omega^{\pm}_{\rm sound}(k)$

They can be mapped to each other,

$$A = \sum_{n=0}^{\infty} a_n \left(-\partial^2\right)^n \qquad a_n = [k^{2n+2}](isT\omega_{\rm shear})$$ & similarly for B,C

(this can be regarded as a matching calculation)

Consider a shear mode $\delta\epsilon = 0,\ \delta\mathbf{u} = (u_1(t,x), 0, \ldots, 0) \qquad x \equiv x^{d-1}$

$$\Pi_{1,d-1}(t,x) = -\sum_{n=0}^{\infty} a_n (-1)^n \partial_x^{2n+1} u_1(t,x).$$

Convergence via root test
(C<1 conv., C>1 div.)

$$C = \limsup_{n\to\infty} \left| a_n \partial_x^{2n+1} u_1 \right|^{\frac{1}{n}}$$

Since $a_n = [k^{2n+2}](isT\omega_{\text{shear}})$ w/ shear branch point k_*

Geometric growth $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = |k_*|^{-2}$

if $u_1(t,x)$ is compactly supported in momentum space to k_{max}

by *Paley-Wiener theorem* $\limsup_{n\to\infty} \left|\partial_x^{2n+1} u_1\right|^{\frac{1}{n}} = |k_{max}|^2$

Consider a shear mode $\delta\epsilon = 0,\ \delta\mathbf{u} = (u_1(t,x), 0, \ldots, 0) \qquad x \equiv x^{d-1}$

$$\Pi_{1,d-1}(t,x) = -\sum_{n=0}^{\infty} a_n (-1)^n \partial_x^{2n+1} u_1(t,x).$$

Convergence via root test
(C<1 conv., C>1 div.)

$$C = \limsup_{n\to\infty} \left| a_n \partial_x^{2n+1} u_1 \right|^{\frac{1}{n}}$$

$$= \frac{|k_{max}|^2}{|k_*|^2}$$

i.e. convergent if $|k_{max}|^2 < |k_*|^2$

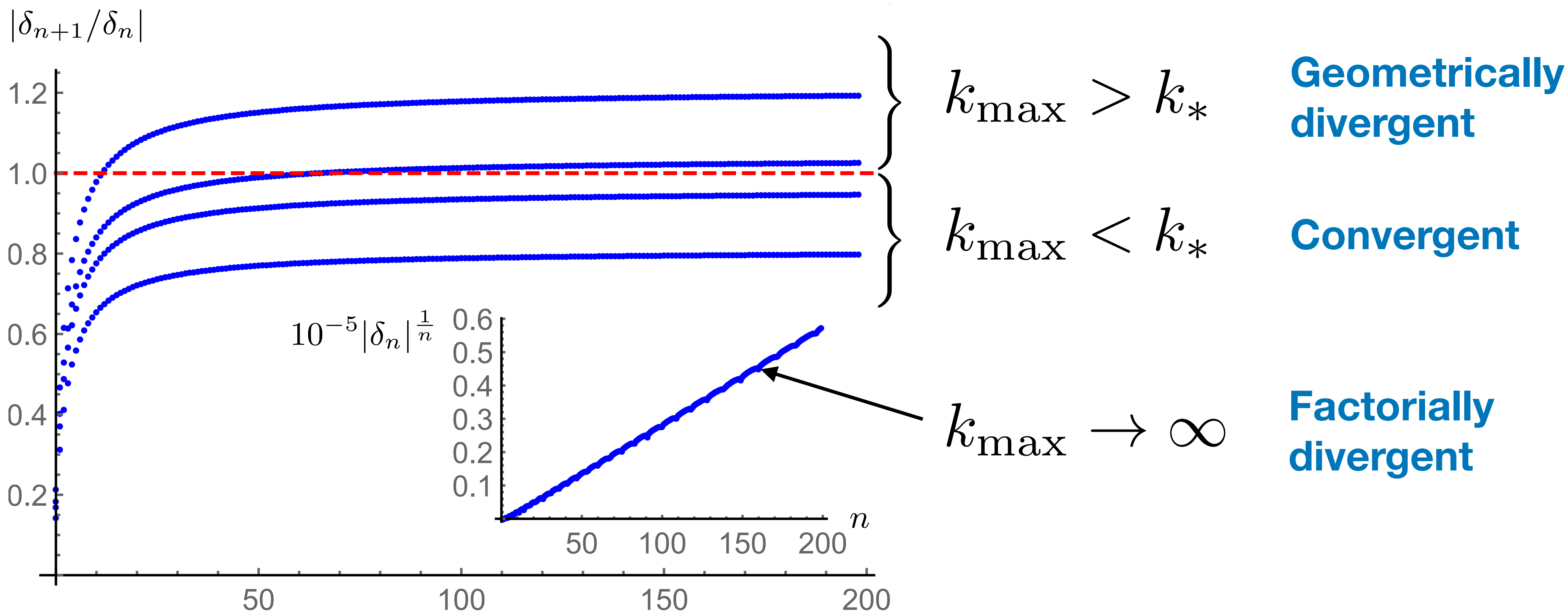
k-support is independent of time
can be evaluated on any time-slice
e.g. initial data

An example in Muller-Israel-Stewart theory

Toy microscopic model (2nd order, one transient) $a_n = sT\mathcal{C}_n D^{n+1}\tau_\pi^n$

Initial data $\hat{u}_1(0,k) = 0 \qquad \partial_t \hat{u}_1(0,k) = \frac{1}{2\pi} e^{-\frac{1}{2}\gamma^2 k^2}\Theta(k_{max}^2 - k^2)$

Ratio test



Summary

- Useful to view $\omega(q)$ as a multi-sheeted Riemann surface
- **Radius =** $|q|$ **of closest singularity to q=0 on hydro sheet**
- Appear to be set by mode collisions
  (branch point singularities of the Riemann surface)

Summary

- Explored consequences for real space

- Provided a new formulation of linearised hydrodynamics

$$\Pi_{jl} = -A(\partial^2)\,\sigma_{jl} - B(\partial^2)\,\pi^u_{jl} - C(\partial^2)\,\pi^\epsilon_{jl},$$

- Coefficients mapped to dispersion relations $\omega_{\rm shear}(k), \omega^{\pm}_{\rm sound}(k)$

- Connects real-space convergence with branch points of ω

- Allows us to definitively answer the main question:

  Convergence requires momentum support not exceed branch point location

- Convergence $\neq$ applicability

Outlook

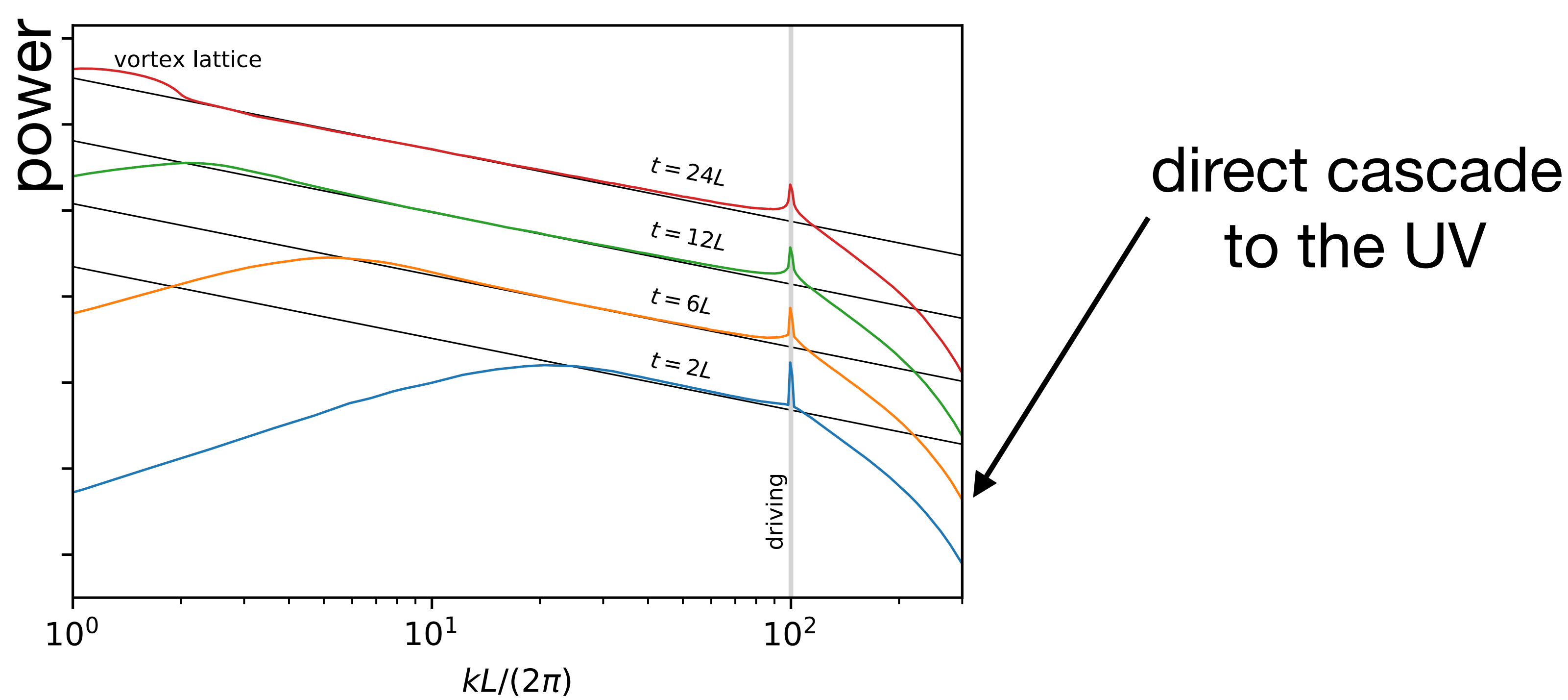
- **Question 2** in real space: new complex non-pt saddles of a Fourier integral
  [M. Heller, A. Serantes, M. Spaliński, V. Svensson, BW (upcoming)]

- Interactions! Can there be a mechanism for compact support?

  Lattices?

  Turbulence?

  **e.g.
  C. Pantelidou
  on Holotube,
  27 Oct. 2020**



Thank you for your attention!