Navier-Stokes to Maxwell via Einstein

Cindy Keeler

Arizona State University

November 10, 2020

arXiv:2005.04242 with T. Manton and N. Monga

Overview

Outline

- From Navier-Stokes to Einstein: fluid-gravity duality via a cutoff
- From Einstein to Maxwell: the classical double copy via Weyl
- Algebraic Speciality in Fluids
- Type D Fluids: constant vorticity
- Type N Fluids: potential flows
- Towards a general fluid?

Overview

Outline

- From Navier-Stokes to Einstein: fluid-gravity duality via a cutoff
- From Einstein to Maxwell: the classical double copy via Weyl
- Algebraic Speciality in Fluids
- Type D Fluids: constant vorticity
- Type N Fluids: potential flows
- Towards a general fluid?

History of Fluid/Gravity Duality

Membrane Paradigm

- Began with prescient thesis of Damour in 1978
- Consider fluctuations of a black hole horizon; these act like a viscous fluid
- Fluid viscosity is computed to be $\eta = 1/16\pi G$
- Dividing by the entropy density $s = 1/4G$ gives $\eta/s = 1/4\pi$
- Always considers fluctuations at the black hole horizon $r = r_h$ itself; produces Damour-Navier Stokes equation

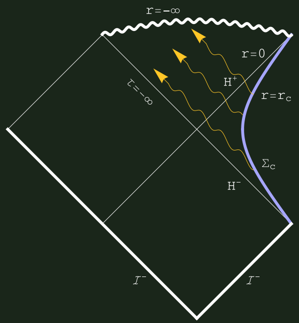
History of Fluid/Gravity Duality

AdS/CFT Method

- Policastro, Son, Starinets hep-th/0205052 considered the hydrodynamics of $\mathcal{N} = 4\, SU(N)$ SYM via AdS/CFT
- Again find $\eta/s = 1/4\pi$
- Performed at AdS spatial infinity $r = \infty$
- Result requires string theory, SUSY gauge theory, and AdS/CFT
- $\eta/s = 1/4\pi$ conjectured to be a bound on the viscosity to entropy ratio/ however can break via higher derivative corrections (e.g. Kats, Petrov, Buchel, Myers, Sinha, Cremonini, Vrigante, Liu, Liu, Shenker, Yaidi, Cai, Nie, Ohta, Sun, Banerjee, Dutta, Paulos, Escobedo, Smolkin, Dasgupta, Mia, Gale, Jeon . . .)
- Bound may still be satisfied by theories that have good causality (Camanho, Edelstein, Maldacena, Zhiboedov 1407.5597) but hyperbolicity may limit the causal restrictions (Papallo, Reall 1508.05303, Andrade, Caceres, CAK 1610.06078)

A 'Wilsonian' Approach

Fluid-gravity duality in the cutoff approach relates solutions of the incompressible Navier-Stokes equation



$$\partial^i v_i = 0, \quad \partial_\tau v_i - \bar{\eta}\partial^2 v_i + \partial_i P + v^j \partial_j v_i = 0$$

to solutions of the Einstein equation:

$$G_{\mu\nu} = 0$$

Bredberg, CAK, Lysov, Strominger, 1006.1902 and 1103.2355

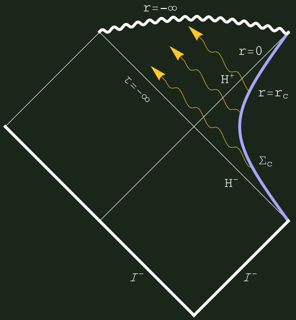
A ‘Wilsonian’ Approach

Fluid-gravity duality in the cutoff approach relates solutions of the incompressible Navier-Stokes equation

$$\partial^i v_i = 0, \quad \partial_\tau v_i - \bar{\eta}\partial^2 v_i + \partial_i P + v^j \partial_j v_i = 0$$

to solutions of the Einstein equation:

$$G_{\mu\nu} = 0$$



Fixing cutoff surface $r = r_c$, then perturbing:

- induced metric at $r = r_c$ is Ricci flat
- waves are infalling at $r = r_h$
- extrinsic curvature at $r = r_c$ becomes fluid stress tensor ...

Bredberg, CAK, Lysov, Strominger, 1006.1902 and 1103.2355

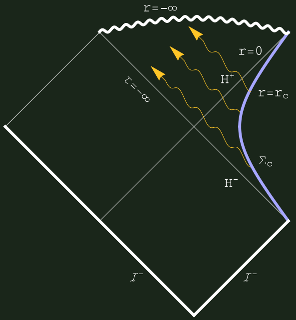
A ‘Wilsonian’ Approach

Fluid-gravity duality in the cutoff approach relates solutions of the incompressible Navier-Stokes equation

$$\partial^i v_i = 0, \quad \partial_\tau v_i - \bar{\eta}\partial^2 v_i + \partial_i P + v^j \partial_j v_i = 0$$

to solutions of the Einstein equation:

$$G_{\mu\nu} = 0$$



Fixing cutoff surface $r = r_c$, then perturbing:

- induced metric at $r = r_c$ is Ricci flat
- waves are infalling at $r = r_h$
- extrinsic curvature at $r = r_c$ becomes fluid stress tensor . . .
- in a hydrodynamic limit

Bredberg, CAK, Lysov, Strominger, 1006.1902 and 1103.2355

The Hydrodynamic Limit

Consider a solution of incompressible Navier-Stokes (v_i, P).
Now, rescale:

$$v_i^\epsilon(x^i, \tau) = \epsilon v_i(\epsilon x^i, \epsilon^2 \tau)$$
$$P_i^\epsilon(x^i, \tau) = \epsilon^2 P(\epsilon x^i, \epsilon^2, \tau)$$

These new quantities solve

$$\partial_\tau v_i^\epsilon - \bar{\eta} \partial^2 v_i^\epsilon + \partial_i P^\epsilon + v^{\epsilon j} \partial_j v_i^\epsilon = 0$$

which is again just Navier-Stokes.

To produce the hydrodynamic limit, we take $\epsilon \to 0$. This procedure will remove any corrections to N-S, as well as inducing incompressibility.

Satisfying the Einstein Constraints

The Nonlinear Metric in the Hydrodynamic Limit

$$
\begin{aligned}
ds^2 = & -r d\tau^2 + 2 d\tau dr + dx_i dx^i \\
& - 2\left(1 - \frac{r}{r_c}\right) v_i dx^i d\tau - 2\frac{v_i}{r_c} dx^i dr \\
& + \left(1 - \frac{r}{r_c}\right)\left[\left(v^2 + 2P\right) d\tau^2 + \frac{v_i v_j}{r_c} dx^i dx^j\right] + \left(\frac{v^2}{r_c} + \frac{2P}{r_c}\right) d\tau dr \\
& - \frac{\left(r^2 - r_c^2\right)}{r_c} \partial^2 v_i dx^i d\tau + \ldots \mathcal{O}(\epsilon^3)
\end{aligned}
$$

with $v_i \sim \mathcal{O}(\epsilon), \quad P \sim \mathcal{O}(\epsilon^2), \quad \partial_i \sim \mathcal{O}(\epsilon), \quad \partial_\tau \sim \mathcal{O}(\epsilon^2)$.

Satisfying the Einstein Constraints

The Nonlinear Metric in the Hydrodynamic Limit

$$
\begin{aligned}
ds^2 = & -r d\tau^2 + 2 d\tau dr + dx_i dx^i \\
& - 2\left(1 - \frac{r}{r_c}\right) v_i dx^i d\tau - 2\frac{v_i}{r_c} dx^i dr \\
& + \left(1 - \frac{r}{r_c}\right)\left[\left(v^2 + 2P\right) d\tau^2 + \frac{v_i v_j}{r_c} dx^i dx^j\right] + \left(\frac{v^2}{r_c} + \frac{2P}{r_c}\right) d\tau dr \\
& - \frac{\left(r^2 - r_c^2\right)}{r_c} \partial^2 v_i dx^i d\tau + \ldots \mathcal{O}(\epsilon^3)
\end{aligned}
$$

with $v_i \sim \mathcal{O}(\epsilon), \quad P \sim \mathcal{O}(\epsilon^2), \quad \partial_i \sim \mathcal{O}(\epsilon), \quad \partial_\tau \sim \mathcal{O}(\epsilon^2)$.

- Induced metric at $r = r_c$ cutoff is flat

Satisfying the Einstein Constraints

The Nonlinear Metric in the Hydrodynamic Limit

$$
\begin{aligned}
ds^2 = & -rd\tau^2 + 2d\tau dr + dx_i dx^i \\
& - 2\left(1 - \frac{r}{r_c}\right) v_i dx^i d\tau - 2\frac{v_i}{r_c} dx^i dr \\
& + \left(1 - \frac{r}{r_c}\right)\left[\left(v^2 + 2P\right) d\tau^2 + \frac{v_i v_j}{r_c} dx^i dx^j\right] + \left(\frac{v^2}{r_c} + \frac{2P}{r_c}\right) d\tau dr \\
& - \frac{\left(r^2 - r_c^2\right)}{r_c} \partial^2 v_i dx^i d\tau + \ldots \mathcal{O}(\epsilon^3)
\end{aligned}
$$

with $v_i \sim \mathcal{O}(\epsilon), \quad P \sim \mathcal{O}(\epsilon^2), \quad \partial_i \sim \mathcal{O}(\epsilon), \quad \partial_\tau \sim \mathcal{O}(\epsilon^2)$.

- Induced metric at $r = r_c$ cutoff is flat
- constraint eqns at $\mathcal{O}(\epsilon^2)$ are $\partial^i v_i = 0$

Satisfying the Einstein Constraints

The Nonlinear Metric in the Hydrodynamic Limit

$$
\begin{aligned}
ds^2 = &- r d\tau^2 + 2 d\tau dr + dx_i dx^i \\
&- 2\left(1 - \frac{r}{r_c}\right) v_i dx^i d\tau - 2\frac{v_i}{r_c} dx^i dr \\
&+ \left(1 - \frac{r}{r_c}\right)\left[\left(v^2 + 2P\right) d\tau^2 + \frac{v_i v_j}{r_c} dx^i dx^j\right] + \left(\frac{v^2}{r_c} + \frac{2P}{r_c}\right) d\tau dr \\
&- \frac{\left(r^2 - r_c^2\right)}{r_c} \partial^2 v_i dx^i d\tau + \ldots \mathcal{O}(\epsilon^3)
\end{aligned}
$$

with $v_i \sim \mathcal{O}(\epsilon), \quad P \sim \mathcal{O}(\epsilon^2), \quad \partial_i \sim \mathcal{O}(\epsilon), \quad \partial_\tau \sim \mathcal{O}(\epsilon^2)$.

- Induced metric at $r = r_c$ cutoff is flat
- constraint eqns at $\mathcal{O}(\epsilon^2)$ are $\partial^i v_i = 0$
- constraint eqns at $\mathcal{O}(\epsilon^3)$ are $\partial_\tau v_i - r_c \partial^2 v_i + \partial_i P + v^j \partial_j v_i = 0$, Navier-Stokes with viscosity $\bar{\eta} = r_c$

Satisfying the Einstein Constraints

The Nonlinear Metric in the Hydrodynamic Limit

$$\begin{aligned} ds^2 = & -r d\tau^2 + 2 d\tau dr + dx_i dx^i \\ & -2\left(1-\frac{r}{r_c}\right) v_i dx^i d\tau - 2\frac{v_i}{r_c} dx^i dr \\ & +\left(1-\frac{r}{r_c}\right)\left[\left(v^2+2P\right) d\tau^2 + \frac{v_i v_j}{r_c} dx^i dx^j\right] + \left(\frac{v^2}{r_c}+\frac{2P}{r_c}\right) d\tau dr \\ & -\frac{\left(r^2-r_c^2\right)}{r_c} \partial^2 v_i dx^i d\tau + \ldots \mathcal{O}(\epsilon^3) \end{aligned}$$

with $v_i \sim \mathcal{O}(\epsilon), \quad P \sim \mathcal{O}(\epsilon^2), \quad \partial_i \sim \mathcal{O}(\epsilon), \quad \partial_\tau \sim \mathcal{O}(\epsilon^2)$.

- Induced metric at $r = r_c$ cutoff is flat
- constraint eqns at $\mathcal{O}(\epsilon^2)$ are $\partial^i v_i = 0$
- constraint eqns at $\mathcal{O}(\epsilon^3)$ are $\partial_\tau v_i - r_c \partial^2 v_i + \partial_i P + v^j \partial_j v_i = 0$, Navier-Stokes with viscosity $\bar{\eta} = r_c$
- $G_{ra}, G_{ab}, G_{rr} = \mathcal{O}(\epsilon^4)$

Cutoff Approach

Highlights

- Does not require AdS, but is connectible to the AdS approach (Brattan, Camps, Loganayagam, Rangamani 1106.2577)
- Extendible to higher orders (Compere, McFadden, Skenderis, Taylor, 1103.3022; Pinzani-Fokeeva, Taylor 1401.5975)
- Hydrodynamic limit can be recast as near horizon limit
- Spacetime is algebraically special!

Petrov Type

Categorizes the multiplicities of principal null directions k^μ of the Weyl tensor W:

$$k_\mu k^\mu = 0, \qquad k_{[\sigma} W_{\mu]\nu\rho[\sigma} k_{\lambda]} k^\nu k^\rho = 0$$

- Spacetimes are algebraically special, or of higher Petrov type, when principal null vectors coincide. E.g. for Petrov type II, there exists a real null vector k^μ which satisfies

$$W_{\mu\nu\rho[\sigma} k_{\lambda]} k^\nu k^\rho = 0$$

- Generic 4d fluid-dual spacetimes are Petrov type II through $\mathcal{O}(\epsilon^{14})$ (Bredberg, Keeler, Lysov, Strominger 1101.2451
- More restricted fluids are more special!
- Petrov conditions can replace some boundary conditions in the cutoff approach (Lysov, Strominger 1104.5502)

Overview

Outline

- From Navier-Stokes to Einstein: fluid-gravity duality via a cutoff
- From Einstein to Maxwell: the classical double copy via Weyl
- Algebraic Speciality in Fluids
- Type D Fluids: constant vorticity
- Type N Fluids: potential flows
- Towards a general fluid?

From Einstein to Maxwell: The Classical Double Copy

Yang-Mills amplitudes $\mathcal{A}^{\text{YM}}$ (properly gauged) 'square' to gravity amplitudes $\mathcal{M}^{\text{grav}}$:

$$\mathcal{A}^{\text{YM}} \sim \sum_k \frac{n_k c_k}{props} \quad \longrightarrow \quad \mathcal{M}^{\text{grav}} \sim \sum_k \frac{n_k n_k}{props}$$

From Einstein to Maxwell: The Classical Double Copy

Yang-Mills amplitudes $\mathcal{A}^{\text{YM}}$ (properly gauged) 'square' to gravity amplitudes $\mathcal{M}^{\text{grav}}$:

$$\mathcal{A}^{\text{YM}} \sim \sum_k \frac{n_k c_k}{props} \quad \longrightarrow \quad \mathcal{M}^{\text{grav}} \sim \sum_k \frac{n_k n_k}{props}$$

Also scalar theory with amplitudes $\mathcal{A}^{\text{s}} \sim \sum_k c_k \tilde{c}_k / props$

For review see Bern, Carrasco, Chiodaroli, Johansson, Roiban 1909.01358

From Einstein to Maxwell: The Classical Double Copy

Yang-Mills amplitudes $\mathcal{A}^{\text{YM}}$ (properly gauged) 'square' to gravity amplitudes $\mathcal{M}^{\text{grav}}$:

$$\mathcal{A}^{\text{YM}} \sim \sum_k \frac{n_k c_k}{props} \quad \longrightarrow \quad \mathcal{M}^{\text{grav}} \sim \sum_k \frac{n_k n_k}{props}$$

Also scalar theory with amplitudes $\mathcal{A}^{\text{s}} \sim \sum_k c_k \tilde{c}_k / props$

For review see Bern, Carrasco, Chiodaroli, Johansson, Roiban 1909.01358

Kerr-Schild Map (Monteiro, O'Connell, White 1410.0239)

Pick metric in Kerr-Schild coordinates (with $k^2 = 0$):

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu \qquad \longrightarrow \qquad G_{\mu\nu} = 0$$

From Einstein to Maxwell: The Classical Double Copy

Yang-Mills amplitudes $\mathcal{A}^{\text{YM}}$ (properly gauged) 'square' to gravity amplitudes $\mathcal{M}^{\text{grav}}$:

$$\mathcal{A}^{\text{YM}} \sim \sum_k \frac{n_k c_k}{props} \quad \longrightarrow \quad \mathcal{M}^{\text{grav}} \sim \sum_k \frac{n_k n_k}{props}$$

Also scalar theory with amplitudes $\mathcal{A}^{\text{s}} \sim \sum_k c_k \tilde{c}_k / props$

For review see Bern, Carrasco, Chiodaroli, Johansson, Roiban 1909.01358

Kerr-Schild Map (Monteiro, O'Connell, White 1410.0239)

Pick metric in Kerr-Schild coordinates (with $k^2 = 0$):

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu \quad \longrightarrow \quad G_{\mu\nu} = 0$$

$$A_\mu = \phi k_\mu \quad \longrightarrow \quad \nabla_\nu F^{\mu\nu} = 0$$

From Einstein to Maxwell: The Classical Double Copy

Yang-Mills amplitudes $\mathcal{A}^{\text{YM}}$ (properly gauged) 'square' to gravity amplitudes $\mathcal{M}^{\text{grav}}$:

$$\mathcal{A}^{\text{YM}} \sim \sum_k \frac{n_k c_k}{props} \quad \longrightarrow \quad \mathcal{M}^{\text{grav}} \sim \sum_k \frac{n_k n_k}{props}$$

Also scalar theory with amplitudes $\mathcal{A}^{\text{s}} \sim \sum_k c_k \tilde{c}_k / props$

For review see Bern, Carrasco, Chiodaroli, Johansson, Roiban 1909.01358

Kerr-Schild Map (Monteiro, O'Connell, White 1410.0239)

Pick metric in Kerr-Schild coordinates (with $k^2 = 0$):

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu \quad \longrightarrow \quad G_{\mu\nu} = 0$$

$$A_\mu = \phi k_\mu \quad \longrightarrow \quad \nabla_\nu F^{\mu\nu} = 0$$

$$\phi \quad \longrightarrow \quad \nabla^2 \phi = 0$$

From Einstein to Maxwell: The Weyl Classical Double Copy

Luna, Monteiro, Nicholson, O'Connell 1810.08183

For Type D/N spacetimes with principal null vectors aligning in pairs/all four align

- Rewrite Weyl tensor in spinor notation: $C_{ABCD} = \frac{1}{4} W_{\mu\nu\lambda\gamma} \sigma^{\mu\nu}_{AB} \sigma^{\lambda\gamma}_{CD}$
- Decompose in principle spinors $C_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$
- $C^D_{ABCD} \sim \alpha_{(A}\alpha_B\beta_C\beta_{D)}$, $C^N_{ABCD} \sim \alpha_{(A}\alpha_B\alpha_C\alpha_{D)}$

From Einstein to Maxwell: The Weyl Classical Double Copy

Luna, Monteiro, Nicholson, O'Connell 1810.08183

For Type D/N spacetimes with principal null vectors aligning in pairs/all four align

- Rewrite Weyl tensor in spinor notation: $C_{ABCD} = \frac{1}{4} W_{\mu\nu\lambda\gamma} \sigma^{\mu\nu}_{AB} \sigma^{\lambda\gamma}_{CD}$
- Decompose in principle spinors $C_{ABCD} = \alpha_{(A} \beta_B \gamma_C \delta_{D)}$
- $C^D_{ABCD} \sim \alpha_{(A} \alpha_B \beta_C \beta_{D)}$, $C^N_{ABCD} \sim \alpha_{(A} \alpha_B \alpha_C \alpha_{D)}$

For these special spacetimes, can 'square root' the Weyl tensor:

$$C_{ABCD} = \frac{1}{S} f_{(AB} f_{CD)}$$

with $\nabla_0^2 S = 0$ and $f_{AB} \to F_{\mu\nu}$ satisfying $\nabla_0^\mu F_{\mu\nu} = 0$.

Overview

Outline

- From Navier-Stokes to Einstein: fluid-gravity duality via a cutoff
- From Einstein to Maxwell: the classical double copy via Weyl
- Algebraic Speciality in Fluids
- Type D Fluids: constant vorticity
- Type N Fluids: potential flows
- Towards a general fluid?

Algebraic Speciality in Fluids

Can prove algebraic speciality by writing

$$\begin{aligned} C_{ABCD} = & \Psi_0 \iota_A \iota_B \iota_C \iota_D - 4\Psi_1 o_{(A} \iota_B \iota_C \iota_{D)} + 6\Psi_2 o_{(A} o_B \iota_C \iota_{D)} \\ & - 4\Psi_3 o_{(A} o_B o_C \iota_{D)} + \Psi_4 o_A o_B o_C o_D \end{aligned}$$

If only Ψ_2 is nonzero, then the spacetime is type D.
If only Ψ_4 is nonzero, then the spacetime is type N.

Algebraic Speciality in Fluids

Can prove algebraic speciality by writing

$$
\begin{aligned}
C_{ABCD} =& \Psi_0 \iota_A \iota_B \iota_C \iota_D - 4\Psi_1 o_{(A} \iota_B \iota_C \iota_{D)} + 6\Psi_2 o_{(A} o_B \iota_C \iota_{D)} \\
& - 4\Psi_3 o_{(A} o_B o_C \iota_{D)} + \Psi_4 o_A o_B o_C o_D
\end{aligned}
$$

If only Ψ_2 is nonzero, then the spacetime is type D.
If only Ψ_4 is nonzero, then the spacetime is type N.
For general fluid-dual spacetimes, $\Psi_0, \Psi_1, \Psi_3 = 0 + \mathcal{O}(\epsilon^3)$,

$$
\begin{aligned}
\Psi_2 &= -i\epsilon^2 \left(\partial_x v_y - \partial_y v_x\right) / 4r_c + \mathcal{O}(\epsilon^3) \\
\Psi_4 &= -\epsilon^2 \left(\partial_x v_x - \partial_y v_y + i(\partial_x v_y + \partial_y v_x)\right) / 2r + \mathcal{O}(\epsilon^3).
\end{aligned}
$$

Algebraic Speciality in Fluids

Can prove algebraic speciality by writing

$$C_{ABCD} = \Psi_0 \iota_A \iota_B \iota_C \iota_D - 4\Psi_1 o_{(A}\iota_B \iota_C \iota_{D)} + 6\Psi_2 o_{(A} o_B \iota_C \iota_{D)} - 4\Psi_3 o_{(A} o_B o_C \iota_{D)} + \Psi_4 o_A o_B o_C o_D$$

If only Ψ_2 is nonzero, then the spacetime is type D.
If only Ψ_4 is nonzero, then the spacetime is type N.
For general fluid-dual spacetimes, $\Psi_0, \Psi_1, \Psi_3 = 0 + \mathcal{O}(\epsilon^3)$,

$$\Psi_2 = -i\epsilon^2 \left(\partial_x v_y - \partial_y v_x\right)/4r_c + \mathcal{O}(\epsilon^3)$$
$$\Psi_4 = -\epsilon^2 \left(\partial_x v_x - \partial_y v_y + i(\partial_x v_y + \partial_y v_x)\right)/2r + \mathcal{O}(\epsilon^3).$$

Type D fluid-dual spacetimes

$$\Psi_4 = 0 \quad \longrightarrow \quad v_x = -\omega y, \quad v_y = \omega x, \quad P = \omega^2 (x^2 + y^2)/2$$

if we choose τ independence. This fluid has constant vorticity ω.

Algebraic Speciality in Fluids

Can prove algebraic speciality by writing

$$C_{ABCD} = \Psi_0 \iota_A \iota_B \iota_C \iota_D - 4\Psi_1 o_{(A} \iota_B \iota_C \iota_{D)} + 6\Psi_2 o_{(A} o_B \iota_C \iota_{D)} - 4\Psi_3 o_{(A} o_B o_C \iota_{D)} + \Psi_4 o_A o_B o_C o_D$$

If only Ψ_2 is nonzero, then the spacetime is type D.
If only Ψ_4 is nonzero, then the spacetime is type N.
For general fluid-dual spacetimes, $\Psi_0, \Psi_1, \Psi_3 = 0 + \mathcal{O}(\epsilon^3)$,

$$\Psi_2 = -i\epsilon^2 \left(\partial_x v_y - \partial_y v_x\right)/4r_c + \mathcal{O}(\epsilon^3)$$

$$\Psi_4 = -\epsilon^2 \left(\partial_x v_x - \partial_y v_y + i(\partial_x v_y + \partial_y v_x)\right)/2r + \mathcal{O}(\epsilon^3).$$

Type N fluid-dual spacetimes

$\Psi_2 = 0 \quad \longrightarrow \partial_x v_y - \partial_y v_x = 0$ so vorticity vanishes.

Also incompressible so ‘potential flow’:

$$v_i = \partial_i \phi, \quad \partial_i P = -\partial_i \partial_\tau \phi - \partial^j \phi \partial_i \partial_j \phi.$$

Algebraic Speciality in Fluids

Can prove algebraic speciality by writing

$$\begin{aligned} C_{ABCD} = & \Psi_0 \iota_A \iota_B \iota_C \iota_D - 4\Psi_1 o_{(A}\iota_B \iota_C \iota_{D)} + 6\Psi_2 o_{(A} o_B \iota_C \iota_{D)} \\ & - 4\Psi_3 o_{(A} o_B o_C \iota_{D)} + \Psi_4 o_A o_B o_C o_D \end{aligned}$$

If only Ψ_2 is nonzero, then the spacetime is type D.
If only Ψ_4 is nonzero, then the spacetime is type N.
For general fluid-dual spacetimes, $\Psi_0, \Psi_1, \Psi_3 = 0 + \mathcal{O}(\epsilon^3)$,

$$\begin{aligned} \Psi_2 &= -i\epsilon^2 \left(\partial_x v_y - \partial_y v_x\right)/4r_c + \mathcal{O}(\epsilon^3) \\ \Psi_4 &= -\epsilon^2 \left(\partial_x v_x - \partial_y v_y + i(\partial_x v_y + \partial_y v_x)\right)/2r + \mathcal{O}(\epsilon^3). \end{aligned}$$

Algebraically special fluid-dual spacetimes

- Type D fluids have constant vorticity
- Type N fluids are potential flows

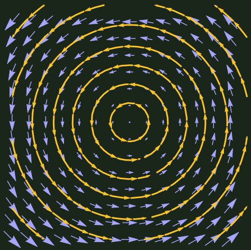
Type D fluids: Constant Vorticity

Only nonzero Ψ_I is

$$\Psi_2 = -i\epsilon^2\omega/2r_c + \mathcal{O}(\epsilon^3)$$

with natural background

$$ds^2_{(0)} = -r d\tau^2 + 2 dr d\tau + dx^2 + dy^2$$



Single and Zeroth Copies

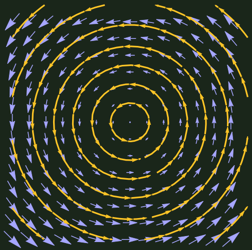
$$S = i\omega r_c e^{2i\theta}, \quad f_{AB} = e^{i\theta}\omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \begin{cases} F^{\tau r} = -\omega\cos\theta \\ F^{xy} = -\omega\sin\theta \end{cases}$$

- all other $F^{\mu\nu}$ components are zero
- S is constant so trivially solves $\nabla^2_{(0)} S = 0$
- $\nabla^{(0)}_\nu F^{\mu\nu} = 0, \quad \nabla^{(0)}_{[\mu} F_{\rho\sigma]} = 0$

Type D fluid single copy: A giant solenoid

Choosing $\theta = 3\pi/2$ we have

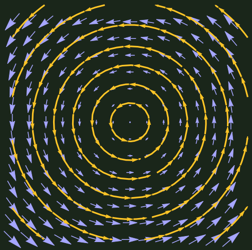
$$v_x = -\omega y, \quad v_y = \omega x$$
$$F^{\tau r} = 0, \quad F^{xy} = \omega$$
$$E_\mu = 0, \quad B_\mu = \omega \delta^r_\mu$$



Type D fluid single copy: A giant solenoid

Choosing $\theta = 3\pi/2$ we have

$$v_x = -\omega y, \quad v_y = \omega x$$
$$F^{\tau r} = 0, \quad F^{xy} = \omega$$
$$E_\mu = 0, \quad B_\mu = \omega \delta^r_\mu$$



Type D Fluid Double Copy Summary

- Fluid is solution inside of slowly rotating cylinder with no-slip conditions at the wall
- Magnetic field $\vec{B} = \omega \hat{r}$ is uniform field inside a big solenoid with current proportional to ω
- zeroth copy field S is constant and thus plays a passive role
- Fluid only in hydro regime for $x, y \sim \epsilon^{-1}$; can fix by going to near-horizon expansion instead

Type N fluids: Planar Extensional Flow

The simplest Type N fluid has $\phi = \frac{\alpha}{2}(y^2 - x^2)$, so

$$v_x = \partial_x \phi = -\alpha x, \; v_y = \partial_y \phi = \alpha y$$

The zeroth and single copy fields become

$$S = \frac{e^{2i\theta}}{\alpha}, \quad f_{AB} = \frac{e^{i\theta}}{\sqrt{r}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Again choosing $\theta = 3\pi/2$ the nonzero components of F become

$$F^{rx} = 1, \; F^{\tau x} = \frac{2}{r} \quad \longrightarrow \quad \vec{E} = -\hat{x}, \; \vec{B} = \hat{y}.$$

On the background $ds^2_{(0)} = -r d\tau^2 + 2 dr d\tau + dx^2 + dy^2$ again both Klein-Gordon and Maxwell's are solved.
Poynting vector is

$$\vec{S} = -\hat{r}.$$

Gauge field is single copy necessary to build up any fluid with a potential component.

Type N fluids: Planar Extensional Flow

The simplest Type N fluid has $\phi = \frac{\alpha}{2}(y^2 - x^2)$, so

$$v_x = \partial_x \phi = -\alpha x,\ v_y = \partial_y \phi = \alpha y$$

The zeroth and single copy fields become

$$S = \frac{e^{2i\theta}}{\alpha}, \quad f_{AB} = \frac{e^{i\theta}}{\sqrt{r}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Again choosing $\theta = 3\pi/2$ the nonzero components of F become

$$F^{rx} = 1,\ F^{\tau x} = \frac{2}{r} \quad \longrightarrow \quad \vec{E} = -\hat{x},\ \vec{B} = \hat{y}.$$

On the background $ds^2_{(0)} = -r d\tau^2 + 2 dr d\tau + dx^2 + dy^2$ again both Klein-Gordon and Maxwell's are solved.
Poynting vector is

$$\vec{S} = -\hat{r}.$$

Gauge field is single copy necessary to build up any fluid with a potential component.
What if we consider a different potential ϕ?

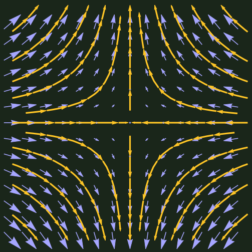
Type N fluids: Potential flow: The Double Copy Story

We already studied extensional flow: $\phi = \frac{\alpha}{2}(y^2 - x^2)$, so

$$v_x = \partial_x \phi = -\alpha x,\ v_y = \partial_y \phi = \alpha y$$

$$\vec{E} = -\hat{x}, \vec{B} = \hat{y}$$

But there are many other potential flow fluids!

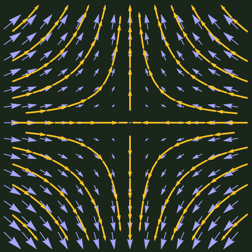


Type N fluids: Potential flow: The Double Copy Story

We already studied extensional flow: $\phi = \frac{\alpha}{2}(y^2 - x^2)$, so

$$v_x = \partial_x \phi = -\alpha x,\ v_y = \partial_y \phi = \alpha y$$

$$\vec{E} = -\hat{x}, \vec{B} = \hat{y}$$



But there are many other potential flow fluids!

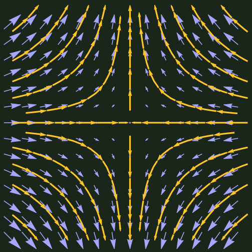
	Potential ϕ	v_x	v_y
Ext. flow	$-\frac{\alpha}{2}(x^2 - y^2)$	$-\alpha x$	αy
Source/Sink	$\ln(x^2 + y^2)$	$2x/(x^2 + y^2)$	$2y/(x^2 + y^2)$
Dipole	$x/(x^2 + y^2)$	$(y^2 - x^2)/(x^2 + y^2)^2$	$-2xy/(x^2 + y^2)^2$
Line Vortex	$\arctan(y/x)$	$-y/(x^2 + y^2)$	$x/(x^2 + y^2)$

Type N fluids: Potential flow: The Double Copy Story

We already studied extensional flow: $\phi = \frac{\alpha}{2}(y^2 - x^2)$, so

$$v_x = \partial_x \phi = -\alpha x,\ v_y = \partial_y \phi = \alpha y$$

$$\vec{E} = -\hat{x}, \vec{B} = \hat{y}$$



But there are many other potential flow fluids!

	Potential ϕ	v_x	v_y
Ext. flow	$-\frac{\alpha}{2}(x^2 - y^2)$	$-\alpha x$	αy
Source/Sink	$\ln(x^2 + y^2)$	$2x/(x^2 + y^2)$	$2y/(x^2 + y^2)$
Dipole	$x/(x^2 + y^2)$	$(y^2 - x^2)/(x^2 + y^2)^2$	$-2xy/(x^2 + y^2)^2$
Line Vortex	$\arctan(y/x)$	$-y/(x^2 + y^2)$	$x/(x^2 + y^2)$

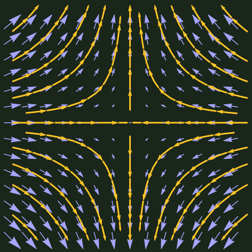
If $F_{\mu\nu}$ is just a 'support' single copy, then what distinguishes these fluids from each other?

Type N fluids: Potential flow: The Double Copy Story

We already studied extensional flow: $\phi = \frac{\alpha}{2}(y^2 - x^2)$, so

$$v_x = \partial_x \phi = -\alpha x,\ v_y = \partial_y \phi = \alpha y$$

$$\vec{E} = -\hat{x}, \vec{B} = \hat{y}$$



But there are many other potential flow fluids!

	Potential ϕ	v_x	v_y
Ext. flow	$-\frac{\alpha}{2}(x^2 - y^2)$	$-\alpha x$	αy
Source/Sink	$\ln(x^2 + y^2)$	$2x/(x^2 + y^2)$	$2y/(x^2 + y^2)$
Dipole	$x/(x^2 + y^2)$	$(y^2 - x^2)/(x^2 + y^2)^2$	$-2xy/(x^2 + y^2)^2$
Line Vortex	$\arctan(y/x)$	$-y/(x^2 + y^2)$	$x/(x^2 + y^2)$

If $F_{\mu\nu}$ is just a 'support' single copy, then what distinguishes these fluids from each other? S!

Type N fluids: Potential flow: The Double Copy Story

The potential ϕ resides in the zeroth copy scalar S .
We have

$$v_x = \partial_x \phi, \quad v_y = \partial_y \phi \text{ with } \phi = f(z) + \bar{f}(\bar{z})$$

The zeroth and single copy fields become

$$S = -\frac{e^{2i\theta}}{2\partial_{\bar{z}}^2 \bar{f}(\bar{z})}, \quad f_{AB} = \frac{e^{i\theta}}{\sqrt{r}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Type N fluids: Potential flow: The Double Copy Story

The potential ϕ resides in the zeroth copy scalar S .
We have

$$v_x = \partial_x \phi, \quad v_y = \partial_y \phi \text{ with } \phi = f(z) + \bar{f}(\bar{z})$$

The zeroth and single copy fields become

$$S = -\frac{e^{2i\theta}}{2\partial_{\bar{z}}^2 \bar{f}(\bar{z})}, \quad f_{AB} = \frac{e^{i\theta}}{\sqrt{r}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Type N Fluid Double Copy Summary

- $\nabla^2_{(0)} S = 0$ nontrivially; because $\phi = f(z) + \bar{f}(\bar{z})$
- ‘Background’ single copy field is still $\vec{E} = -\hat{x}$, $\vec{B} = \hat{y}$
- Poynting vector of single copy is $\vec{S} = -\hat{r}$.
- Gauge field is single copy necessary to build up any fluid with a potential component.

Algebraically Special Fluid Double Copy Summary

Type D Fluid Double Copy Summary

- Fluid: inside of slowly rotating cylinder with no-slip conditions
- Magnetic field $\vec{B} = \omega\hat{r}$ is uniform field inside a big solenoid with current proportional to ω
- zeroth copy field S is constant and thus plays a passive role
- Fluid only in hydro regime for $x, y \sim \epsilon^{-1}$; can fix by going to near-horizon expansion instead

Type N Fluid Double Copy Summary

- $\nabla^2_{(0)} S = 0$ nontrivially; because $\phi = f(z) + \bar{f}(\bar{z})$
- 'background' single copy field is still $\vec{E} = -\hat{x}$, $\vec{B} = \hat{y}$
- Poynting vector of single copy is $\vec{S} = -\hat{r}$.
- Gauge field is single copy necessary to build up any fluid with a potential component.

Future Directions

Future Questions

- higher orders in ϵ?
- generic incompressible fluids? Helmholtz decomposition $v_i = \partial_i \phi + \epsilon_{ij} \partial_j A_z$
- solution generating mechanisms
- relate to other fluid-gravity dualities, such as large D or AdS/CFT