Karlsruhe Institute of Technology

Superconductivity in SYK models

Gian Andrea Inkof, Jörg Schmalian, Koenraad Schalm

Based on PRB 100, 115132 (2019) and on unpublished results

Motivation

Goal: characterize a model for superconductivity in non Fermi liquid metals.

Why: conventional theory of superconductivity is inapplicable in non FL metals, notably for strongly interacting systems (e.g. cuprates).

Methods:

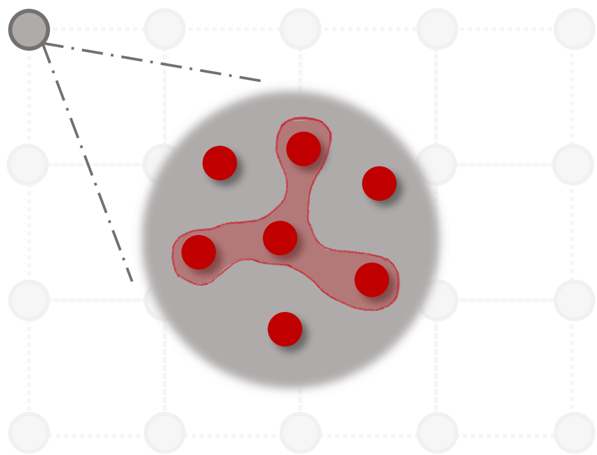
- Model with non-local disordered interaction (SYK)
- Holographic models

Outline

1. **Review of SYK**
    - The dot
2. **Superconducting SYK models**
    - The phonon– fermion dot
    - Higher dimensional generalization
3. **Conclusions and holographic perspective**

Karlsruhe Institute of Technology

The SYK dot



- 0+1 dimensional model with random all-to-all interaction among *N* fermion modes

$$H_{\text{int}} = J_{ijkl}\, \hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l$$

- Charged fermions → μ
- *J* are gaussian distributed random numbers

$$\overline{J_{ijkl}} = 0 \quad \overline{J_{ijkl}^2} = J^2/N^3$$

- Solvable in the large-*N* limit
- non-FL with emergent conformal symmetry an no quasiparticles

[Maldacena, Stanford (2016)]
[Davison, Fu, Georges, Gu, Jensen, Sachdev, (2017)]

The SYK dot effective action

Goal: thermodynamics ~ $\ln Z$ → Average log-terms is hard → **replica trick**

$$\overline{\ln Z} = \lim_{m\to 0} \frac{\overline{Z^m} - 1}{m}$$

$$\overline{Z^m} = \int \mathcal{D}\psi\, e^{-\sum_a S_0[\psi_a]}\left(\int dJ\, \varrho(J)\, e^{-\sum_a S_J[\psi_a]}\right) = \int \mathcal{D}[G,\Sigma] e^{-S_{\text{eff}}[G,\Sigma]}$$

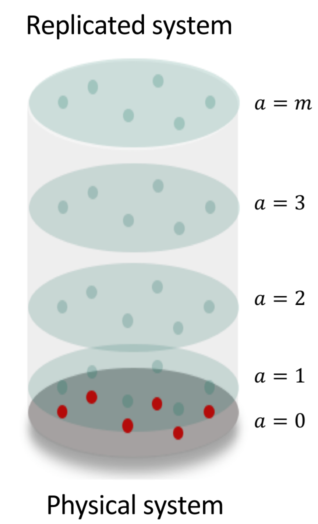
$$\frac{S_{\text{eff}}}{N} = -\mathrm{Tr}\ln[G_0^{-1}(\tau,\tau') - \Sigma(\tau,\tau')] - \int d^2\tau\, [\Sigma(\tau,\tau')G(\tau',\tau) + J^2 G^2(\tau,\tau')G^2(\tau',\tau)]$$

$$G_{ab}(\tau,\tau') = \frac{1}{N}\sum_i \hat{c}^\dagger_{i\,b}(\tau')\hat{c}_{i\,a}(\tau)$$

- G: cumulative field
- Σ: Lagrange multiplier
- Matrices in imaginary time and replica space

- At large N: saddle point approximation (= Dyson)

$$\Sigma(\tau) = -4J^2 G^2(\tau)G(-\tau)$$
$$G^{-1}(i\,\omega_n) = i\omega_n - \Sigma(i\omega_n)$$



Low energy effective theory

- Emergent *Diff* and $\mathrm{U}(1)$ symmetries in the low T and IR limit: $|\omega| \ll J$

$$G(\tau_1, \tau_2) = [f_1' f_2']^{\Delta} e^{i\phi_1 - i\phi_2}\, G(f_1, f_2)$$

Similar for Σ

- At $T\neq 0$ the saddle G_s is not invariant → SSB
- Action for the Goldstones: acting on G_s + quadratic expansion in f and ϕ

$$\frac{S[\phi,\epsilon]}{N} = \frac{K}{2}\int d\tau[\partial_\tau\phi + i\,(2\pi\,\mathcal{E}\,T)\,\partial_\tau\epsilon]^2 - \frac{\gamma}{2}\int d\tau\{\tan(\pi T[\tau + \epsilon(\tau)]), \tau\}$$

- $\partial_\tau\phi \;\leftrightarrow\;$ density fluctuations
- $\partial_\tau\epsilon \;\leftrightarrow\;$ energy fluctuations
- K charge compressibility
- γ specific heat

- **Higher dimension** extensions: diffusive behavior, FL to incoherent metal crossover, linear T-resistivity

[Davison, Fu, Georges, Gu, Jensen, Sachdev, (2017)]
[Song, Jian, Balents, (2017)]

SYK and superconductivity

Goal: model with anomalous terms

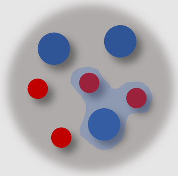
$$F_{\sigma\sigma'}(\tau,\tau') = \frac{1}{N}\sum_i \hat{c}_{i\sigma'}(\tau')\hat{c}_{i\sigma}(\tau)$$

$$G_{\sigma\sigma'}(\tau,\tau') = \frac{1}{N}\sum_i \hat{c}^\dagger_{i\sigma'}(\tau')\hat{c}_{i\sigma}(\tau)$$

Upgrade SYK-dot to spin-full fermions with an extra mechanism of attraction

- special correlations between matrix elements
- negative Hubbard
- phonons

$$H_{\text{int}} = g_{ijk}\hat{c}^\dagger_{i\sigma}\hat{c}_{j\sigma}\hat{\Phi}_k$$



Disorder average

$$O_{ijk} \sim \hat{c}^\dagger{}_{i\sigma a}(\tau)\hat{c}_{j\sigma a}(\tau)\hat{\Phi}_{ka}(\tau)$$

- $\overline{e^{-g_{ijk}O_{ijk}}}\Big|_{\text{GUE}} = e^{2g^2 O^\dagger_{ijk}O_{ijk}}$
- $\overline{e^{-g_{ijk}O_{ijk}}}\Big|_{\text{GOE}} = e^{g^2(O^\dagger_{ijk}+O_{ijk})^2}$

Anomalous F-terms are allowed

[Patel et al. (2018) – Chowdhury, Berg (2019) – Wang et al. (2020)]
[Esterlis, Schmalian (2019)]

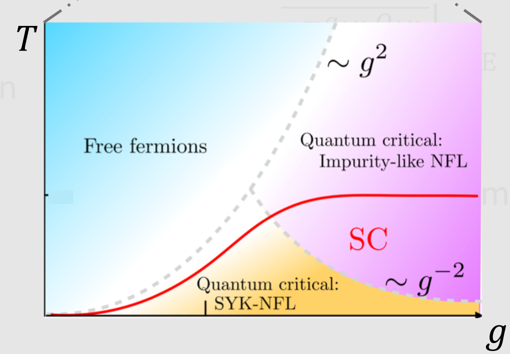
SYK and superconductivity

$$\frac{S_{\text{eff}}}{N} = -\operatorname{Tr}\ln[\bar{G}_0^{-1} - \bar{\Sigma}] + \frac{1}{2}\operatorname{Tr}\ln[D_0^{-1} - \Pi]$$

$$-2\int d^2\tau\, G(\tau',\tau)\,\Sigma(\tau,\tau') + \tfrac{1}{2}\int d^2\tau\, D(\tau',\tau)\,\Pi(\tau,\tau')$$

$$-2\int d^2\tau \left(F(\tau',\tau)\Phi^\dagger(\tau,\tau') + F^\dagger(\tau',\tau)\Phi(\tau,\tau')\right)$$

$$+g^2\int d^2\tau \left(G(\tau',\tau)G(\tau,\tau') - F^\dagger(\tau,\tau')\mathrm{F}(\tau',\tau)\right) D(\tau,\tau')$$



[Patel et al. (2018) – Chowdhury, Berg (2019) – Wang et al. (2020)]
Figure from [Esterlis, Schmalian (2019)]

Testing superconductivity

- Perfect conductor: **finite Drude weight**
- Perfect diamagnet: **Meissner effect**

Need for spatial dimensions

$$J_\alpha(\omega, \boldsymbol{q}) = K_{\alpha\beta}(\omega, \boldsymbol{q}) A_\beta(\omega, \boldsymbol{q})$$

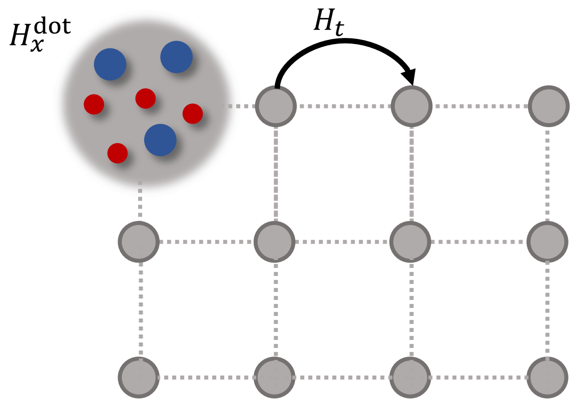
- $\sigma_l(\omega, \boldsymbol{q}) = -i \frac{K_l(\omega, \boldsymbol{q})}{\omega + i\, 0^+}$

$\lim_{\omega \to 0} K_l(\omega, \boldsymbol{q} = \boldsymbol{0}) \neq 0$

- $\boldsymbol{B}(\boldsymbol{r}) \sim \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \frac{i\, \boldsymbol{q} \times \boldsymbol{j}_{\mathrm{ext}}}{q^2 - K_t(q, 0)}\, e^{i\, \boldsymbol{q} \cdot \boldsymbol{r}}$

$K_t(\omega = 0, \boldsymbol{q})$ vanishes faster than $\boldsymbol{q}^2$

Higher dimensional SYK



SYK_2- hopping

$$H_t = \sum_{\langle \boldsymbol{r},\boldsymbol{r}' \rangle} t_{ij,\boldsymbol{r}\boldsymbol{r}'} \; \hat{c}^{\dagger}_{j\sigma\boldsymbol{r}'} \hat{c}_{i\sigma\boldsymbol{r}}$$

- $t \in \mathbb{R}$ **superconducting state**
- $t \in \mathbb{C}$ **normal state**

$$S_{\mathrm{eff}} = \sum_{\boldsymbol{r}} S_{\mathrm{dot},\boldsymbol{r}} + N t_0^2 \sum_{\langle \boldsymbol{r},\boldsymbol{r}' \rangle} \int d^2\tau \left(G_{\boldsymbol{r}}(\tau',\tau) G_{\boldsymbol{r}'}(\tau,\tau') - F^{\dagger}_{\boldsymbol{r}}(\tau,\tau') F_{\boldsymbol{r}'}(\tau',\tau) \right)$$

Phase mode action

- $G_{\boldsymbol{r}}(\tau,\tau') = e^{i(\phi(\tau,r)-\phi(\tau',r))} G_s(\tau-\tau')$
- $\Sigma_{\boldsymbol{r}}(\tau,\tau') = e^{i(\phi(\tau,r)-\phi(\tau',r))} \Sigma_s(\tau-\tau')$
- $F_{\boldsymbol{r}}(\tau,\tau') = e^{i(\phi(\tau,r)+\phi(\tau',r))} F_s(\tau-\tau')$
- $\Phi_{\boldsymbol{r}}(\tau,\tau') = e^{i(\phi(\tau,r)+\phi(\tau',r))} F_s(\tau-\tau')$

$$\frac{S[\phi]}{N} = T \sum_{p,\omega_n} D^{-1}_{p,\omega_n} \phi_{-p\,\omega_n} \phi_{p\,-\omega_n}$$

$$D^{-1}_{p,\omega_n} = \omega_n^2 \Lambda_{\omega_n} - p^2 t_0^2\, L_{\omega_n}$$

$$\Lambda_G(\tau) = G(\tau)G(-\tau)$$
$$\Lambda_F(\tau) = F^\dagger(\tau)F(-\tau)$$
$$\Lambda(\tau) = \Lambda_G(\tau) + 4\Lambda_F(\tau)$$

$$L(i\omega_n) = t_0^2(\Lambda_G(0) - \Lambda_G(i\,\omega_n) - \Lambda_F(0) - \Lambda_F(i\,\omega_n))$$

The kernel

- Peierls' substitution

$$\hat{c}_{\boldsymbol{r}i\sigma}(\tau) \mapsto \hat{c}_{\boldsymbol{r}i\sigma}(\tau) e^{-ie\int_{\boldsymbol{r_0}}^{\boldsymbol{r}} d\boldsymbol{x}\cdot \boldsymbol{A}(\tau,\boldsymbol{x})}$$

- Phase mode mapping

$$i\, p_\alpha \phi(p,\tau) \mapsto i\, p_\alpha \phi(p,\tau) - e\, A_\alpha(\boldsymbol{r},\tau)$$

$$\frac{S[\phi,\boldsymbol{A}]}{N} = \frac{S[\phi]}{N} - \int_p j_\alpha(p) A_\alpha(-p) + \frac{1}{2}\int_p A_\alpha(p) m_{\alpha\beta}(\omega_n) A_\beta(-p)$$

$$Z = \int \mathcal{D}\phi\, e^{-S[\phi,\boldsymbol{A}]} \qquad K_{\alpha\beta}(\boldsymbol{p}, i\omega_n) = \left.\frac{\delta^2 \ln Z}{\delta A_\beta(p)\delta A_\alpha(-p)}\right|_{A=0}$$

$$K_l(\boldsymbol{p}, i\omega_n) = 2\,N\,e^2 L(i\omega_n)\left(1 - \frac{2L(i\omega_n)\boldsymbol{p}^2}{\omega_n^2\Lambda(i\omega_n) - \boldsymbol{p}^2 L(i\omega_n)}\right)$$

$$K_t(\boldsymbol{p}, i\omega_n) = 2\,N\,e^2 L(i\omega_n)$$

$$K_l(\boldsymbol{p}=0, i\omega_n \to 0) \propto L(0)$$

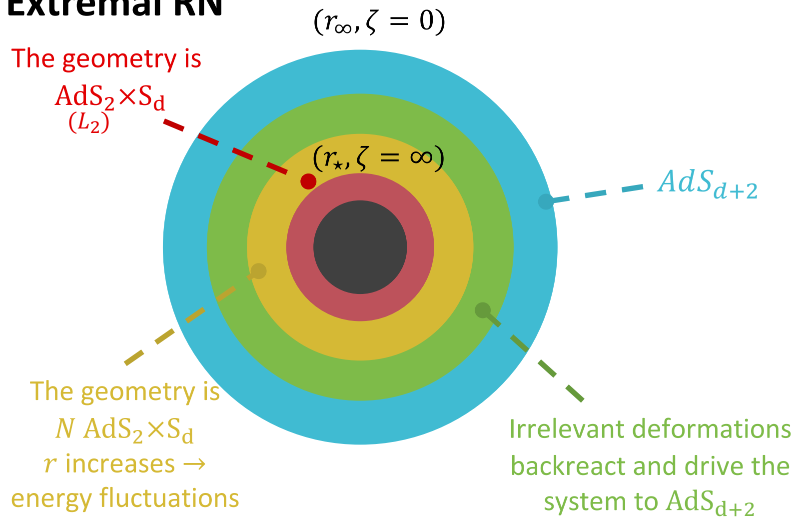
$$K_t(\boldsymbol{p}\to 0, i\omega_n = 0) \propto L(0)$$

- $L(0) = -2t_0^2\Lambda_F(0) \propto \int d\tau\, F^\dagger(\tau)F(-\tau)$ finite only when $t \in \mathbb{R}$
- The model has finite Drude weight and stiffness only in the superconducting state

Holography

SYK $\approx$ **Extremal RN**

- Maximally chaotic
- Strongly coupled
- Emergent conformal invariance



Holography

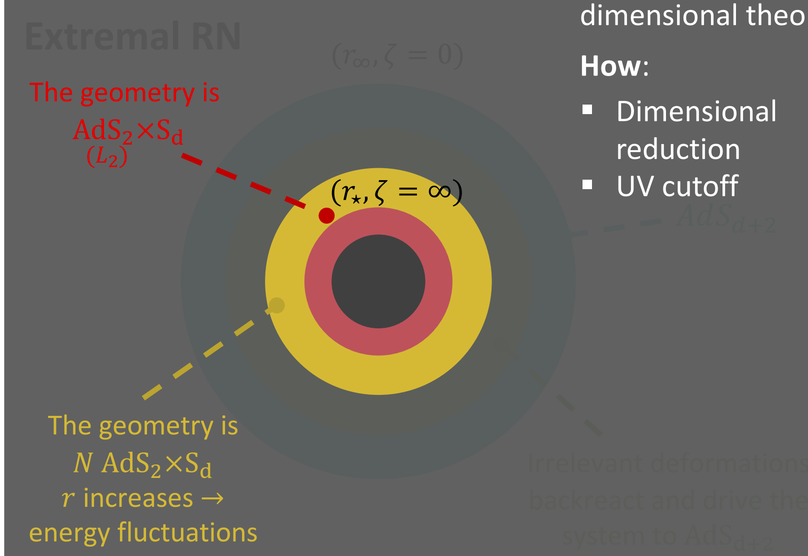
SYK ≈

- Maximally chaotic
- Strongly coupled
- Emergent conformal invariance

Task: write the 2-dimensional theory

How:

- Dimensional reduction
- UV cutoff



Holography

SYK

- Maximally chaotic
- Strongly coupled
- Emergent conformal invariance

Dimensional reduction

$$S_{EM} = S_{EM}[g_{\mu\nu}, A_\mu]$$

$$ds^2 = \frac{ds_2^2}{\Phi^{d-1}} + \Phi^2 d\Omega_d^2$$

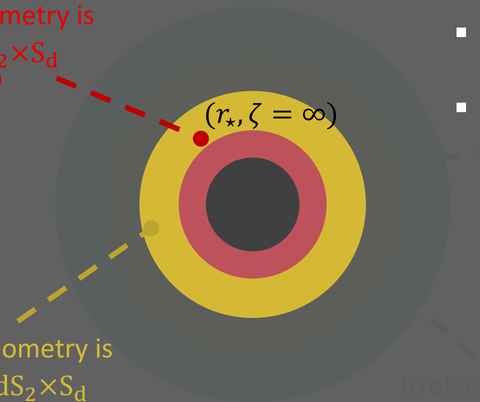
$$\Phi(\zeta) = r_\star + L_2^2/\zeta$$

red	$\frac{1}{2\pi T} > \zeta \gg r_\star$	Φ const
yellow	$\frac{1}{T} \gg \zeta \gg r_\star$	Φ slowly varying
blue	$\zeta \sim r_\star$	Φ big

Task: write the 2-dimensional theory

How:

- Dimensional reduction
- UV cutoff



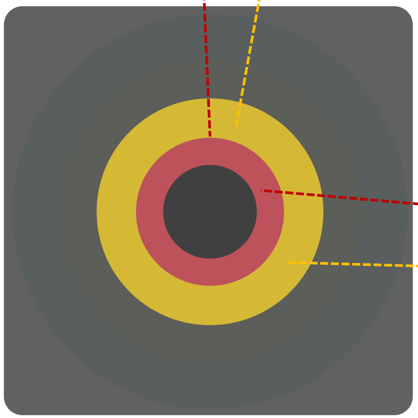
$$S_{EM} = \int d^2x \sqrt{g_2} \left[-\frac{s_d}{2\kappa^2} \Phi^2 R_2 + U(\Phi) + \frac{Z(\Phi)}{4 g_F^2} F^2 \right]$$

$$S_{GH} = -\frac{s_d}{\kappa^2} \int_\partial dx \sqrt{\gamma} \Phi^2 \mathcal{K}_1$$

Holography

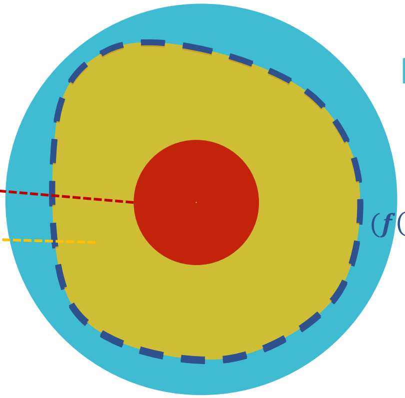
- To probe the universal region we make an **expansion in the dilaton**

$$\phi = \phi_0 + \phi_1(\zeta)$$



- The metric in the universal region is well described by AdS_2
- We work with cutoff versions of AdS_2
- The cutoff is a boundary curve embedded in AdS_2
- Symmetry breaking down to $SL(2,\mathbb{R})$

AdS_2 is the hyperbolic disk



$$S_{\mathrm{eff}}[f] = -\frac{s_d \phi_1}{\kappa^2} \int d\tau \{\tan(\pi T f(\tau), \tau)\}$$

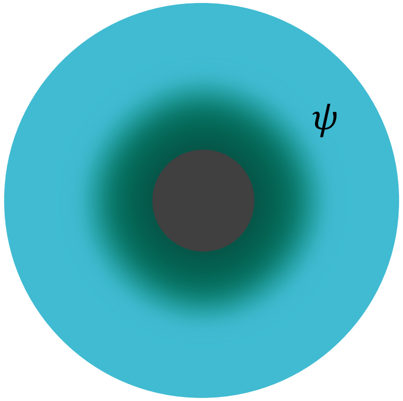
[Maldacena, Stanford, Yang (2016) - Sachdev (2019)]

Abelian-Higgs model

(work in progress)

$$S_{AH} = \int d^{d+2}x \sqrt{g} \left[-\frac{1}{2\kappa^2}(R_{d+2} - 2\Lambda) + \frac{1}{4g_F^2}F^2 + |D\psi|^2 + m^2|\psi|^2 \right]$$

- $\psi = 0$: Reissner-Nordström
- $m_{\text{eff}}^2 = m^2 - |g^{tt}|q^2 A_t^2$
- For appropriate m and q formation of a scalar hair
- **Conjecture**: scalar hair geometry IR and SYK IR match



[Gubser (2008) - Hartnoll, Herzog, Horowitz (2008) - Zaanen, Liu, Sun, Schalm (2015)]

Abelian-Higgs model

(work in progress)

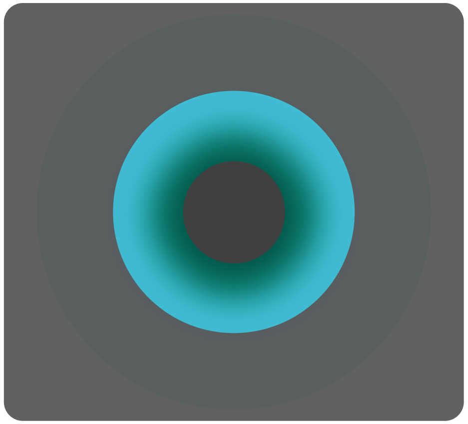
- **Dimensional reduction**

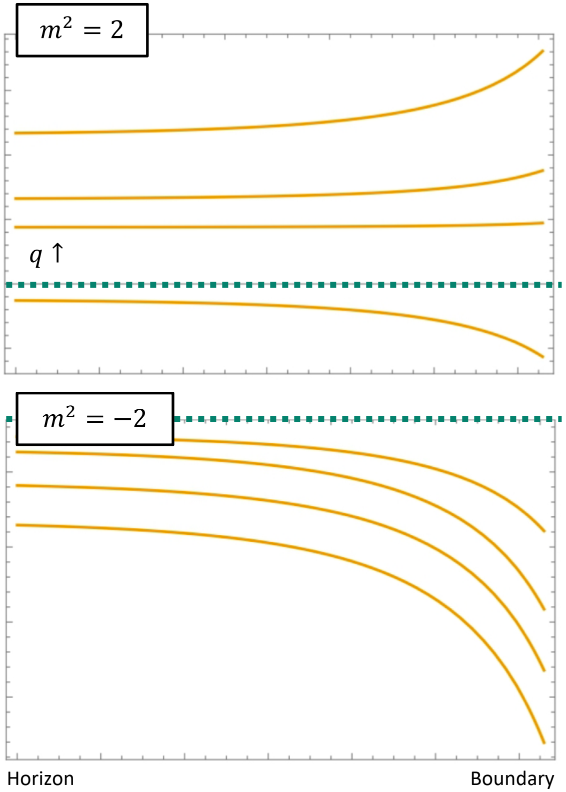
$$S_{AH} = \int d^2x \sqrt{g_2}\left[-\frac{s_d}{2\kappa^2}\Phi^2 R_2 + U(\Phi) + \frac{Z(\Phi)}{4g_F^2}F^2 + s_d \Phi^2 |D\psi|^2 + m^2(\Phi)|\psi|^2\right]$$

$$\partial_a\left(\sqrt{g_2} s_d \Phi^2 \partial^a \psi\right) = \sqrt{g_2}\, m_{\text{eff}}^2(\Phi)\psi$$

- $\boldsymbol{E = 0}$ **Schrödinger problem**

$$-\partial_s^2 \psi(s) + V(s)\psi(s) = 0$$





[Faulkner, Liu, McGreevy, Vegh, (2011)]

Abelian-Higgs model

(work in progress)

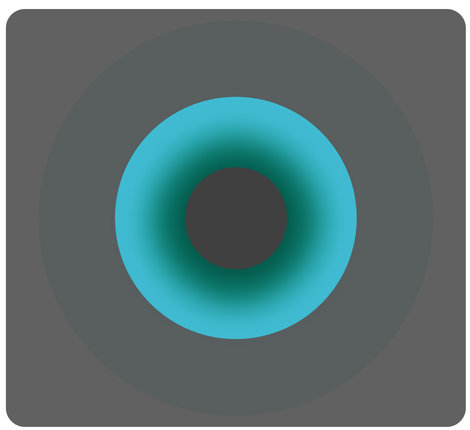
- **Dimensional reduction**

$$S_{AH} = \int d^2x\,\sqrt{g_2}\left[-\frac{s_d}{2\kappa^2}\Phi^2 R_2 + U(\Phi) + \frac{Z(\Phi)}{4g_F^2}F^2 + s_d\Phi^2|D\psi|^2 + m^2(\Phi)|\psi|^2\right]$$

$$\partial_a\left(\sqrt{g_2}s_d\Phi^2\partial^a\psi\right) = \sqrt{g_2}m_{\text{eff}}^2(\Phi)\psi$$

- $\boldsymbol{E=0}$ **Schrödinger problem**

$$-\partial_s^2\psi(s) + V(s)\psi(s) = 0$$



BKT scaling

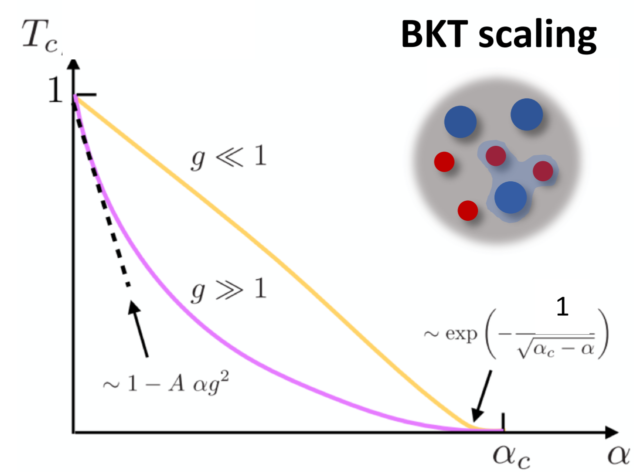


Figure from [Hauck, Klug, Esterlis, Schmalian, (2019)]
[Kaplan, Lee, Son, Stephanov (2009)]

Conclusions and outlook

- Extension to **higher dimensions** of the fermion phonon dot.
- Analysis of the kernel *K:* the system has **finite Drude weight** and **superconducting stiffness**
- Numerical treatment of the model saddle point equations
- Match holographic and QFT approach: derive the **same** low energy **effective action**
- Reproduce **BKT scaling** from holography
- Torus reduction to evaluate the stiffness

Thanks for your attention

Karlsruhe Institute of Technology