The complex life of hydrodynamic modes

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P.Kovtun, S.Grozdanov, A.Starinets and P.Tadic
1904.12862 [hep-th]
P.Kovtun, S.Grozdanov, A.Starinets and P.Tadic
1904.01018 [hep-th]
and
S.Grozdanov, N.Kaplis and A.Starinets
1605.02173 [hep-th] JHEP 1607 (2016) 151

HOLO-TUBE SEMINAR

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Motivation: ongoing experimental programs and theoretical advances of the last two decades

Experiments:

Experiments on heavy ion collisions at RHIC (2000-current), LHC (2010-current) and future (FAIR, NICA) colliders
(relativistic, "many-body", ***strongly interacting***, non-equilibrium "hot" system)

Romatschke and Romatschke, ``Relativistic Fluid Dynamics Out of Equilibrium : Ten Years of Progress in Theory and Numerical Simulations of Nuclear Collisions,'' arXiv:1712.05815 [nucl-th] – NOW A MAJOR ~~MOTION PICTURE~~ BOOK FROM Cambridge U.Press!
Busza, Rajagopal and van der Schee, ``Heavy Ion Collisions: The Big Picture, and the Big Questions,'' arXiv:1802.04801 [hep-ph].

Experimental realization (1995-1999) of new classes of quantum "many-body" systems (e.g. ultra-cold atomic Bose and Fermi gases),
current extensive study of their collective behavior
(non-relativistic, "many-body", ***strongly interacting***, non-equilibrium "cold" system)

Theory:

Gauge-string duality: A "new" (1997) non-perturbative tool to study ***strongly interacting*** quantum systems
(zero or finite temperature/density, relativistic and non-relativistic, equilibrium and non-equilibrium – but for limited class of theories/parameters)

Hydrodynamics is an effective theory valid at long times & large distances

Small fluctuations of an equilibrium state (here: homogeneous, isotropic, neutral, relativistic) are the hydrodynamic (gapless) modes with dispersion relations (in Fourier space $\sim e^{-i\omega t + iqz}$):

Shear mode: $$\mathfrak{w} = \mathfrak{w}(\mathfrak{q}) = -i \frac{\eta}{\epsilon + P}\, \mathfrak{q}^2 + \cdots$$

Sound mode: $$\mathfrak{w} = \mathfrak{w}_{\pm}(\mathfrak{q}) = \pm v_s \mathfrak{q} - i \frac{\zeta + \frac{4}{3}\eta}{\epsilon + P}\, \mathfrak{q}^2 + \cdots$$

$\mathfrak{w} = \frac{\omega}{2\pi T}$, $\mathfrak{q} = \frac{q}{2\pi T}$; η, ζ – shear & bulk viscosities; v_s – speed of sound; ϵ, P – energy density & pressure

1) Do the series above converge? If so, what determines their radii of convergence? Does the effective theory "know" its limits? Why hydro is so effective at strong coupling?

$$\omega = \frac{p^2}{2m} - \frac{p^4}{8m^3} + \frac{p^6}{16m^5} + \cdots = \sqrt{p^2 + m^2} - m\,, \quad p = \pm i m$$

2) How do transport coefficients change when the coupling in an underlying microscopic theory changes? Can we interpolate between weak and strong coupling?

Thanks to holographic duality, these questions can be investigated for some QFTs.

Motivational Slide - I

Interpolation between weak and strong coupling:
exact results are rare (even at T=0)…

Example (old & beautiful): expectation value of a circular Wilson loop in

$\mathcal{N} = 4\ SU(N_c)$ SYM in $d = 4$ in the limit $N_c \to \infty, \lambda \equiv g_{YM}^2 N_C$

$$\langle W_C \rangle = \frac{2}{\sqrt{\lambda}} I_1 \left(2\sqrt{\lambda} \right)$$

$$\langle W_C \rangle = 1 + \frac{\lambda}{4} + \frac{\lambda^2}{48} + \cdots \quad \lambda \ll 1$$

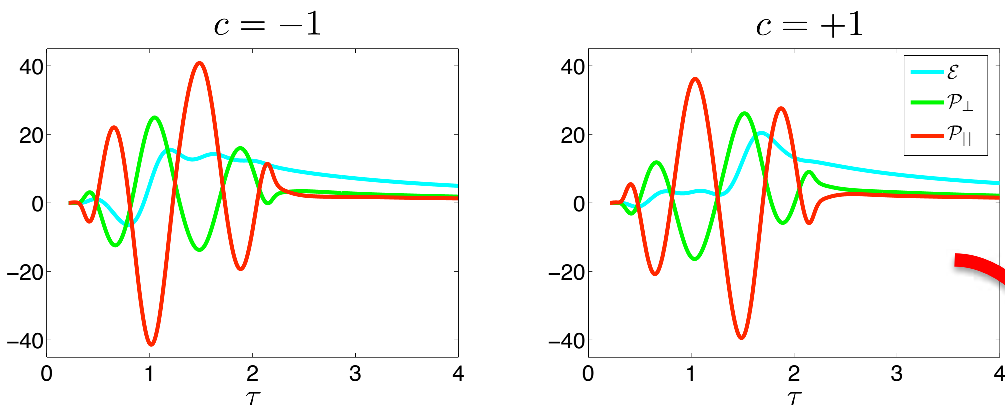
$$\langle W_C \rangle \sim \sqrt{\frac{2}{\pi}} \frac{e^{\sqrt{2\lambda}}}{(2\lambda)^{3/4}} + \cdots \quad \lambda \gg 1$$

Erickson, Semenoff and Zarembo (2000)

The “unreasonable effectiveness of hydrodynamics “…

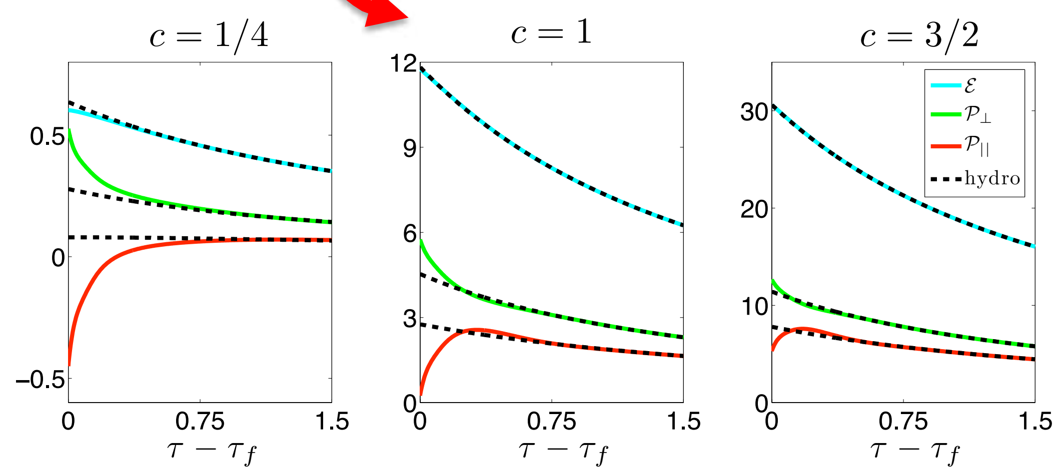
A paraphrase (one of many) of Wigner’s "The Unreasonable Effectiveness of Mathematics in the Natural Sciences” (1960)

Approach to equilibrium – expect : $\langle T^{\mu\nu} \rangle \rightarrow \mathrm{diag}\,(\varepsilon, P, P, P)$



Full non-linear evolution after “quench”
P.Chesler and L.G.Yaffe (2009)

Dashed black lines –
hydrodynamic approximation
to different stress-tensor components



Hydrodynamics seems to work remarkably well even when the gradients are not small…

Relevant issues *not* discussed in this talk

Causality problems in relativistic hydrodynamics and their resolution

Kovtun, [1907.08191](https://arxiv.org/abs/1907.08191) [hep-th]

Hoult and Kovtun, [2004.04102](https://arxiv.org/abs/2004.04102) [hep-th]

Bemfica, Disconzi, Noronha, [1907.12695](https://arxiv.org/abs/1907.12695) [gr-qc]; [2009.11388](https://arxiv.org/abs/2009.11388) [gr-qc]

Convergence/divergence/resurgence/insurgence in the position space

Heller, Serantes, Spalinski, Svensson, Withers, [2007.05524](https://arxiv.org/abs/2007.05524) [hep-th]

Berges, Heller, Mazeliauskas, Venugopalan, [2005.12299](https://arxiv.org/abs/2005.12299) [hep-th]

Relation to the singularity theorems in GR and stability analysis

Dafermos, Holzegel, Rodnianski, [1601.06467](https://arxiv.org/abs/1601.06467) [gr-qc]

Fluid dynamics is an effective theory valid in the long-wavelength, long-time limit

Fundamental degrees of freedom = densities of conserved charges

Equations of motion = conservation laws + constitutive relations^*

Example I

$$\left.\begin{aligned} &\partial_a J^a = 0 \\ &J^i = -D\,\nabla^i J^0 + \cdots \end{aligned}\right\} \quad \begin{aligned} &\partial_t J^0 = D\,\nabla^2 J^0 + \cdots \\ &\omega = -iDq^2 + \cdots \end{aligned}$$

Example II

$$\left.\begin{aligned} &\partial_a T^{ab} = 0 \\ &T^{ab} = \varepsilon u^a u^b + P(\varepsilon)\left(g^{ab} + u^a u^b\right) + \Pi^{ab} + \cdots \end{aligned}\right\}$$

Navier-Stokes eqs
Burnett eqs
…..

* Modulo assumptions e.g. analyticity

** E.o.m. universal, transport coefficients depend on underlying microscopic theory

Consider relativistic neutral conformal fluid in a d-dimensional (curved) space-time

$$T^{ab} = \varepsilon u^a u^b + P(\varepsilon)\left(g^{ab} + u^a u^b\right) + \Pi^{ab} + \cdots$$

Including only terms with first and second derivatives of fluid velocity:

$$\begin{aligned}\Pi^{ab} = & -\eta\sigma^{ab} \\ & + \eta\tau_\Pi\left[{}^{\langle}D\sigma^{ab\rangle} + \frac{1}{d-1}\sigma^{ab}\left(\nabla\cdot u\right)\right] \\ & + \kappa\left[R^{\langle ab\rangle} - (d-2)u_c R^{c\langle ab\rangle d}u_d\right] \\ & + \lambda_1\sigma^{\langle a}{}_c\sigma^{b\rangle c} + \lambda_2\sigma^{\langle a}{}_c\Omega^{b\rangle c} + \lambda_3\Omega^{\langle a}{}_c\Omega^{b\rangle c}\end{aligned}$$

Transport coefficients (in conformal case): $\eta\,,\,\tau_\Pi\,,\,\kappa\,,\,\lambda_1\,,\,\lambda_2\,,\,\lambda_3$

Non-conformal case: 2 first order coefficients, 15 (10) second order coefficients
(see S.Bhattacharyya, 1201.4654 [hep-th])

Beyond second order hydrodynamics

Tensors structures appearing in the derivative expansion have been analyzed using computer algebra in 1507.02461 [hep-th] by Grozdanov & Kaplis.

At third order, there are 20 relevant structures in the conformal case and 68 in the non-conformal one.

This still needs an entropy current analysis similar to the one in S.Bhattacharyya, 1201.4654 [hep-th]

Example: dispersion relations in conformal case

$$\omega = -i\frac{\eta}{\varepsilon+P}k^2 - i\left[\frac{\eta^2\tau_\Pi}{(\varepsilon+P)^2} - \frac{\theta_1}{2(\varepsilon+P)}\right]k^4 + \cdots$$

$$\omega = \pm c_s k - i\Gamma k^2 \mp \frac{\Gamma}{2c_s}\left(\Gamma - 2c_s^2\tau_\Pi\right)k^3 - i\left[\frac{8\eta^2\tau_\Pi}{9(\varepsilon+P)^2} - \frac{\theta_1+\theta_2}{3(\varepsilon+P)}\right]k^4 + \cdots$$

Here $c_s = 1/\sqrt{3}$ $\qquad \Gamma = \eta/(\varepsilon+P)$

Precursors

Misha Lublinsky

Ben Withers

Bu and Lublinsky,
Linearized Fluid/Gravity Correspondence: From Shear Viscosity To All Order Hydrodynamics,
1409.3095 [hep-th] and subsequent papers

Withers,
Short-lived modes from hydrodynamic dispersion relations, 1803.08058 [hep-th]

Relativistic hydrodynamics to all orders

$$\langle T^{\mu\nu}\rangle \equiv T^{\mu\nu}(x,t) = T^{\mu\nu}_{eq} + \delta T^{\mu\nu}(x,t)\,, \quad T^{\mu\nu}_{eq} = \mathrm{diag}\,(\epsilon, P, P, P)$$

Fundamental d.o.f. – densities of conserved charges: $T^{00},\, T^{0i}$

All other components should be expressed through them (and their derivatives) via constitutive relations. To linear order in fluctuations, in Fourier space:

$$\delta T^{ij} = -iA\left(q^i\delta T^{0j} + q^j\delta T^{0i}\right) + \delta T^{00}\left(Bq^iq^j + C\delta^{ij}\right) + iq_k\delta T^{0k}\left(Dq^iq^j + E\delta^{ij}\right)$$

Here A,B,C,D,E are functions of ω, q^2 Expanding them around (0,0) to order k, we get the usual derivative expansion in k-th order hydrodynamics, e.g. for k=1:

$$\delta T^{ij} = \delta^{ij}v_s^2\,\delta T^{00} - \frac{i}{\epsilon+P}\left[\eta\left(q^i\delta T^{0j} + q^j\delta T^{0i} - \frac{2}{3}\delta^{ij}q_k\delta T^{0k}\right) + \zeta\delta^{ij}q_k\delta T^{0k}\right]$$

Combining this with conservation equation $\partial_\mu T^{\mu\nu} = 0$, we get a 4x4 matrix (M) equation for fluctuations $\delta T^{00}, \delta T^{0i}$

Hydrodynamic spectral curve

Non-trivial solution for fluctuations $\delta T^{00}, \delta T^{0i}$: $\quad \det M \equiv P(q^2,\omega) = 0$

For relativistic homogeneous & isotropic neutral fluid in d+1 dimensions

$$P(q^2,\omega) = \left(\omega + i q^2 \gamma_\eta(\omega,q^2)\right)^{d-1} \left(\omega^2 + i\omega q^2 \gamma_s(\omega,q^2) - q^2 H(\omega,q^2)\right) = 0$$

Here the functions γ_η, γ_s, H are simply related to A,B,C,D,E in the constitutive relations

$$\delta T^{ij} = -iA\left(q^i \delta T^{0j} + q^j \delta T^{0i}\right) + \delta T^{00}\left(B q^i q^j + C\delta^{ij}\right) + i q_k \delta T^{0k}\left(D q^i q^j + E\delta^{ij}\right)$$

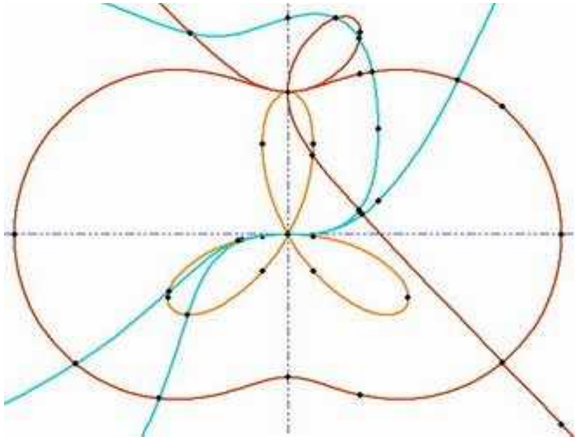
So we have 2 hydrodynamic spectral curves:

$$F_{\text{shear}} = \omega + i q^2 \gamma_\eta(\omega,q^2) = 0$$
$$F_{\text{sound}} = \omega^2 + i\omega q^2 \gamma_s(\omega,q^2) - q^2 H(\omega,q^2) = 0$$

We treat them as complex curves $F(x,y)=0$ in the space of $(x \equiv q^2, y \equiv \omega) \in \mathbb{C}^2$

Note: at any finite order of the gradient expansion, F(x,y) is a polynomial

Complex curves F(x,y)=0 are interesting objects. Here, we are interested in solutions y=y(x) and their properties (we assume the curve is analytic or algebraic).



Example:

$$f(x,y) = y^5 - 4y^4 + 4y^3 + 2x^2y^2 - xy^2 + 2x^2y + 2xy + x^4 + x^3 = 0$$

Regular points: $F(x_*, y_*) = 0\,, F_y(x_*, y_*) \neq 0$ $\quad y = \sum_{n>n_0}^{\infty} a_n (x - x_*)^n$

Critical points: $F(x_*, y_*) = 0\,, F_y(x_*, y_*) = 0\,, \ldots F_y^{(p)}(x_*, y_*) \neq 0$

Puiseux series: $y = \sum_{n>n_0}^{\infty} a_n (x - x_*)^{\frac{n}{m_j}}\,, m_j = 1, \ldots p$

Example: Kepler's equation at complex eccentricity

Kepler's Third Law contains non-analyticity: $T \propto a^{3/2}$ This is not an accident.

Motion of a planet with eccentricity e in parametric form:

$$r = a\,(1 - e\cos\psi)$$

$$t = \frac{T}{2\pi}\,(\psi - e\sin\psi)$$

Solving for $\psi(t)$ (eccentric anomaly) determines the position of the planet

Kepler's equation: $$F = \tau - \psi + e\sin\psi = 0$$

Solution (Lagrange, 1771): $$\psi(\tau, e) = \tau + \sum_{n=1}^{\infty} a_n(\tau)\frac{e^n}{n!}$$

The series converges for $|e| \le e_L \approx 0.662743\ldots$ (Laplace, 1823)

Example: Kepler's equation at complex eccentricity (continued)

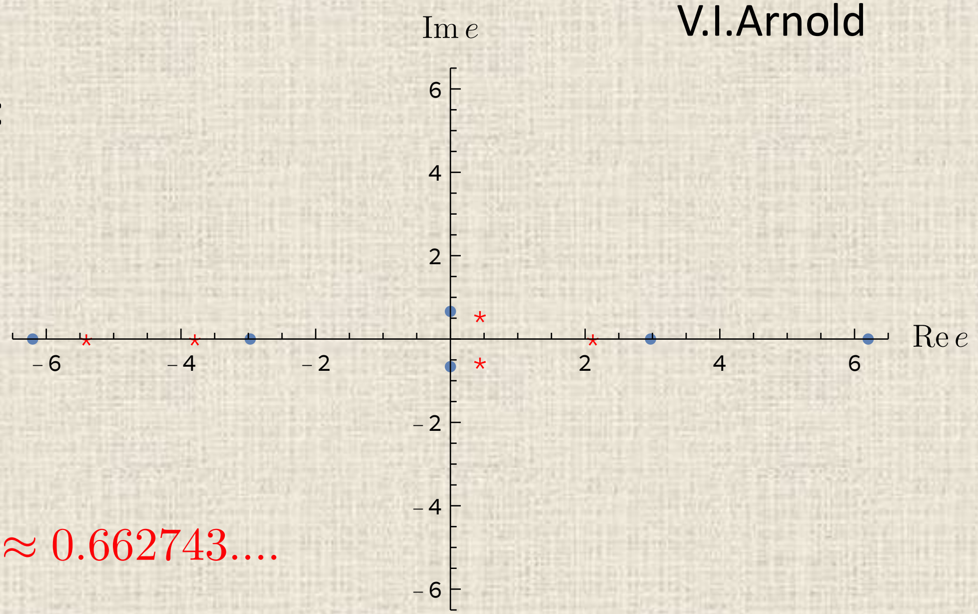
``This equation plays an important role in the history of mathematics. From the time of Newton, the solution has been sought in the form of a series in powers of the eccentricity e. The series converges when |e|< 0.662743.... The investigation of the origin of this mysterious constant led Cauchy to the creation of complex analysis. Such fundamental mathematical concepts and results as Bessel functions, Fourier series, the topological index of a vector field, and the ``principle of the argument'' of the theory of functions of a complex variable also first appeared in the investigation of Kepler's equation''.

V.I.Arnold

Critical points of the Kepler's curve:

$$F = \tau - \psi + e \sin \psi = 0$$

$$\frac{\partial F}{\partial \psi} = e \cos \psi - 1 = 0$$



The series converges for $|e| \leq e_L \approx 0.662743....$

The critical points closest to the origin are located at $e = \pm 0.662743i$

Applying these theorems to hydrodynamic spectral curves

$$F_{\text{shear}} = \omega + i q^2 \gamma_\eta(\omega, q^2) = 0$$
$$F_{\text{sound}} = \omega^2 + i\omega q^2 \gamma_s(\omega, q^2) - q^2 H(\omega, q^2) = 0$$

we conclude that solutions are given by Taylor or Puiseux series converging in the vicinity of q=0

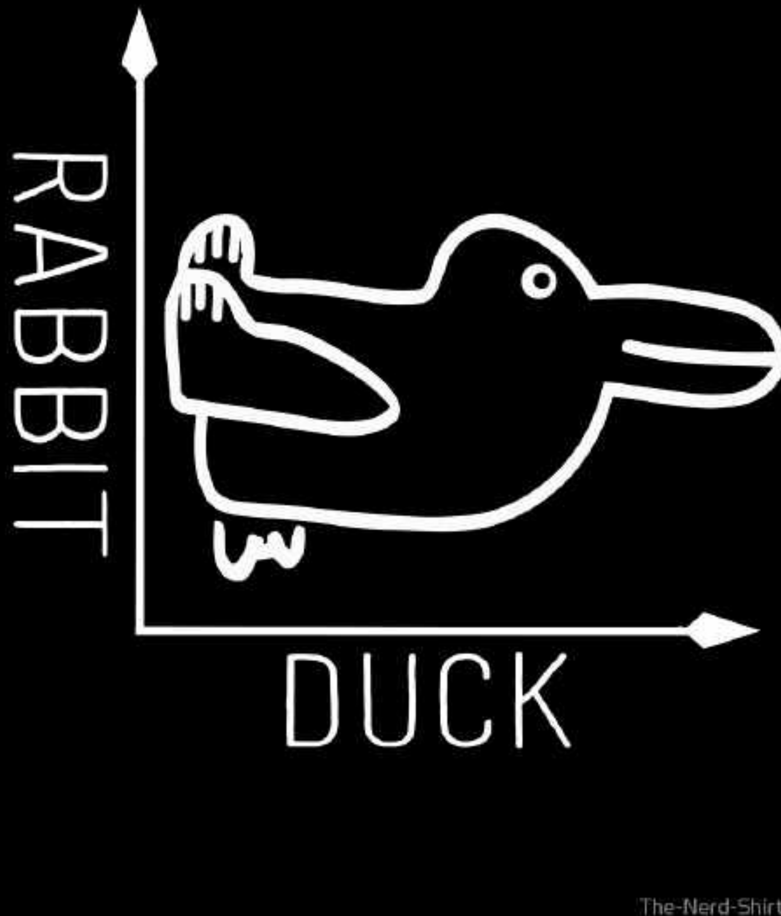
Shear mode: $$\omega_\pm = -i \sum_{n=1}^{\infty} c_n (q^2)^n$$

Sound mode: $$\omega_\pm = -i \sum_{n=1}^{\infty} a_n e^{\pm \frac{i\pi n}{2}} (q^2)^{\frac{n}{2}}$$

Obstruction to the convergence of these series is the next critical point of the spectral curve

Spectral curves can be easily found in theories with dual gravity description

Ludwig Wittgenstein's view of duality (1892; 1953)



(The analogy stolen from Shamit Kachru's talk at Simons Foundation, New York, Feb 27, 2019)

Spectral curves from holography

Dual black hole fluctuations reduce (in gauge-invariant variables) to ODEs such as

$$\Phi'' - \frac{(\mathfrak{w}^2 - \mathfrak{q}^2 f)f - z\mathfrak{w}^2 f'}{zf(\mathfrak{w}^2 - \mathfrak{q}^2 f)}\Phi' + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{zf^2}\Phi = 0$$

$$\Phi(z) = \mathcal{A}\,\varphi_1(z) + \mathcal{B}\,\varphi_2(z)$$

$$\Phi(z) = \mathcal{A}\,z^{\Delta_-}\left(1+\cdots\right) + \mathcal{B}\,z^{\Delta_+}\left(1+\cdots\right) \quad \text{for} \quad z \to 0$$

I. Computing the retarded correlator: inc. wave b.c. at the horizon, normalized to 1 at the boundary

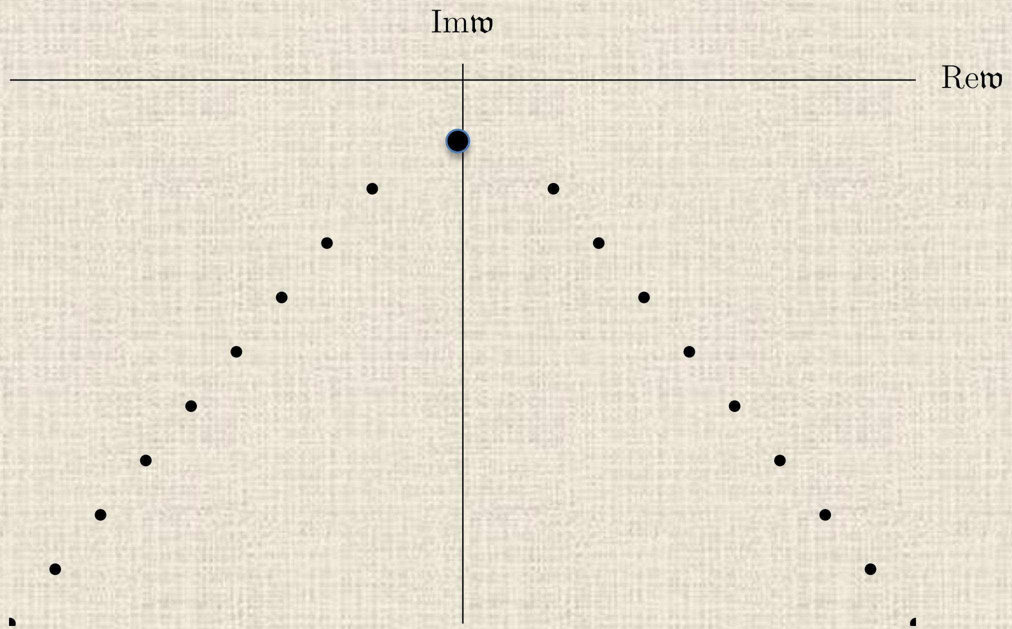
$$G^R \sim \frac{\mathcal{B}}{\mathcal{A}} + \text{contact terms}$$

II. Computing quasinormal spectrum: inc.wave b.c. at the horizon, Dirichlet at the boundary

$$\mathcal{A}\left(\omega, q^2\right) = 0$$

This is the full spectral curve.

Singularities of a (retarded) Green's function in the complex frequency plane



Shear channel

$$\mathfrak{w} = -iD\mathfrak{q}^2 + \cdots$$

Strong (infinite) coupling

Real spatial momentum q^2

Recall the condition for a critical point of a curve

$$F(x_*, y_*) = 0\,, F_y(x_*, y_*) = 0\,, ... F_y^{(p)}(x_*, y_*) \neq 0$$

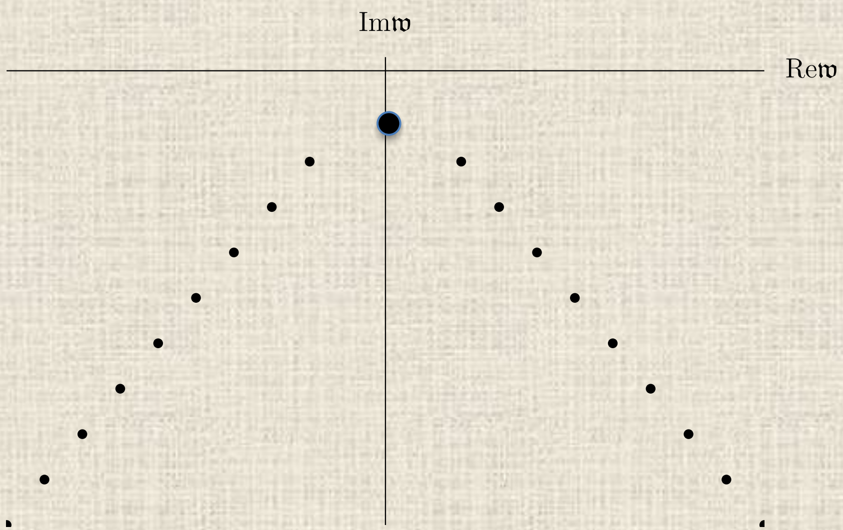
This is actually a "level-crossing" condition

$$F(x,y) = (y - y_*(x))^p (y - y_1(x)) \cdots (y - y_k(x)) = 0$$

Thus for the quasinormal spectrum curve $\mathcal{A}(\omega, q) = 0$ we expect $\omega_1(q_*) = \omega_2(q_*)$

Shear channel

$$\mathfrak{w} = -iD\mathfrak{q}^2 + \cdots$$



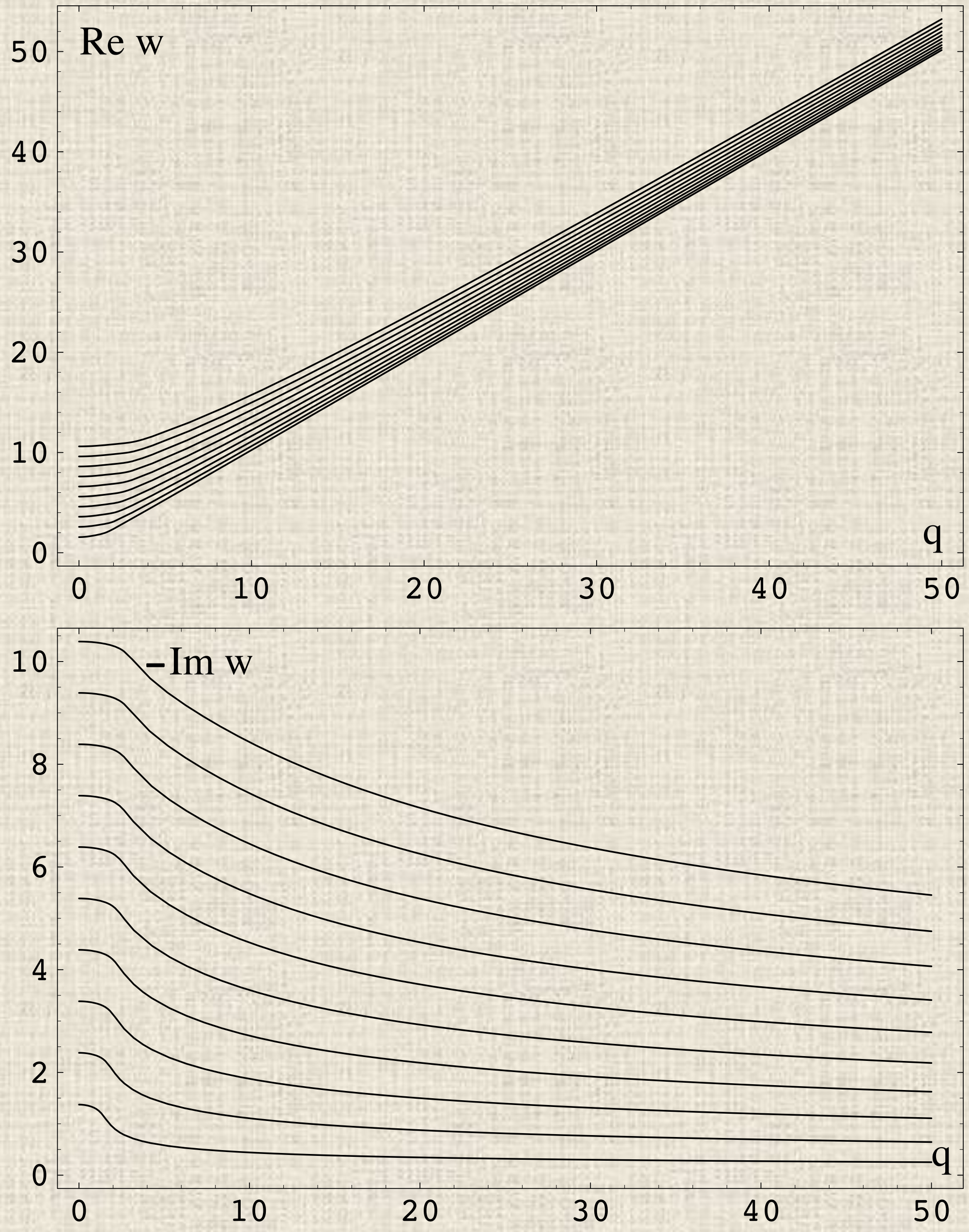
Critical point conditions:

$$\mathcal{A}\left(\omega, q^2\right) = 0$$

$$\frac{\partial \mathcal{A}\left(\omega, q^2\right)}{\partial \omega} = 0$$

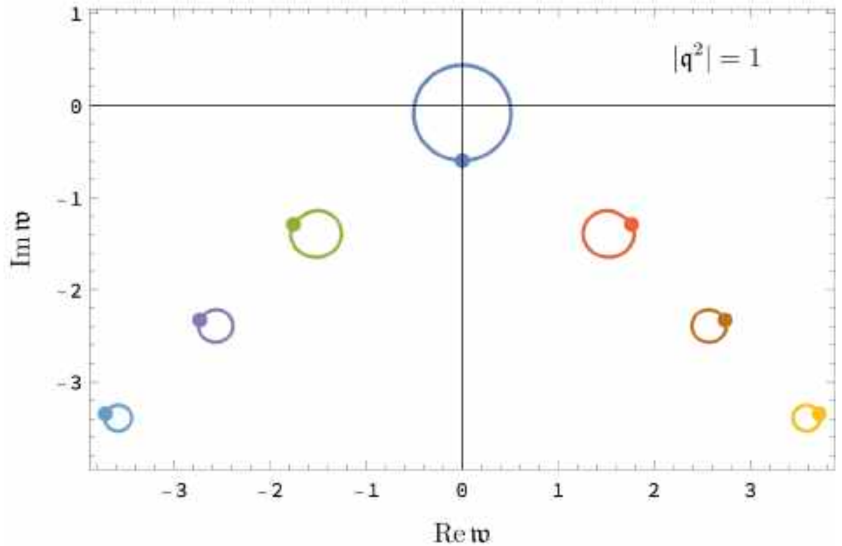
This "quasinormal level crossing" can happen at complex q_*

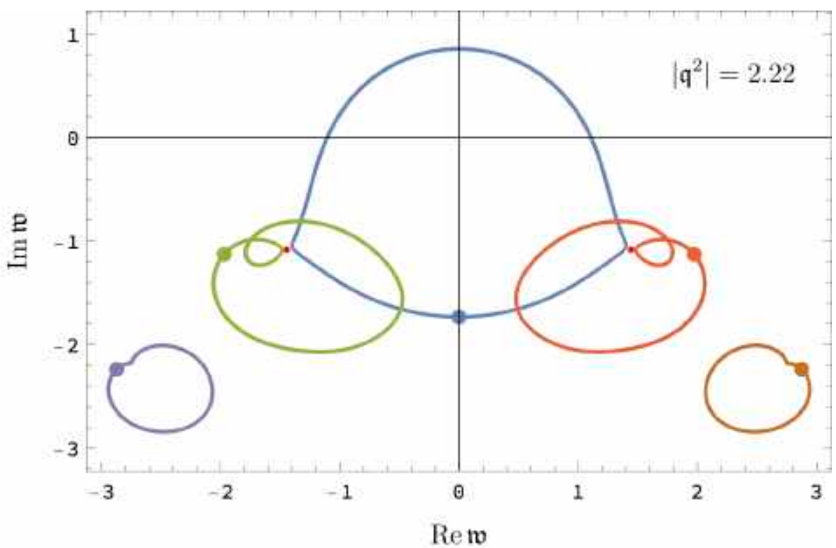
Dispersion relations of the quasinormal modes for real q

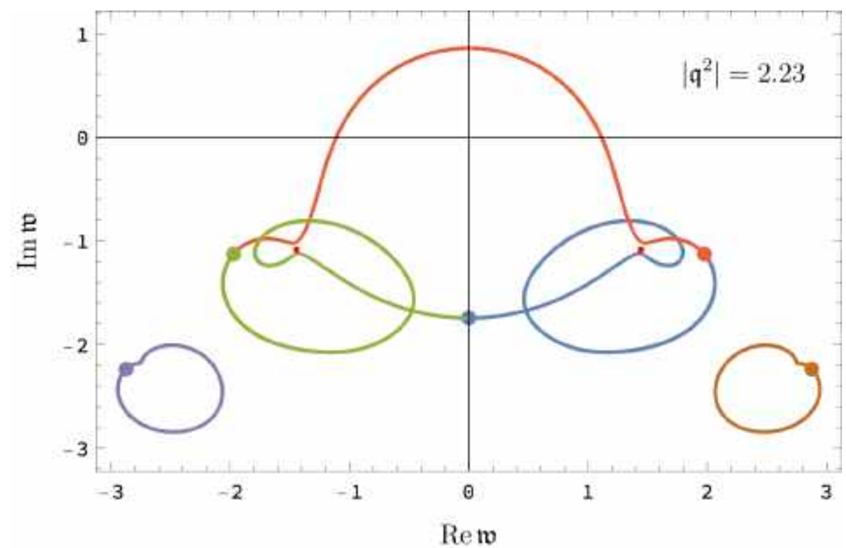


Poles of the retarded energy-momentum tensor correlator in complex $\mathfrak{w}$-plane at complex $\mathfrak{q}^2 = |\mathfrak{q}^2| e^{i\theta}\,, \theta \in [0, 2\pi]$

Shear channel

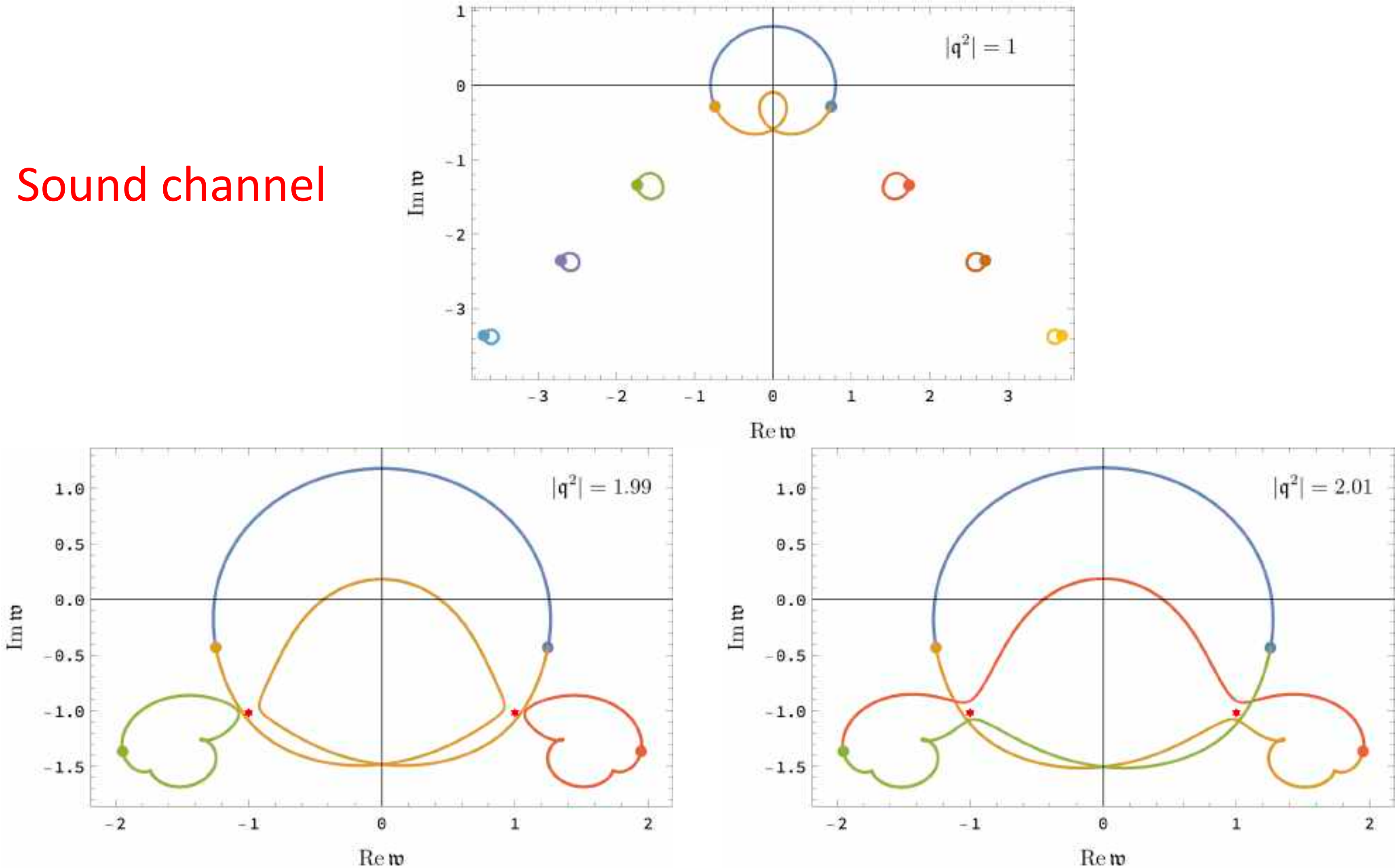




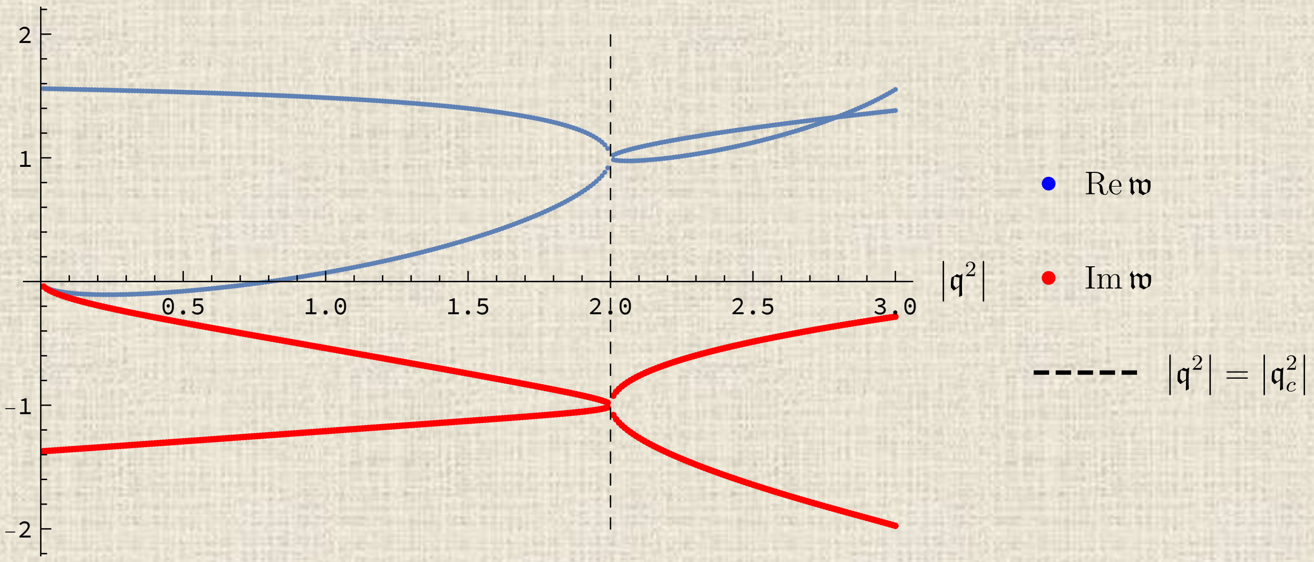


Poles of the retarded energy-momentum tensor correlator in complex $\mathfrak{w}$ -plane at complex $\mathfrak{q}^2 = |\mathfrak{q}^2| e^{i\theta}$, $\theta \in [0, 2\pi]$

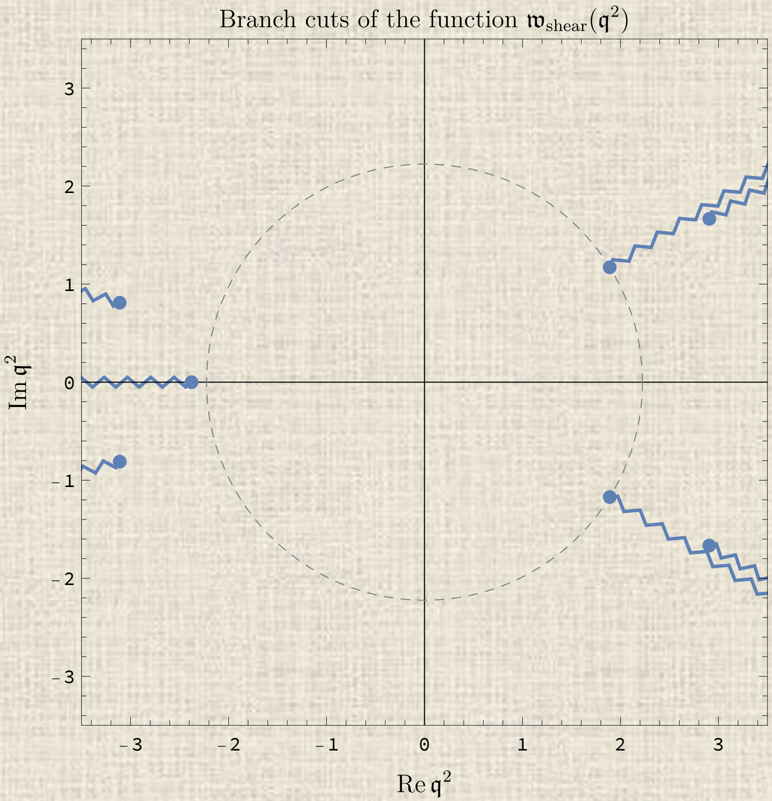
Sound channel

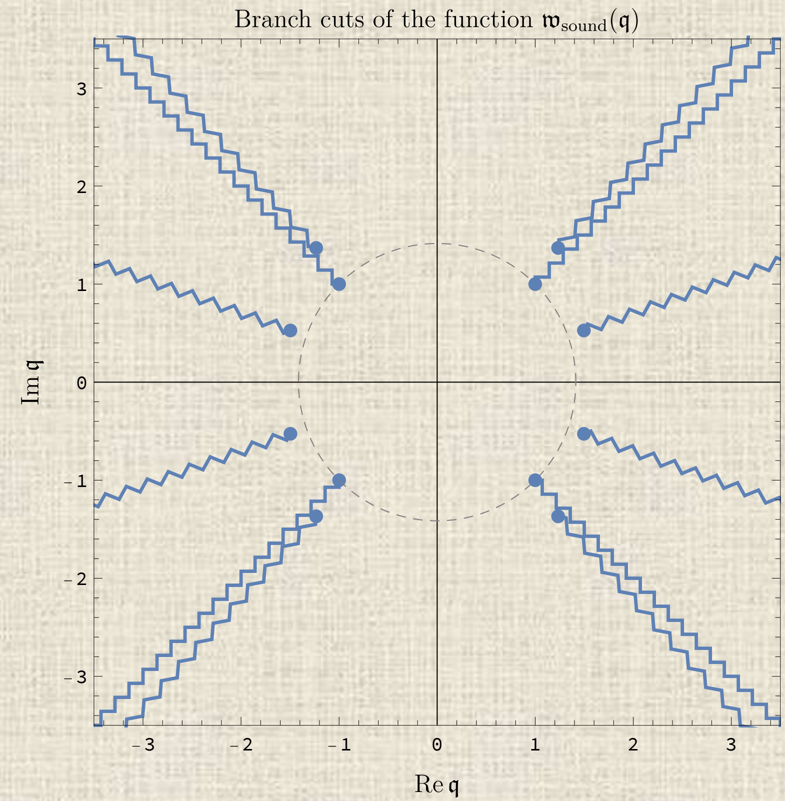


Quasinormal modes level-crossing at complex q



Singularities in the complex q^2-plane





Note: at finite density, the branch point singularity on the negative real axis may be dominant

Withers, [1803.08058] [hep-th]

Abbasi and Tahery, [2007.10024] [hep-th]

Jansen and Pantelidou, [2007.14418] [hep-th]

Radii of convergence of hydrodynamic modes in N=4 SYM
(at infinite 't Hooft coupling, from dual gravity)

Shear mode: $\mathfrak{w} = \mathfrak{w}(\mathfrak{q}) = -i\dfrac{\eta}{\epsilon + P}\,\mathfrak{q}^2 + \cdots$ $\qquad |q_*| \approx 1.49131 \times (2\pi T)$

Sound mode: $\mathfrak{w} = \mathfrak{w}_{\pm}(\mathfrak{q}) = \pm v_s \mathfrak{q} - i\dfrac{\zeta + \frac{4}{3}\eta}{\epsilon + P}\,\mathfrak{q}^2 + \cdots$ $\qquad |q_*| = \sqrt{2} \times (2\pi T)$

$\mathfrak{w} = \dfrac{\omega}{2\pi T}\,,\ \mathfrak{q} = \dfrac{q}{2\pi T}\,; \eta, \zeta -$ shear & bulk viscosities; $v_s -$ speed of sound; $\epsilon, P -$ energy density & pressure

What about finite 't Hooft coupling?

Crude estimate: $\qquad |\mathfrak{q}^{\mathrm{c}}_{\mathrm{sound}}| = \sqrt{3}\left(1 - 15\zeta(3)\lambda^{-3/2} + \cdots\right)$

Thus *it appears* that the radius of convergence is smaller at weaker coupling – hydrodynamics in strongly interacting systems is more "robust"?

END OF PART I

THANK YOU!

Motivation and three examples:

weak-strong coupling interpolation for

1) Zero-temperature observables

2) Thermodynamic observables

3) Transport

$\mathcal{N} = 4$ supersymmetric YM theory

Gliozzi,Scherk,Olive'77
Brink,Schwarz,Scherk'77

- Field content:

$$A_\mu \qquad \Phi_I \qquad \Psi_\alpha^A \qquad \text{all in the adjoint of } SU(N)$$

$I = 1\ldots6 \qquad A = 1\ldots4$

- Action:

$$S = \frac{1}{g_{YM}^2}\int d^4x \,\mathrm{tr}\left\{\frac{1}{2}F_{\mu\nu}^2 + (D_\mu\Phi_I)^2 - \frac{1}{2}[\Phi_I,\Phi_J]^2 + i\bar{\Psi}\Gamma^\mu D_\mu\Psi - \bar{\Psi}\Gamma^I[\Phi_I,\Psi]\right\}$$

- Large N: effective coupling = 't Hooft coupling $\lambda = g_{YM}^2 N$

(super)conformal field theory = coupling doesn't run

Interpolation between weak and strong coupling:
exact results are rare (even at T=0)…

Example (old & beautiful): expectation value of a circular Wilson loop in $\mathcal{N} = 4$ $SU(N_c)$ SYM in $d = 4$ in the limit $N_c \to \infty, \lambda \equiv g_{YM}^2 N_C$

$$\langle W_C \rangle = \frac{2}{\sqrt{\lambda}} I_1\left(2\sqrt{\lambda}\right)$$

$$\langle W_C \rangle = 1 + \frac{\lambda}{4} + \frac{\lambda^2}{48} + \cdots \quad \lambda \ll 1$$

$$\langle W_C \rangle \sim \sqrt{\frac{2}{\pi}} \frac{e^{\sqrt{2\lambda}}}{(2\lambda)^{3/4}} + \cdots \quad \lambda \gg 1$$

Erickson, Semenoff and Zarembo (2000)

Energy density vs temperature for various gauge theories

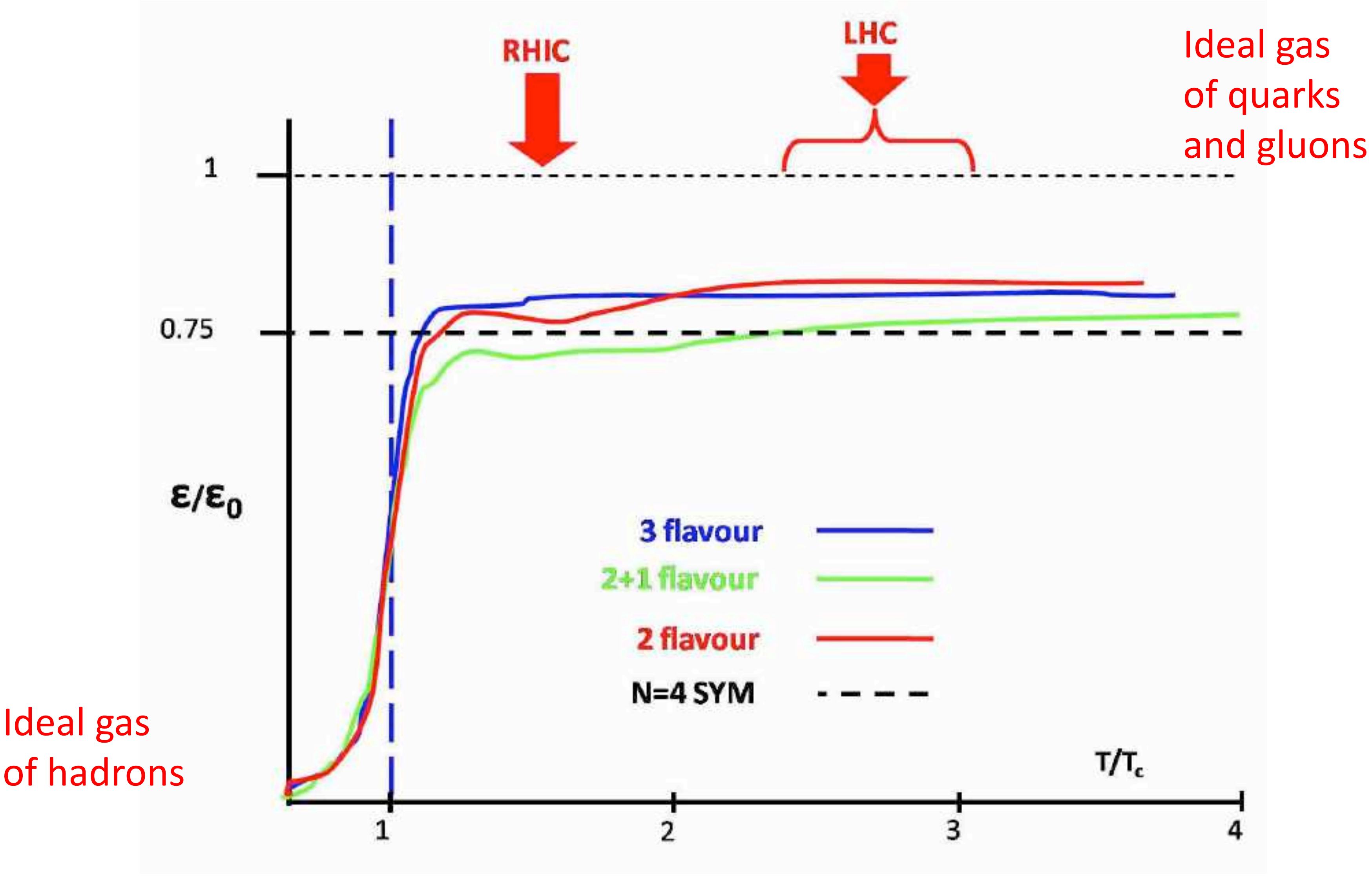
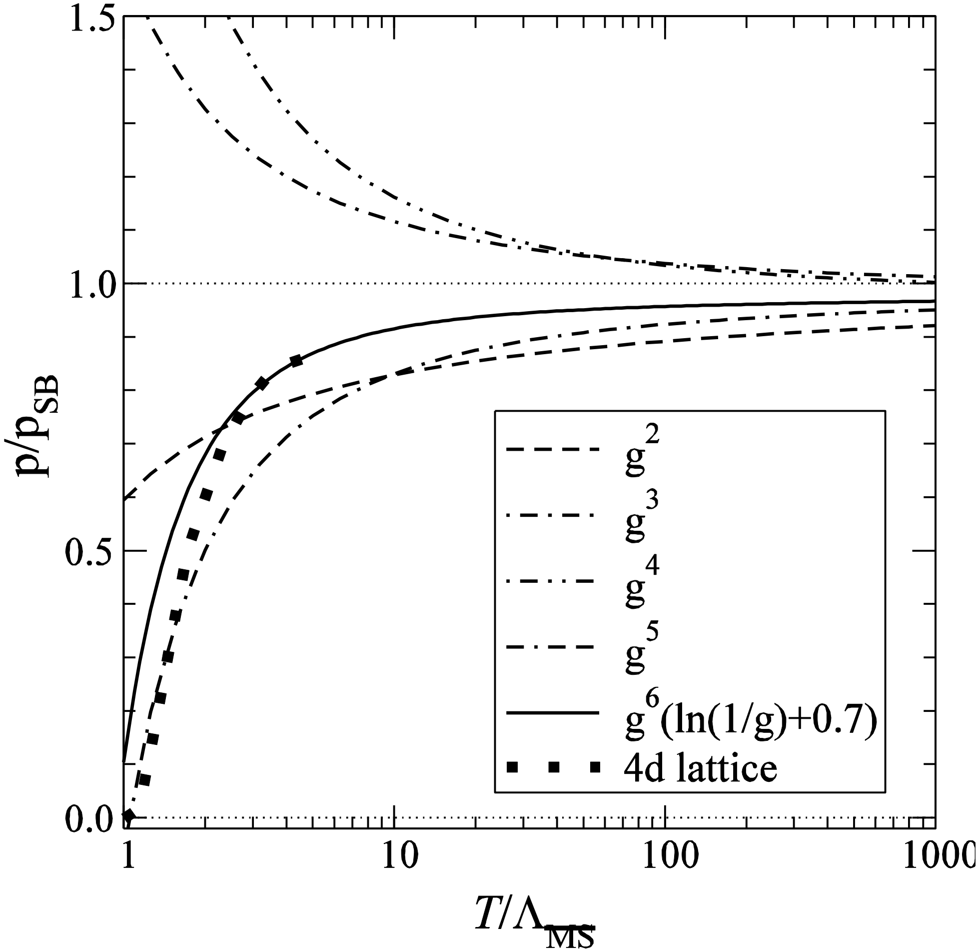


Figure: artist's impression based on LQCD, from Myers and Vazquez, 0804.2423 [hep-th]

Pressure in perturbative QCD



Entropy density of $\mathcal{N}=4$ SYM in the planar limit ($N\to\infty$)

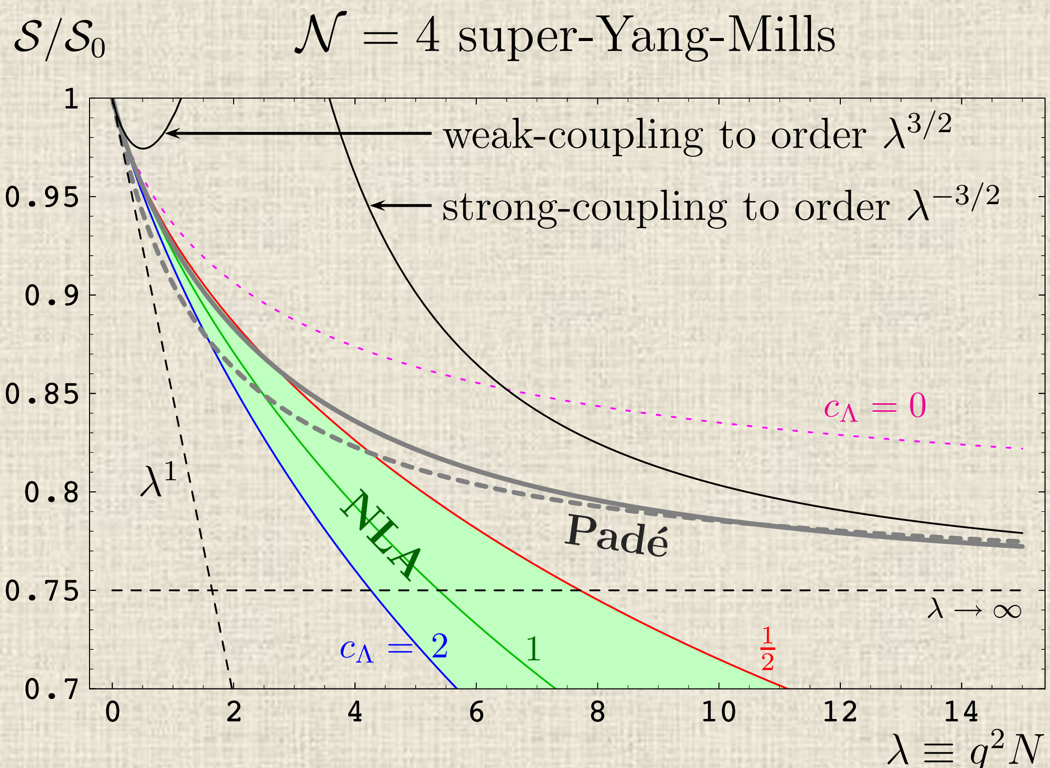


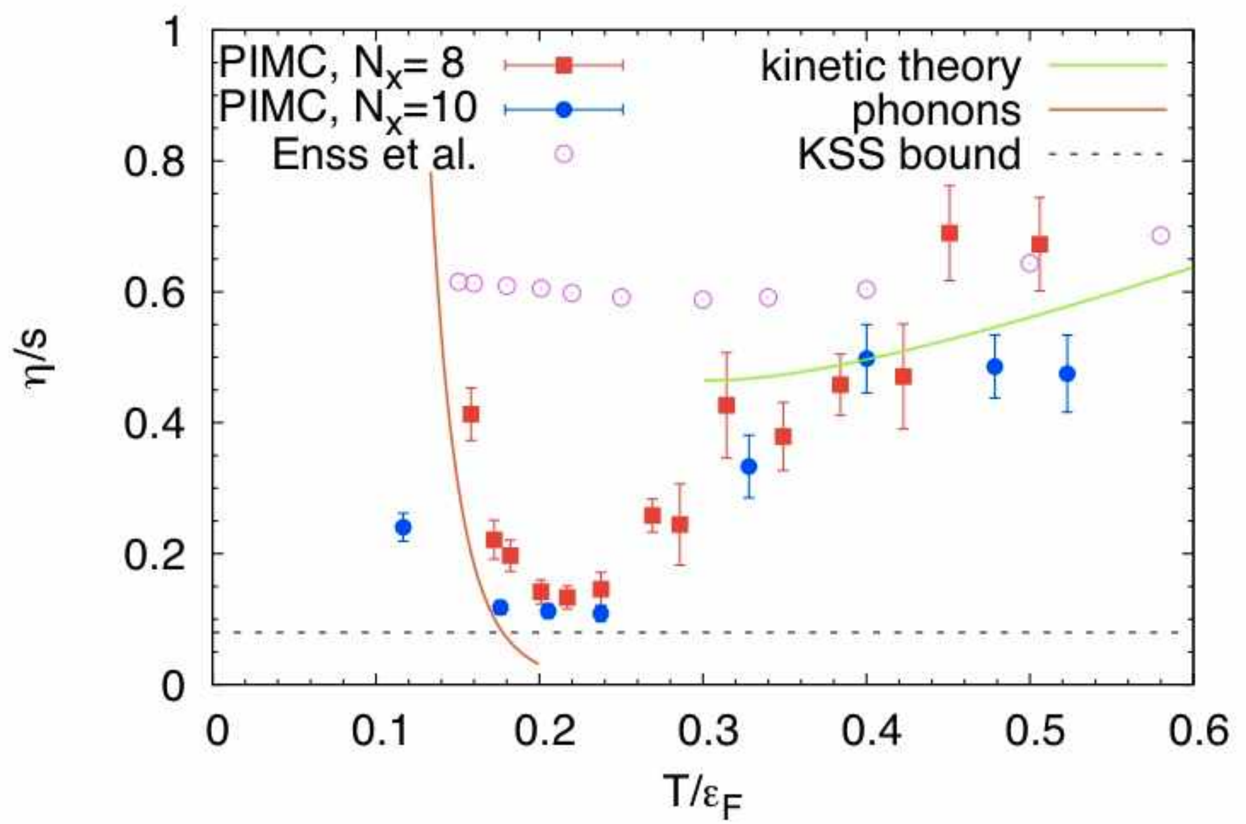
Fig from Blaizot, Iancu, Kraemmer and Rebhan, hep-ph/0611393

$\lambda \ll 1$ $\quad s/s_0 = 1 - \dfrac{3}{2\pi^2}\lambda + \dfrac{\sqrt{2}+3}{\pi^3}\lambda^{3/2} + \cdots$ Fotopoulos and Taylor, hep-th/9811224

$\lambda \gg 1$ $\quad s/s_0 = \dfrac{3}{4} + \dfrac{45}{32}\zeta(3)\,\lambda^{-3/2} + \cdots$ Gubser, Klebanov and Tseytlin, hep-th/9805156

$s_0 = \dfrac{2\pi^2}{3} N_c^2 T^3$ - Stefan-Boltzmann (free gas)

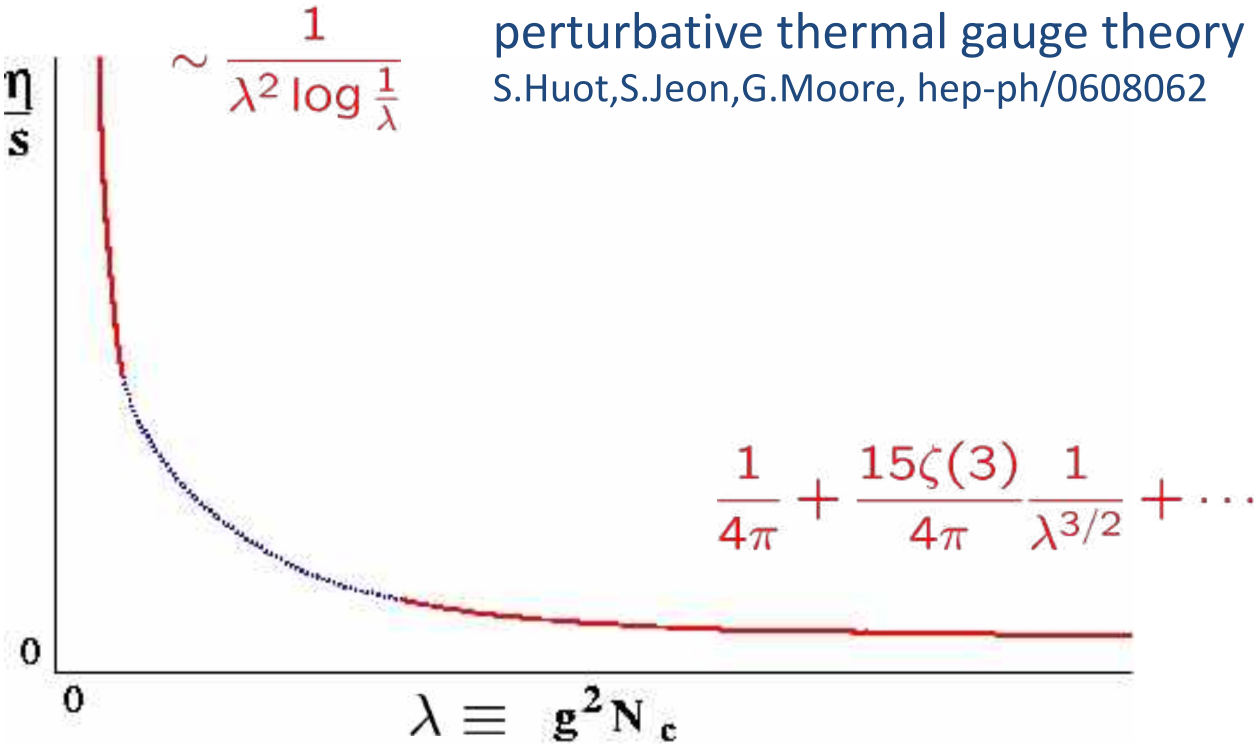
Viscosity-entropy ratio in Unitary Fermi gas



G.Vlazlowski, P.Magierski, J.E.Drut, arXiv:1204.0270 [cond-mat.quant-gas]

$$(\eta/s)_{\min} \sim 1.38 \text{ in units of } \frac{\hbar}{4\pi k_B}$$

Shear viscosity in $\mathcal{N} = 4$ SYM



perturbative thermal gauge theory
S.Huot,S.Jeon,G.Moore, hep-ph/0608062

Correction to $1/4\pi$: Buchel, Liu, A.O.S., hep-th/0406264

Buchel, 0805.2683 [hep-th]; Myers, Paulos, Sinha, 0806.2156 [hep-th]

Plan

Transport properties and analytic structure of correlation functions
in weakly interacting many-body quantum system (particles or quasiparticles)

Transport properties and analytic structure of correlation functions
in strongly interacting many-body quantum systems (from holography - dual gravity)

Real systems are at intermediate coupling (e.g. QGP)

The problem of interpolation between weak and strong coupling is non-trivial

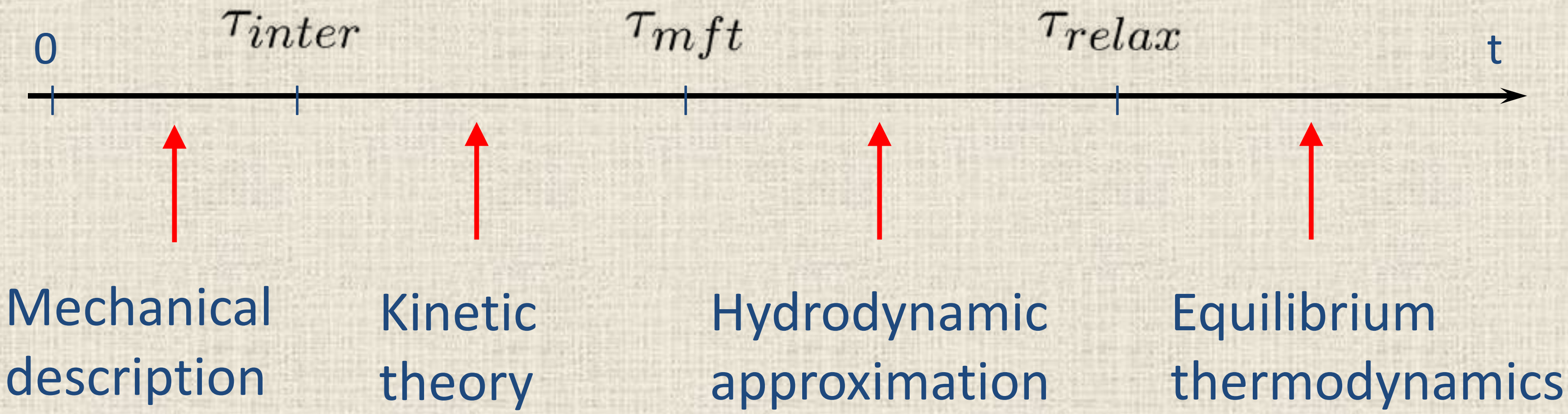
We compute (inverse) coupling corrections using two dual higher-derivative actions - R^2 (Gauss-Bonnet) and R^4 (dual to N=4 SYM)
and argue that results are consistent with expectations from (interpolated) weak coupling calculations

Weak coupling

Hydrodynamic regime in kinetic theory

Hierarchy of times (e.g. in Bogolyubov picture of kinetic theory)

$$\tau_{mft} \ll t \ll T_{existence}$$



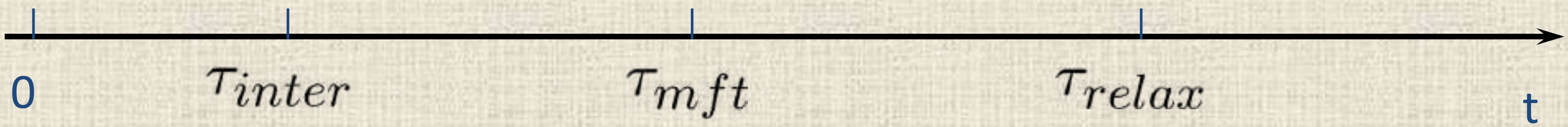
Hierarchy of scales

$$l_{mfp} \ll l \ll L$$

(L is a macroscopic size of a system)

The hydrodynamic regime (continued)

Degrees of freedom



Mechanical description	Kinetic theory	Hydrodynamic approximation	Equilibrium thermodynamics
Coordinates, momenta of individual particles	Coordinate- and time-dependent distribution functions	Local densities of conserved charges $\epsilon(x,t)$	Globally conserved charges E

Hydro regime: $\boxed{\tau_{\text{micro}} \ll \tau \ll t_{\text{global}}}$ $\boxed{l_{\text{micro}} \ll l \ll L_{\text{global}}}$

Relaxation time in kinetic theory

Kinetic equation
$$\frac{\partial F}{\partial t} + \frac{p_i}{m}\frac{\partial F}{\partial r^i} - \frac{\partial U(r)}{\partial r^i}\frac{\partial F}{\partial p_i} = C[F]$$

Linearized by
$$F(t,\mathbf{r},\mathbf{p}) = F_0(\mathbf{r},\mathbf{p})\left[1+\varphi(t,\mathbf{r},\mathbf{p})\right]$$

Leads to
$$\frac{\partial \varphi}{\partial t} = -\frac{p_i}{m}\frac{\partial \varphi}{\partial r^i} + \frac{\partial U(r)}{\partial r^i}\frac{\partial \varphi}{\partial p_i} + L_0[\varphi]$$

For spatially homogeneous distributions:
$$\varphi(t,\mathbf{p}) = e^{-\nu t}h(\mathbf{p})$$

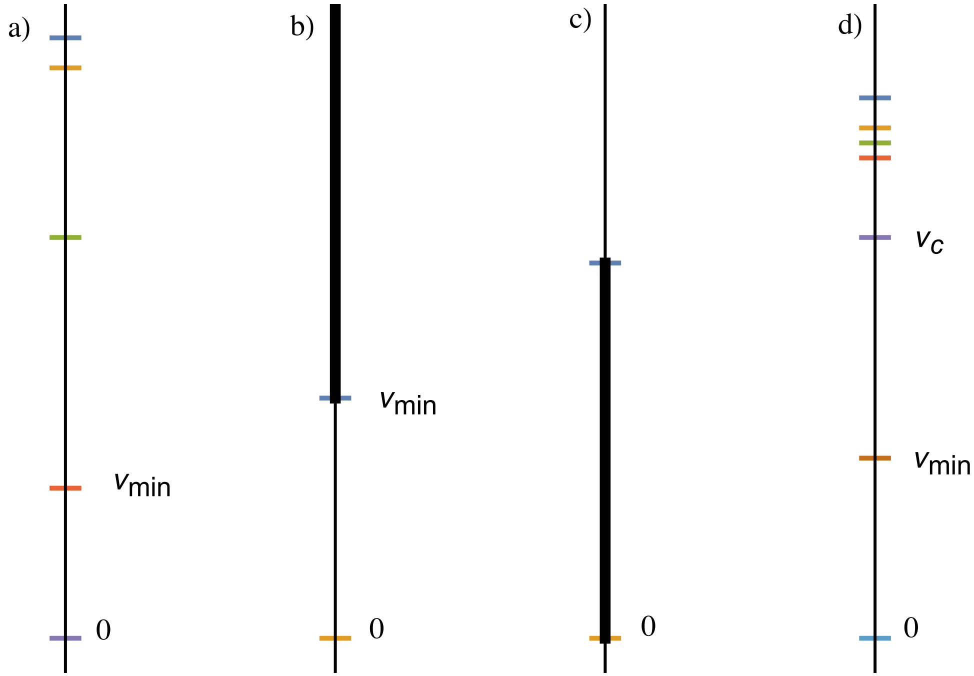
Eigenvalue problem:
$$-\nu h = L_0[h]$$

Solution:
$$\varphi(t,\mathbf{p}) = \sum_n C_n e^{-\nu_n t} h_n(\mathbf{p})$$

Spectrum of linearized kinetic operator

(at zero spatial momentum, i.e. all hydro modes are at 0)

Wang Chang & Uhlenbeck (1952), Grad (1963)



a) Discrete spectrum, $U = \alpha/r^4$

b) Continuous spectrum with a gap, $U = \alpha/r^n, \;\; n > 4$

c) Continuous gapless spectrum, $U = \alpha/r^n, \;\; n < 4$

d) Hod spectrum

Relaxation time in kinetic theory (continued)

$$\varphi(t, \mathbf{p}) = \sum_n C_n e^{-\nu_n t} h_n(\mathbf{p})$$

$$-\nu h = L_0[h]$$

The hierarchy of relaxation times is determined by the spectrum of the linearized kinetic operator

$$\tau_R = 1/\nu_{min}$$

For weakly inhomogeneous systems:

$$\frac{\partial F}{\partial t} + \frac{p_i}{m}\frac{\partial F}{\partial r^i} - \frac{\partial U(r)}{\partial r^i}\frac{\partial F}{\partial p_i} = -\frac{F - F_0}{\tau_R}$$

Krook-Gross-Bhatnagar (KGB) equation (1959) a.k.a. “RTA”

Transport is then essentially determined by the relaxation time, e.g. shear viscosity is

$$\eta = \tau_R\, s\, T$$

Of course, the situation is significantly more complicated for generic weakly interacting quantum systems (relativistic or not) at finite temperature and/or density

Resummations typically lead to effective kinetic theory (AGD, Popov, AMY++). Transport is determined by the spectrum of kinetic operator. Partial results exist, yet e.g. the analytic structure of correlators of gauge-invariant operators is generically unknown (but see recent work by Guy D. Moore, 1803.0073).

G.D.Moore, ``Stress-stress correlator in phi^4 theory: Poles or a Cut?,'' arXiv:1803.00736 [hep-ph].

A.Kurkela and U.A.Wiedemann, ``Analytic structure of nonhydrodynamic modes in kinetic theory,'' arXiv:1712.04376 [hep-ph].

P.Romatschke, ``Retarded correlators in kinetic theory: branch cuts, poles and hydrodynamic onset transitions,'' [arXiv:1512.02641 [hep-th]].

Strong coupling

How to compute second order transport coefficients?

Fluid-gravity correspondence [Bhattacharyya et al, 2007]

Quasinormal spectrum [Baier et al, 2007]

Kubo formulas & three-point functions
[Moore, Sohrabi, Saremi, 2010, 2011; Arnold, Vaman, Wu, Xiao, 2011]

First and second order transport coefficients of *conformal* holographic fluids *to leading order* in supergravity approximation

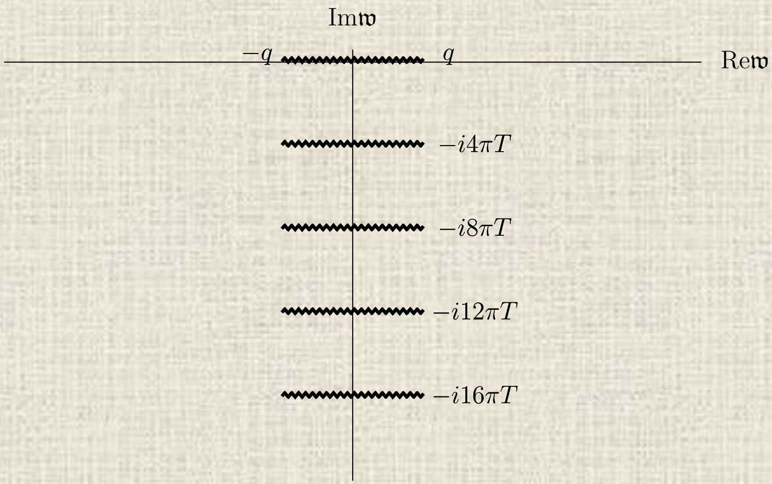
$$
\begin{aligned}
\eta &= s/4\pi\,, \\
\tau_\Pi &= \frac{d}{4\pi T}\left(1+\frac{1}{d}\left[\gamma_E+\psi\left(\frac{2}{d}\right)\right]\right)\,, \\
\kappa &= \frac{d}{d-2}\,\frac{\eta}{2\pi T}\,, \\
\lambda_1 &= \frac{d\eta}{8\pi T}\,, \\
\lambda_2 &= \left[\gamma_E+\psi\left(\frac{2}{d}\right)\right]\,\frac{\eta}{2\pi T}\,, \\
\lambda_3 &= 0
\end{aligned}
$$

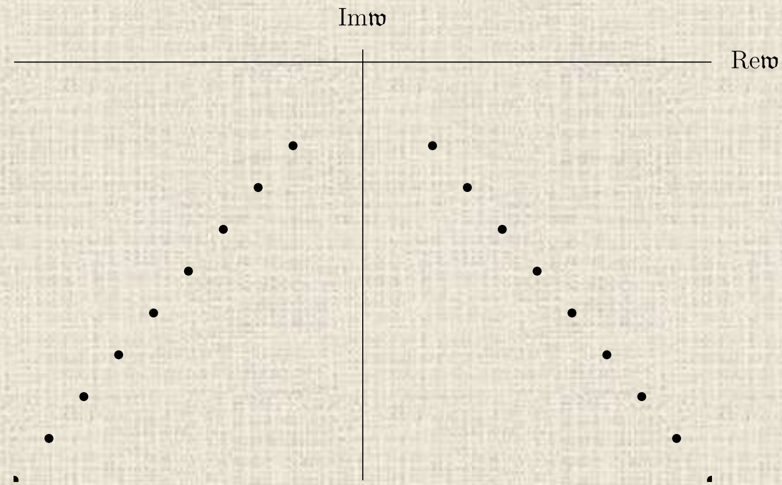
Bhattacharyya et al, 2008

Note: $2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = 0$

Cuts versus poles: a mystery

Singularities of a Green's function in the complex frequency plane





Weak (vanishing) coupling

Hartnoll, Kumar (2005)

Strong (infinite) coupling

AOS (2002)

We should be able to interpolate between the two limits…

Coupling constant corrections to N=4 SYM transport coefficients

$$S = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left(R + \frac{12}{L^2} + \gamma \mathcal{W}\right) \qquad \gamma = \lambda^{-3/2}\zeta(3)/8$$

$$\eta = \frac{\pi}{8}N_c^2 T^3\,(1 + 135\gamma + \dots)$$

$$\tau_\Pi = \frac{(2-\ln 2)}{2\pi T} + \frac{375\gamma}{4\pi T} + \dots$$

$$\kappa = \frac{N_c^2 T^2}{8}(1 - 10\gamma + \dots)$$

$$\lambda_1 = \frac{N_c^2 T^2}{16}(1 + 350\gamma + \dots)$$

$$\lambda_2 = -\frac{N_c^2 T^2}{16}(2\ln 2 + 5\,(97 + 54\ln 2)\,\gamma + \dots)$$

$$\lambda_3 = \frac{25 N_c^2 T^2}{2}\,\gamma + \dots$$

Note: $2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = 0$

Curvature squared corrections to transport coefficients of a (hypothetical) strongly coupled liquid

$$S_{R^2} = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left[R - 2\Lambda + L^2\left(\alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\right]$$

$$\eta = \frac{r_+^3}{2\kappa_5^2}\left(1 - 8\left(5\alpha_1 + \alpha_2\right)\right) + \ldots$$

$$\eta\tau_\Pi = \frac{r_+^2\left(2 - \ln 2\right)}{4\kappa_5^2}\left(1 - \frac{26}{3}\left(5\alpha_1 + \alpha_2\right)\right) - \frac{r_+^2\left(23 + 5\ln 2\right)}{12\kappa_5^2}\alpha_3 + \ldots$$

$$\kappa = \frac{r_+^2}{2\kappa_5^2}\left(1 - \frac{26}{3}\left(5\alpha_1 + \alpha_2\right)\right) - \frac{25 r_+^2}{6\kappa_5^2}\alpha_3 + \ldots$$

$$\lambda_1 = \frac{r_+^2}{4\kappa_5^2}\left(1 - \frac{26}{3}\left(5\alpha_1 + \alpha_2\right)\right) - \frac{r_+^2}{12\kappa_5^2}\alpha_3 + \ldots$$

$$\lambda_2 = -\frac{r_+^2\ln 2}{2\kappa_5^2}\left(1 - \frac{26}{3}\left(5\alpha_1 + \alpha_2\right)\right) - \frac{r_+^2\left(21 + 5\ln 2\right)}{6\kappa_5^2}\alpha_3 + \ldots$$

$$\lambda_3 = -\frac{28 r_+^2}{\kappa_5^2}\alpha_3 + \ldots$$

Note: $2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = 0$

Gauss-Bonnet corrections to transport coefficients of a (hypothetical) strongly coupled liquid

$$S_{GB} = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left[R + \frac{12}{L^2} + \frac{\lambda_{GB}}{2} L^2 \left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right) \right]$$

$$\eta = \frac{r_+^2}{2\kappa_5^2}\left(1 - 4\lambda_{GB}\right)$$

$$\eta\tau_\Pi = \frac{r_+^2\left(2 - \ln 2\right)}{4\kappa_5^2} - \frac{r_+^2\left(25 - 7\ln 2\right)}{8\kappa_5^2}\lambda_{GB} + \dots$$

$$\kappa = \frac{r_+^2}{2\kappa_5^2} - \frac{17 r_+^2}{4\kappa_5^2}\lambda_{GB} + \dots$$

$$\lambda_1 = \frac{r_+^2}{4\kappa_5^2} - \frac{9 r_+^2}{8\kappa_5^2}\lambda_{GB} + \dots$$

$$\lambda_2 = -\frac{r_+^2 \ln 2}{2\kappa_5^2} - \frac{7 r_+^2\left(1 - \ln 2\right)}{4\kappa_5^2}\lambda_{GB} + \dots$$

$$\lambda_3 = -\frac{14 r_+^2}{\kappa_5^2}\lambda_{GB} + \dots$$

Brigante, Myers, H.Liu, Myers, Shenker, Yaida, 2008

Shaverin, Yarom, 2012

Note: $2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = 0$

Non-perturbative Gauss-Bonnet corrections to transport coefficients of a (hypothetical) strongly coupled liquid

$$S_{GB} = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left[R + \frac{12}{L^2} + \frac{\lambda_{GB}}{2}L^2\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\right]$$

$$\eta = \frac{s}{4\pi}\gamma^2 = \frac{s}{4\pi}(1 - 4\lambda_{GB})$$ Brigante et al, 2008

$$\tau_\Pi = \frac{1}{2\pi T}\left(\frac{1}{4}(1+\gamma)\left(5+\gamma-\frac{2}{\gamma}\right) - \frac{1}{2}\log\left[\frac{2(1+\gamma)}{\gamma}\right]\right)$$ Banerjee and Dutta, 2011

$$\lambda_1 = \frac{\eta}{2\pi T}\left(\frac{(1+\gamma)(3-4\gamma+2\gamma^3)}{2\gamma^2}\right)$$ Grozdanov and AOS, 2014

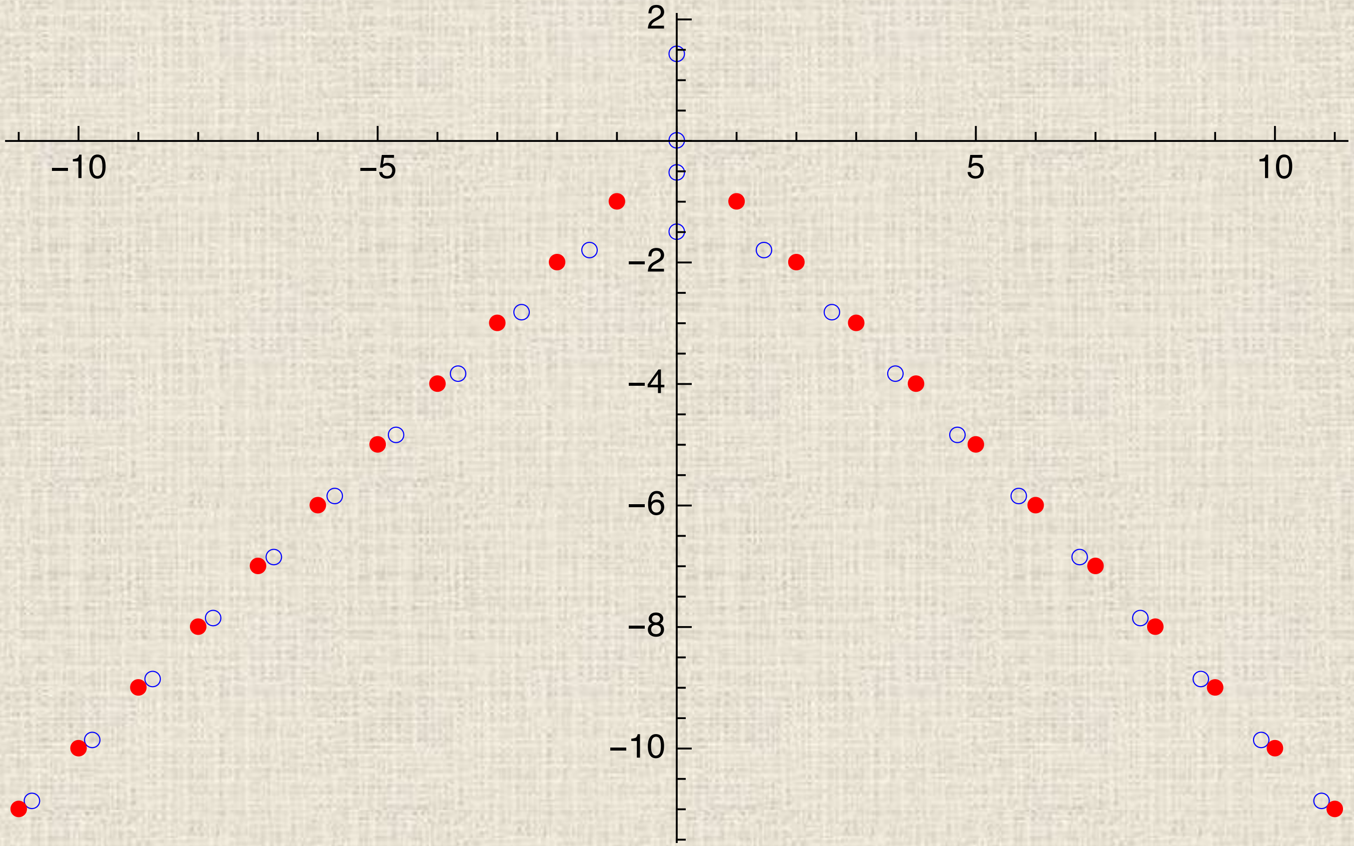
$$\lambda_2 = -\frac{\eta}{\pi T}\left(-\frac{1}{4}(1+\gamma)\left(1+\gamma-\frac{2}{\gamma}\right) + \frac{1}{2}\log\left[\frac{2(1+\gamma)}{\gamma}\right]\right)$$ Grozdanov and AOS, 2014

$$\lambda_3 = -\frac{\eta}{\pi T}\left(\frac{(1+\gamma)(3+\gamma-4\gamma^2)}{\gamma^2}\right)$$ Grozdanov and AOS, 2014

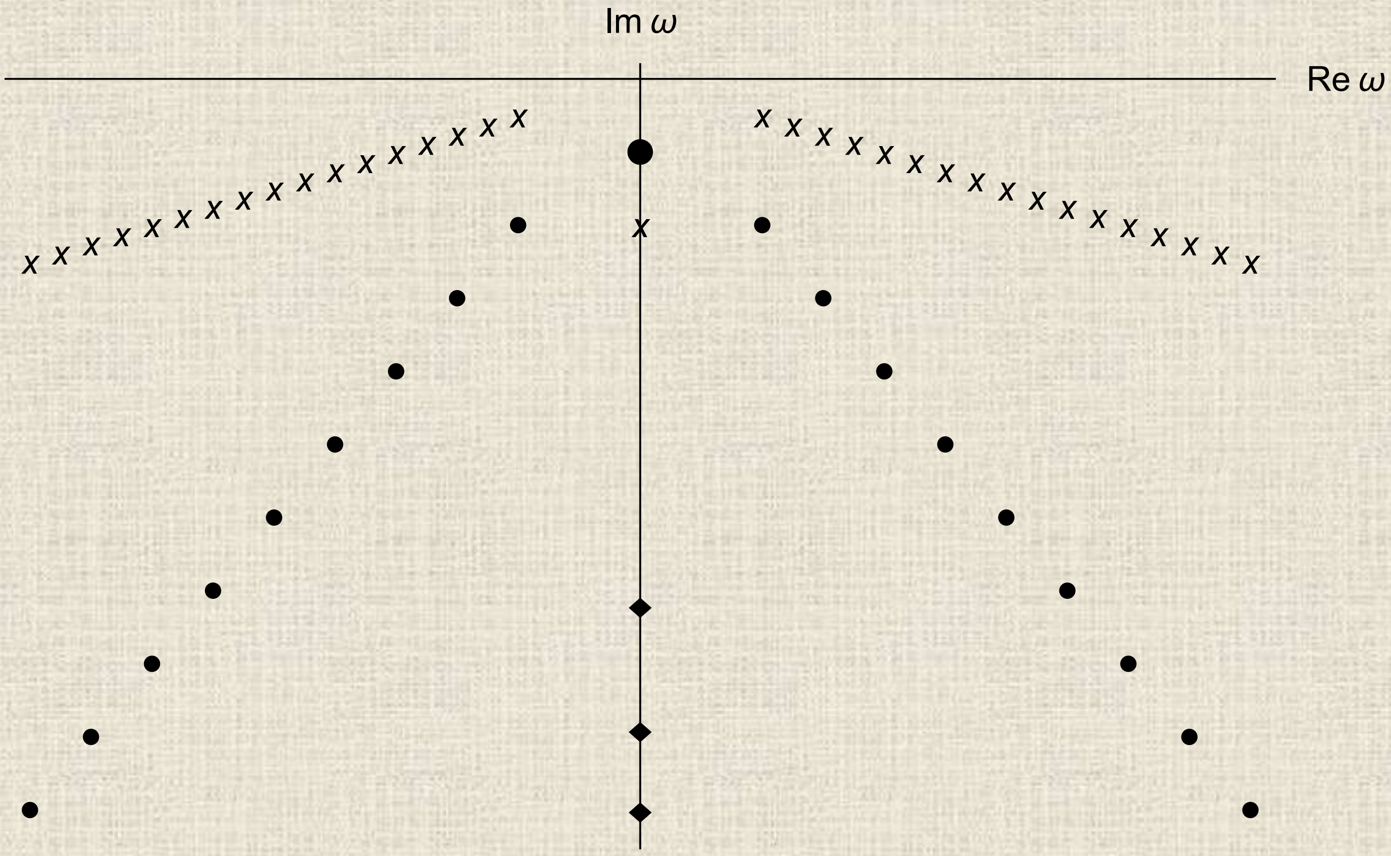
$$\kappa = \frac{\eta}{\pi T}\left(\frac{(1+\gamma)(2\gamma^2-1)}{2\gamma^2}\right)$$ Banerjee and Dutta, 2011

$$H(\lambda_{GB}) = 2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = -\frac{\eta}{\pi T}\frac{(1-\gamma_{GB})(1-\gamma_{GB}^2)(3+2\gamma_{GB})}{\gamma_{GB}^2} = -\frac{40\lambda_{GB}^2\eta}{\pi T} + \ldots$$

Poles (blue) and zeros (red) of a typical retarded correlator at infinite coupling (dual gravity results)

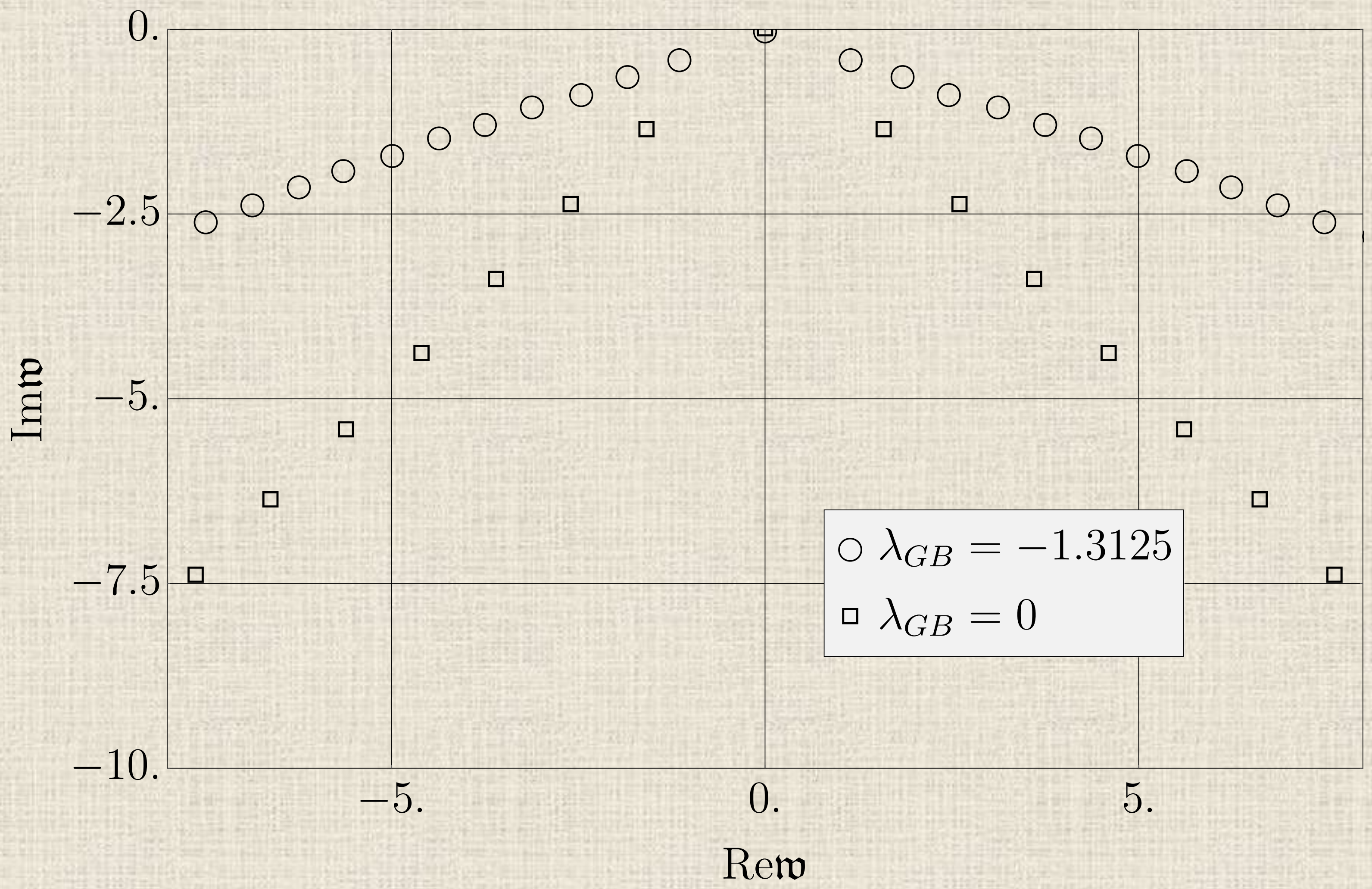


Singularities of stress-energy tensor Green's function at infinite (black dots) and finite (black crosses and diamonds) coupling

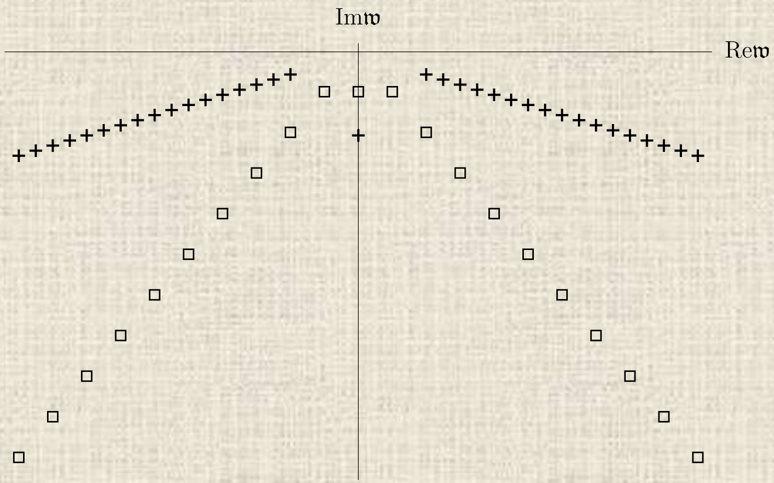


Earlier work: Stricker, 1307.2736 [hep-th]; Waeber, Schäfer, Vuorinen and Yaffe, 1509.02983 [hep-th].

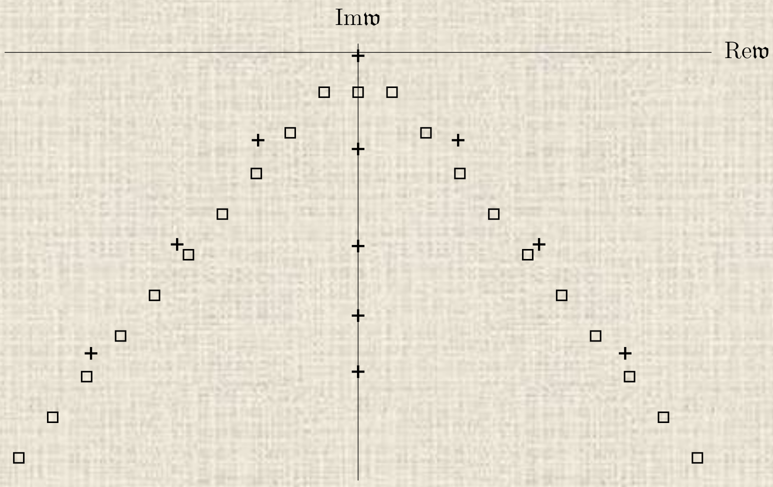
Quasinormal spectrum of Gauss-Bonnet black brane
vs
AdS-Schwarzschild black brane (numerical data)



Singularities of stress-energy tensor Green's function in different regimes of viscosity-entropy ratio (shear channel)



$$\frac{\eta}{s} \geq \frac{1}{4\pi}$$



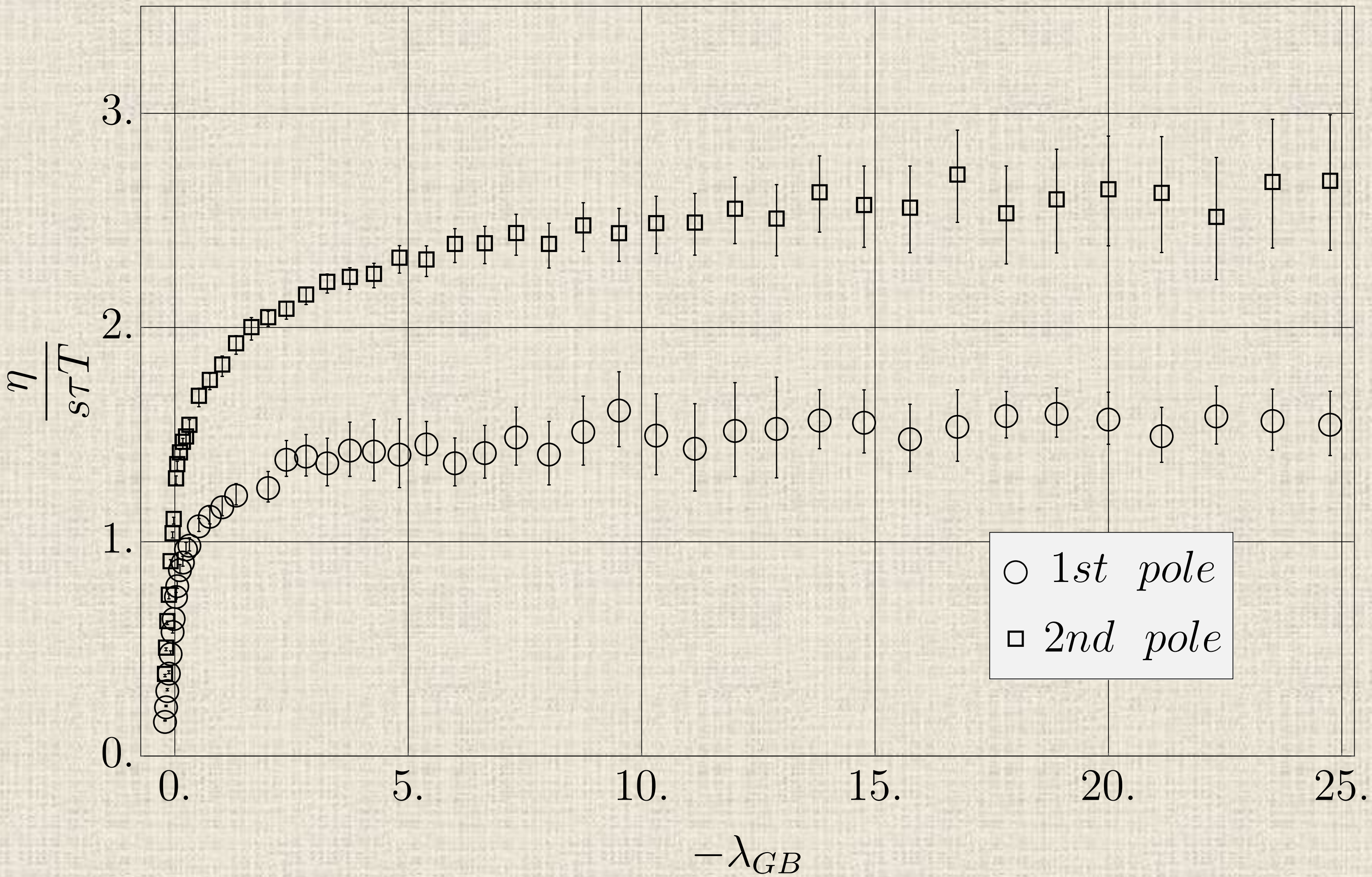
$$\frac{\eta}{s} < \frac{1}{4\pi}$$

White squares: poles at infinite coupling
Crosses: poles at finite coupling

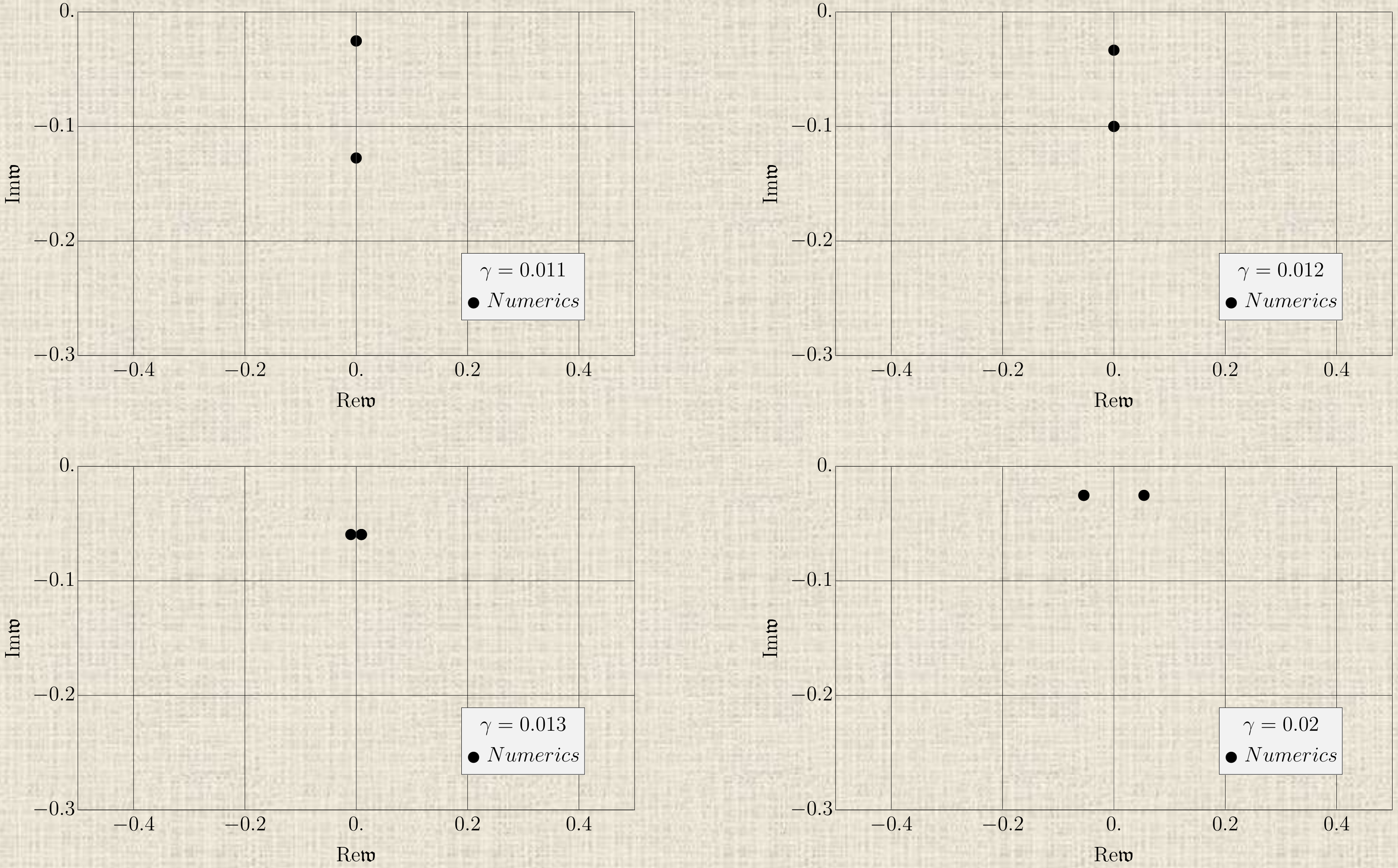
On the "unreasonable effectiveness" of kinetic theory at strong coupling

Recall that in kinetic theory $\eta = const\; s\, \tau_R\, T$

What happens at large but finite coupling, with $\tau_R = 1/|\mathrm{Im}\,\omega_F|$?

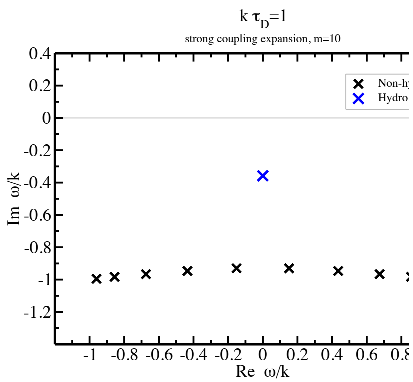


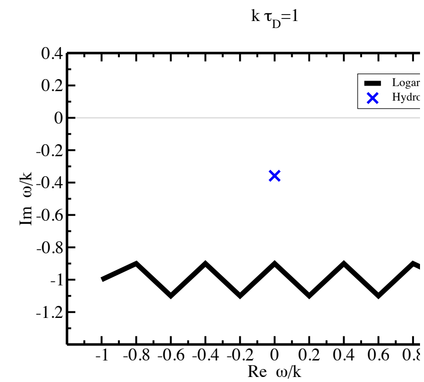
Breakdown of hydrodynamics at (large) finite coupling

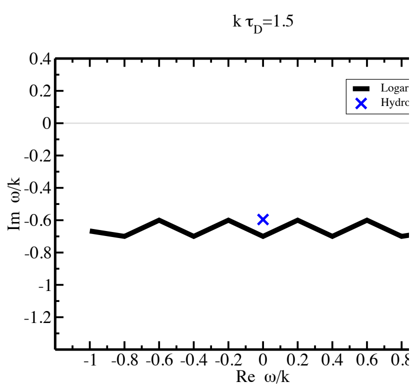


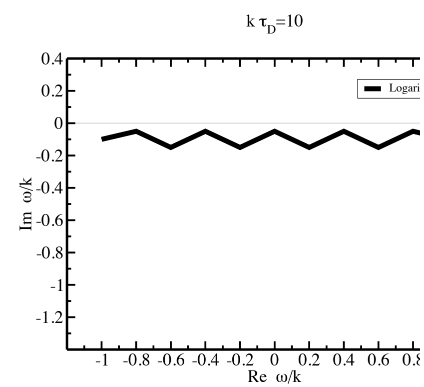
$$q_c = q_c(\lambda)$$

Kinetic theory (relaxation time approximation)





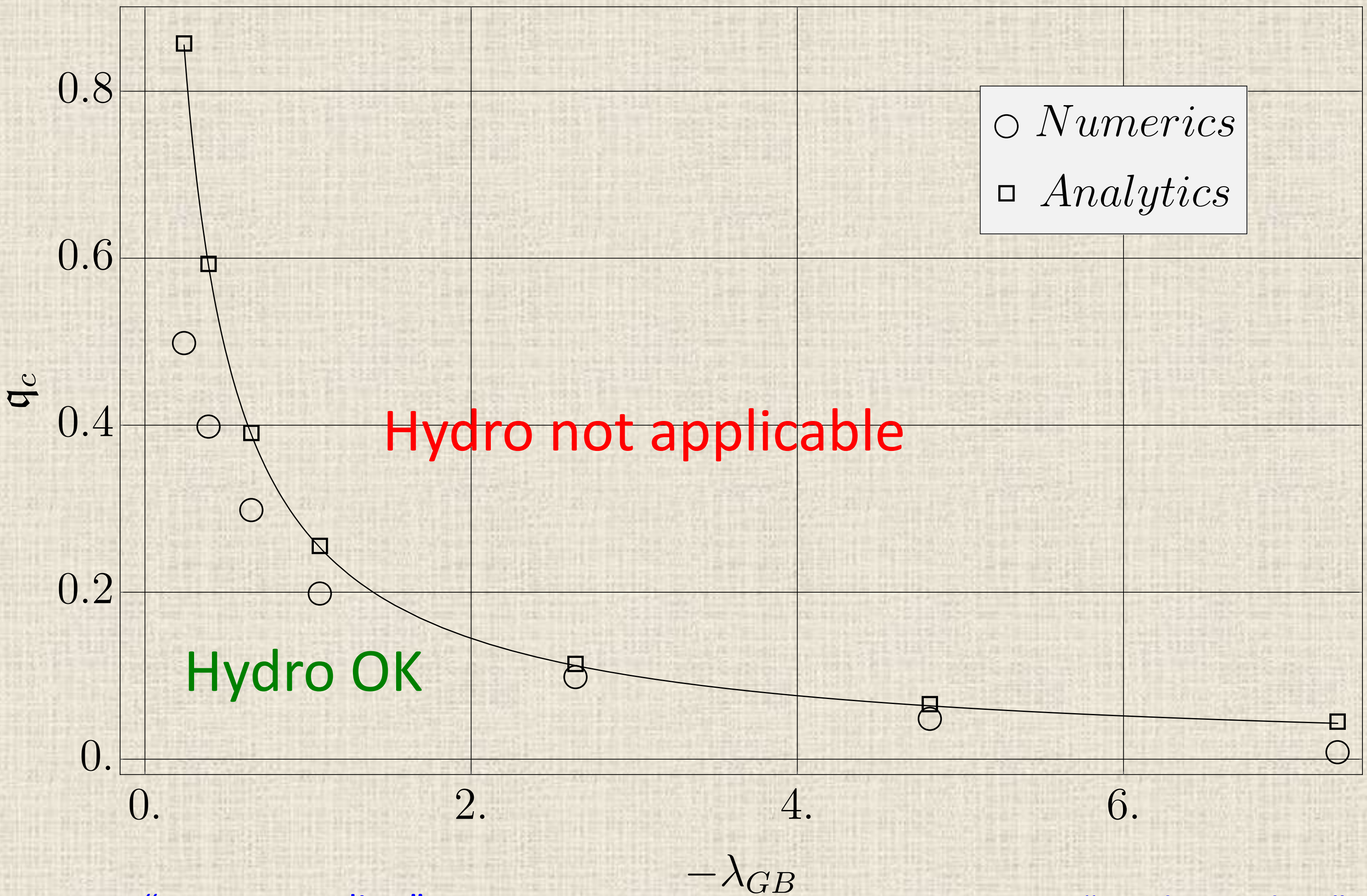




Figs from Paul Romatschke, 1512.02641 [hep-th]

See also A.Kurkela and U.Wiedemann, 1712.04376 [hep-ph]

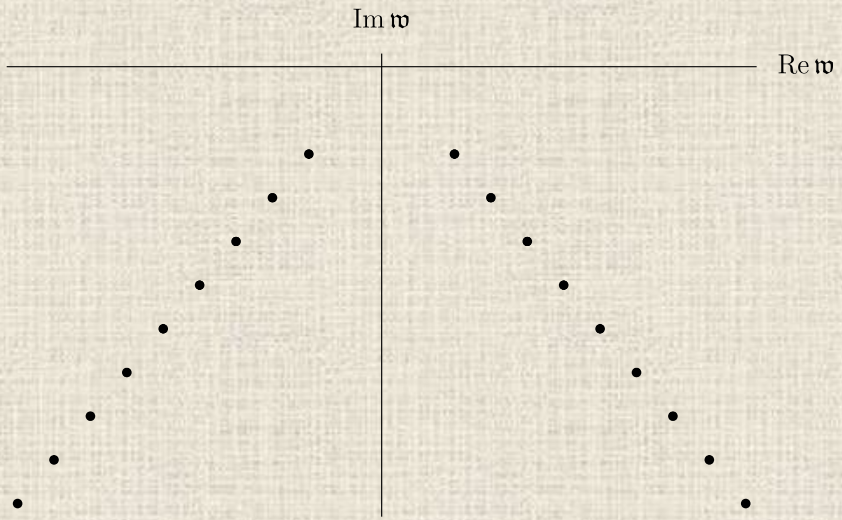
“Applicability of hydrodynamics” as a function of coupling

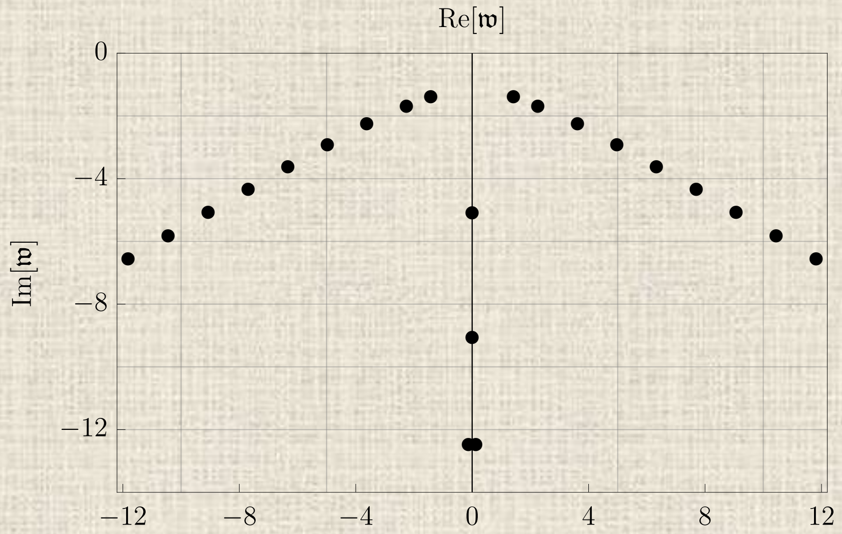


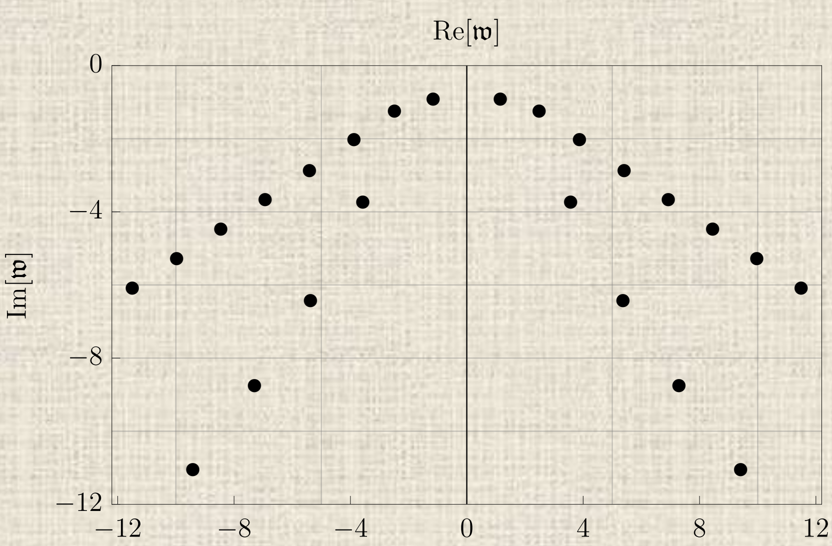
“strong coupling”

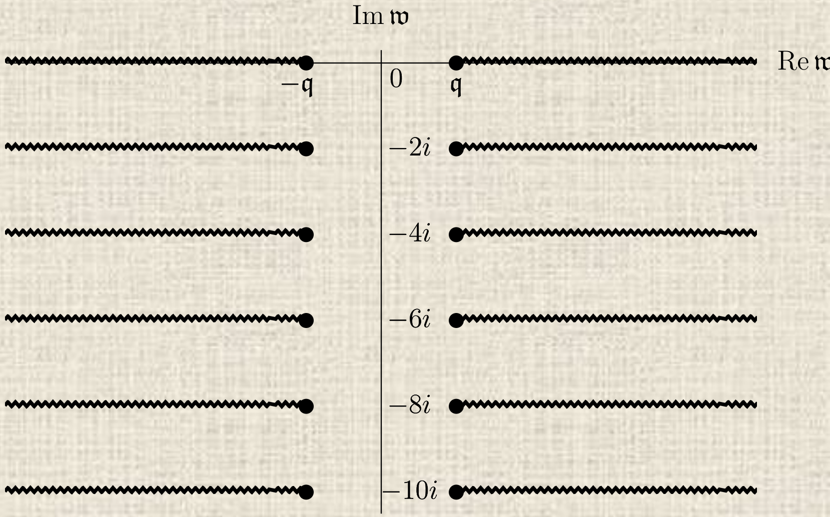
“weak coupling”

POLES VS CUTS: FROM INFINITE TO ZERO COUPLING









Transport peak of spectral functions at large finite coupling

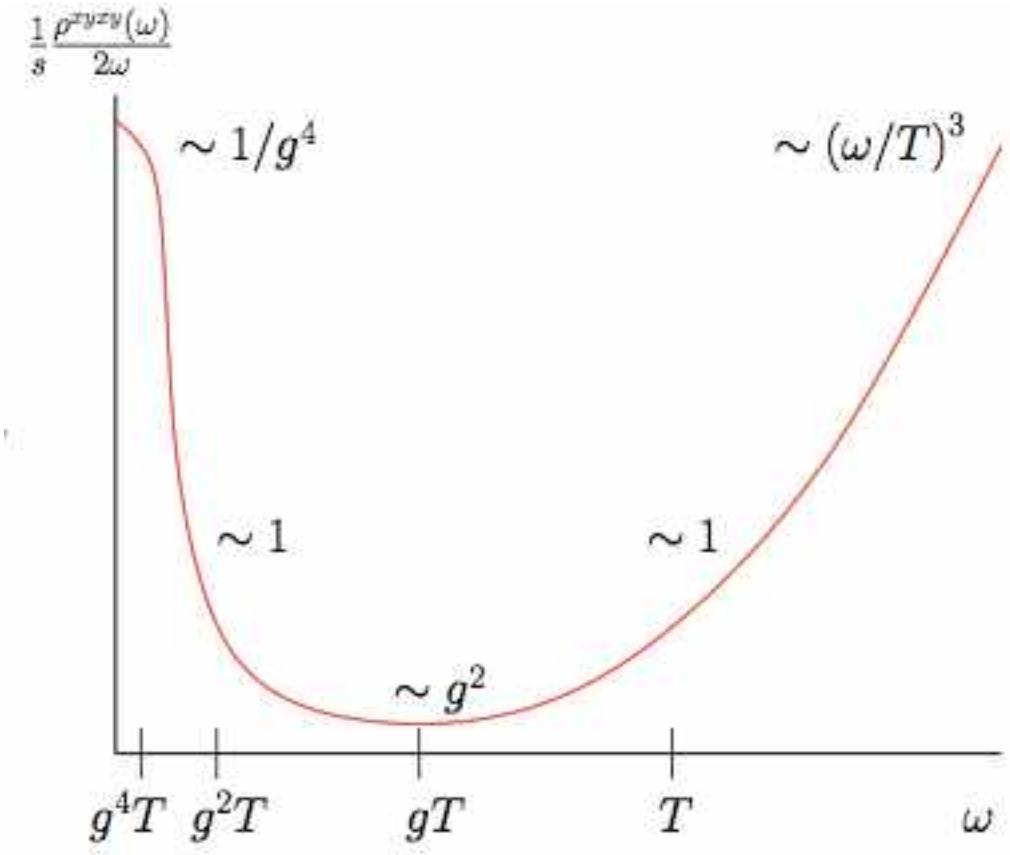
$$\chi_{xy,xy}(k) = \int d^4x e^{-ikx} \langle [T_{xy}(x) T_{xy}(0)] \rangle = -2\, \mathrm{Im}\, G^R_{xy,xy}$$

Viscosity is determined by the height of the peak of the spectral function at w=0.

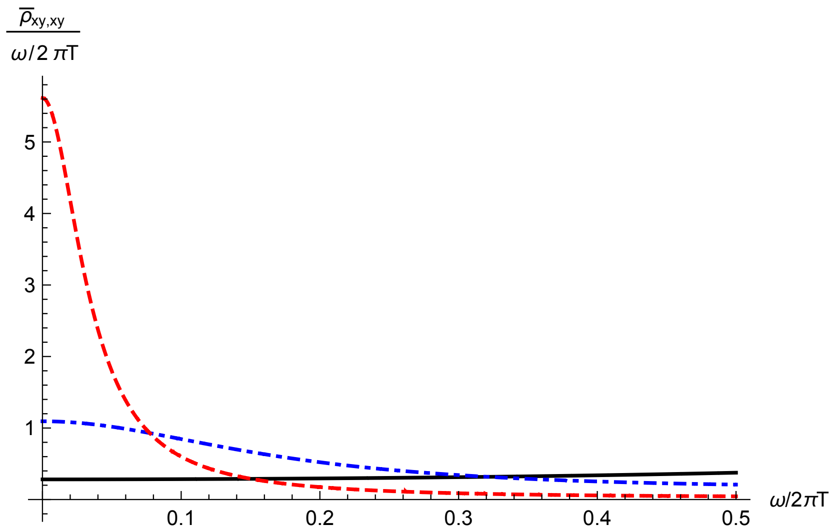
The peak is affected by the singularities of the correlator in the complex w plane.

What kind of singularities? Are they the same at weak and strong coupling?

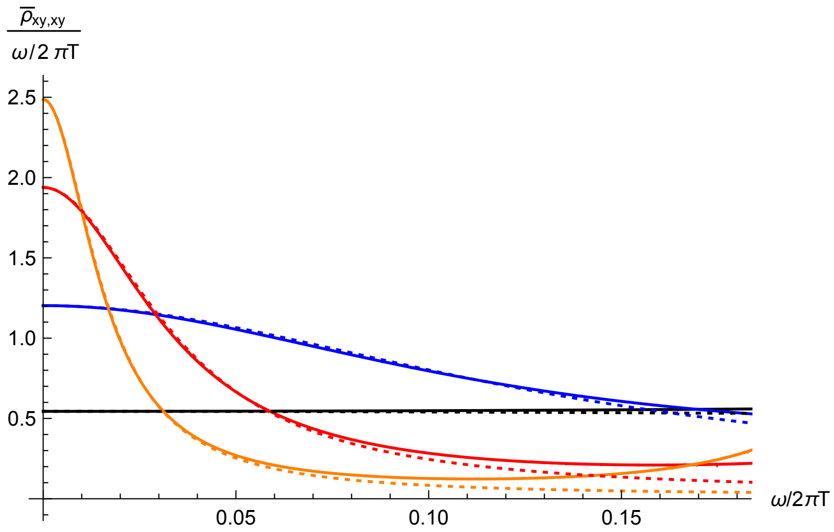
Transport peak in QCD at finite temperature (sketch)



Transport peak of spectral functions at large finite coupling



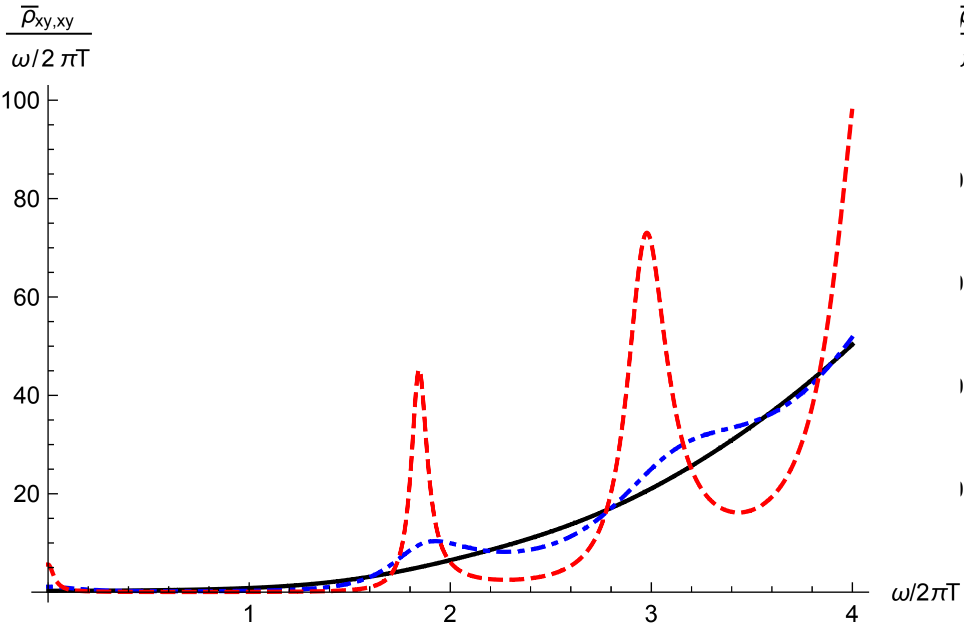
$\mathcal{N} = 4$ SYM



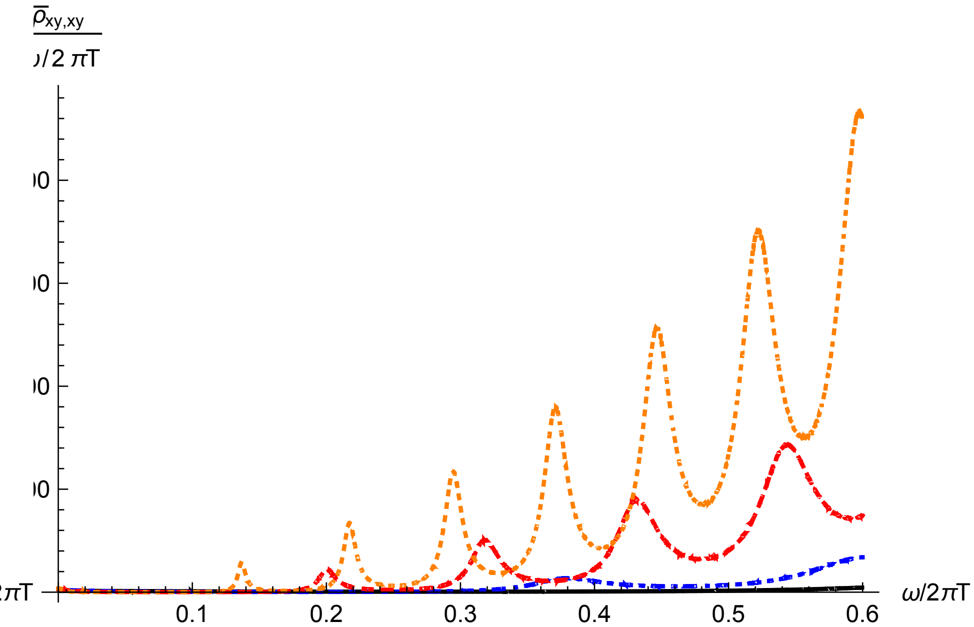
Gauss-Bonnet

Note: Black solid line is the spectral function at infinite coupling

Quasiparticle peaks at high frequency



$\mathcal{N} = 4$ SYM



Gauss-Bonnet

Linear instability of black brane backgrounds in higher-derivative gravity

Coupling constant corrections to the entropy, viscosity, correlators etc are coming from

$$S_{IIB} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(R - \frac{1}{2}\left(\partial\phi\right)^2 - \frac{1}{4\cdot 5!}F_5^2 + \gamma e^{-\frac{3}{2}\phi}\mathcal{W} + \dots\right)$$

$$\mathcal{W} = C^{\alpha\beta\gamma\delta}C_{\mu\beta\gamma\nu}C_{\alpha}{}^{\rho\sigma\mu}C^{\nu}{}_{\rho\sigma\delta} + \frac{1}{2}C^{\alpha\delta\beta\gamma}C_{\mu\nu\beta\gamma}C_{\alpha}{}^{\rho\sigma\mu}C^{\nu}{}_{\rho\sigma\delta} \qquad \gamma = \lambda^{-3/2}\zeta(3)L^6/8$$

The corrected 5d metric is

$$ds^2 = \frac{(\pi T L)^2}{u}\left(-e^{a(u)}f(u)dt^2 + dx^2 + dy^2 + dz^2\right) + e^{b(u)}\frac{L^2 du^2}{4u^2 f}$$

$$a(u) = -15\gamma\left(5u^2 + 5u^4 - 3u^6\right), \qquad b(u) = 15\gamma\left(5u^2 + 5u^4 - 19u^6\right)$$

Gubser, Klebanov and Tseytlin, hep-th/9805156; Pawelczyk and Theisen, hep-th/9808126

Linear metric fluctuations satisfy e.o.m. of the type

$$\partial_u^2 Z_1 - \frac{1+u^2}{u\left(1-u^2\right)}\partial_u Z_1 + \frac{w^2 - q^2\left(1-u^2\right)}{u\left(1-u^2\right)^2}Z_1 = \gamma\, G_1\left[Z_1\right]$$

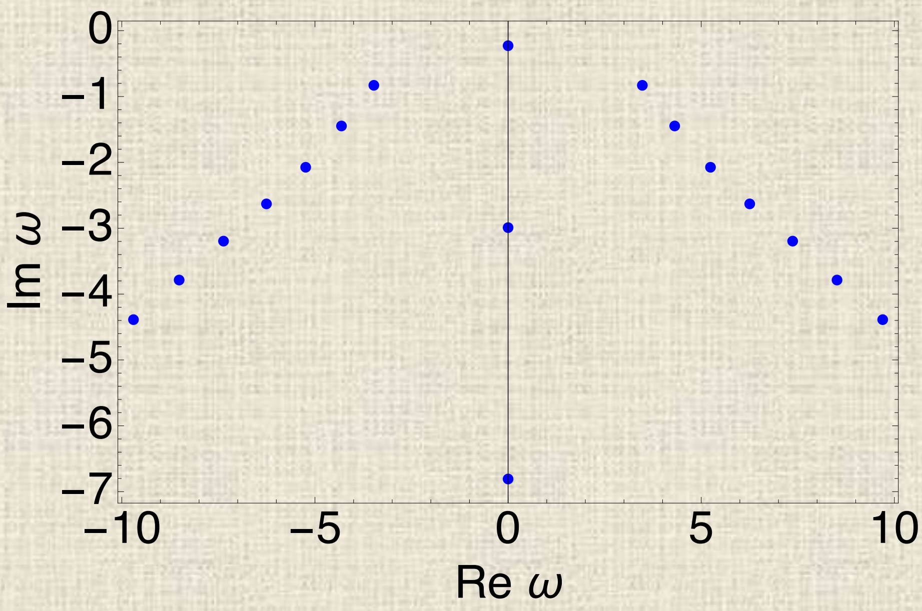
This can be re-written in Eddington-Finkelstein coordinates as

$$-\sqrt{b}\frac{d}{du}\left(\frac{1}{\sqrt{b}}\psi'\right) - 2iw\sqrt{b}\psi' + V_{eff}\psi = 0$$

The sign of the imaginary part of an eigenfrequency is determined by

(Horowitz and Hubeny, 1999)

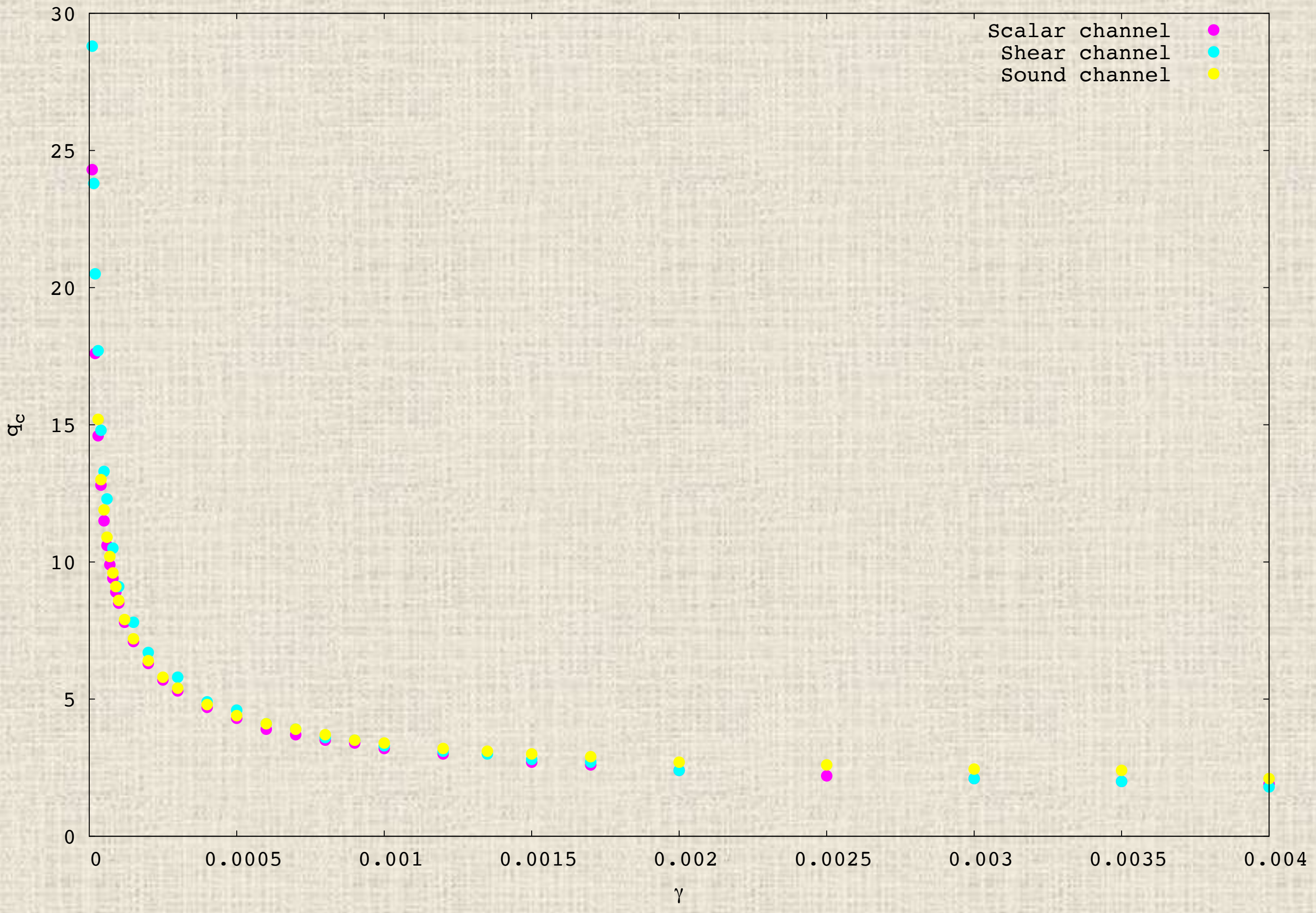
$$\frac{|w|^2|\psi(1)|^2}{\mathrm{Im}\, w} = -\int_0^1 \frac{du}{\sqrt{b}}\left(|\psi'|^2 + V_{eff}|\psi|^2\right)$$



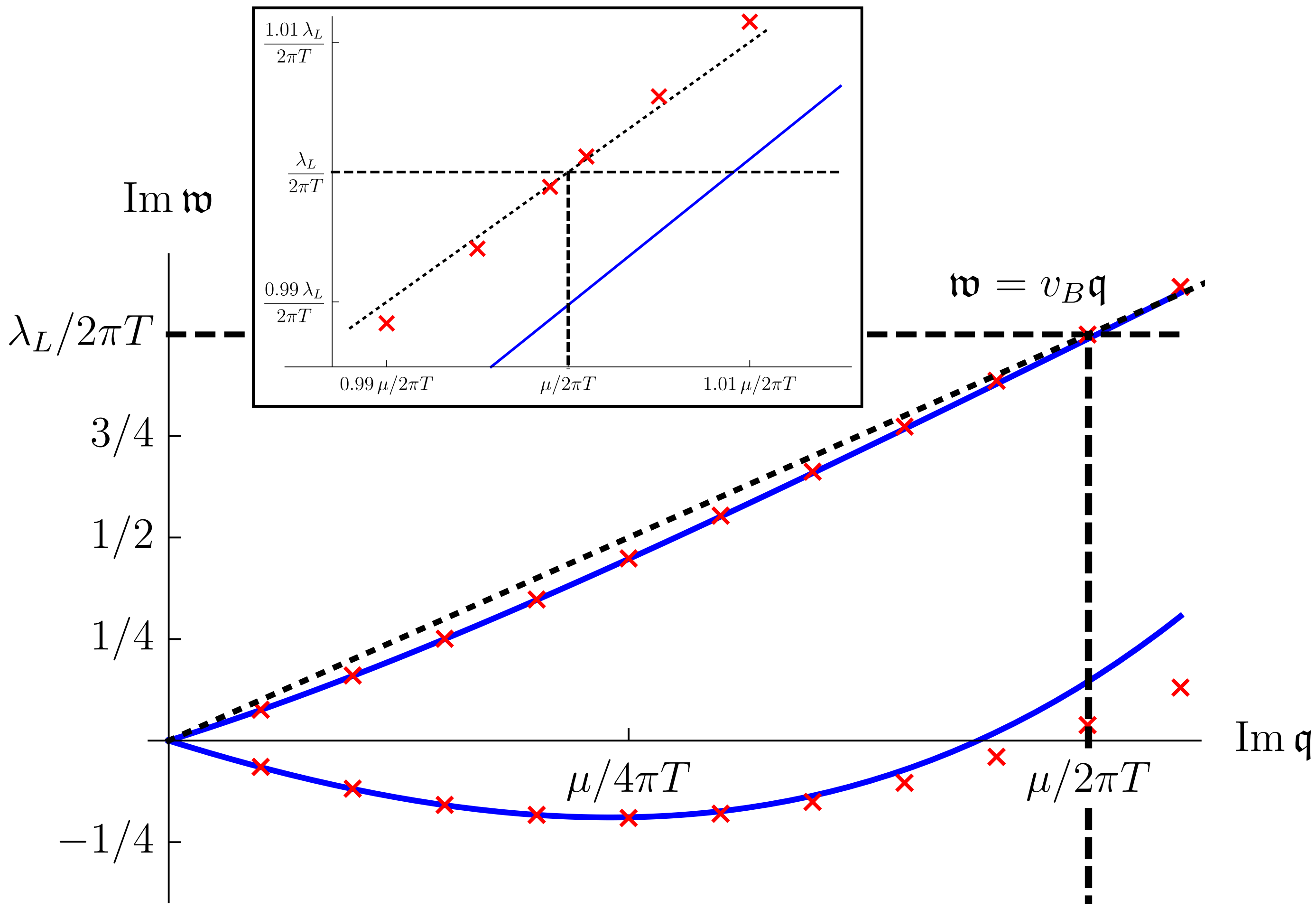
The instability seems to be generic.

Konoplya and Zhidenko, 2017; Grozdanov, Gushterov, AOS, 2018

Critical momentum vs (inverse) coupling in N=4 SYM



Hydrodynamic modes and quantum chaos



Grozdanov, Schalm and V.Scopelliti, 1710.00921 [hep-th]

Conclusions & open questions

Finite coupling corrections seem to show qualitatively similar behavior irrespective of the precise structure of higher derivative terms in dual gravity (we did R^2 and R^4)

How robust are the results (structure of higher derivative expansion)?

We observe breakdown of hydrodynamics at coupling-dependent value of a wave-vector. The dependence on coupling suggests that hydrodynamics has a wider applicability range at stronger coupling

Our results suggest that kinetic theory results may be formally still applicable in the intermediate and strong coupling regime where the use of kinetic theory itself cannot be justified. In particular, transport peak is visible at large finite coupling due to inflow of poles. Compare to pQFT?

We observe qualitatively different analytic structure of correlators depending on whether

$$\eta/s > 1/4\pi \text{ or } \eta/s < 1/4\pi$$

We observe linear instability of the dual metric at finite coupling. Need to explain this.

THANK YOU!