Holographic and Localization Calculations of Boundary F in $\mathcal{N} = 4$ SYM

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$$\mathrm{CFT}_{\mathrm{UV}}$$

$$\downarrow \quad + \int d^d x \, \lambda O$$

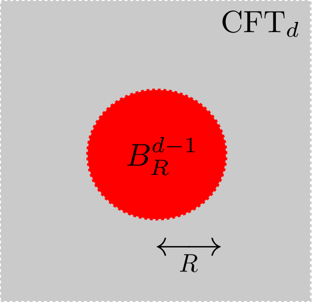
$$\mathrm{CFT}_{\mathrm{IR}}$$

$$c_{\mathrm{UV}} \geq c_{\mathrm{IR}}$$

C-theorems known in

- $d = 2$: "c-theorem" [Zamolodchikov]
- $d = 3$: "F-theorem" [Jafferis, Klebanov, Pufu, Safdi; Casini, Huerta; Klebanov, Pufu, Safdi]
- $d = 4$: "a-theorem" [Cardy; Komargodski, Schwimmer]

Generalized F-Theorem



Candidate C-function: universal term in vacuum entanglement entropy of ball-shaped region ("generalized F")

Generalizes results from $d = 2, 3, 4$

$$S[B_R^{d-1}] = \underbrace{a_{d-2}\,(R/\epsilon)^{d-2} + a_{d-4}\,(R/\epsilon)^{d-4} + \ldots}_{\text{scheme-dependent}} + \underbrace{\begin{cases} 4(-1)^{\frac{d-2}{2}} A \ln(R/\epsilon) & 2 \mid d \\ (-1)^{\frac{d-1}{2}} F & 2 \nmid d \end{cases}}_{\text{universal}}$$

$F_{\mathrm{UV}} \geq F_{\mathrm{IR}} \qquad A_{\mathrm{UV}} \geq A_{\mathrm{IR}}$

$\ln Z[S_R^d]_{\text{univ.}} = S[B_R^{d-1}]_{\text{univ.}}$ (CHM)

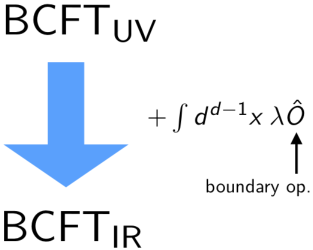
Boundary Conformal Field Theory

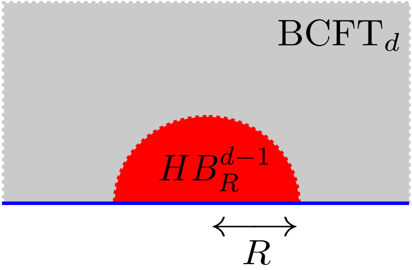
Now take CFT_d and...

- Add a boundary
- Impose boundary conditions which preserve conformal symmetry of CFT_{d-1}
- (Possibly) couple in extra DOF at boundary

$\implies$ boundary conformal field theory (BCFT)!

Boundary $\tilde{F}$-Theorem





Now have UV divergences of both d-dimensional and $(d-1)$-dimensional origin

$$S[HB_R^{d-1}] = \tilde{a}_{d-2}\,(R/\epsilon)^{d-2} + \tilde{a}_{d-3}\,(R/\epsilon)^{d-3} + \ldots + \begin{cases} 4(-1)^{\frac{d-2}{2}} A \ln(R/\epsilon) + (-1)^{\frac{d-2}{2}} \tilde{F} & 2 \mid d \\ 4(-1)^{\frac{d-3}{2}} \tilde{A} \ln(R/\epsilon) + (-1)^{\frac{d-1}{2}} F & 2 \nmid d \end{cases}$$

Want to extract universal contribution

Boundary $\tilde{F}$-Theorem

$$S[HB_R^{d-1}] = \tilde{a}_{d-2}\,(R/\epsilon)^{d-2} + \tilde{a}_{d-3}\,(R/\epsilon)^{d-3} + \ldots + \begin{cases} 4(-1)^{\frac{d-2}{2}} A \ln(R/\epsilon) + (-1)^{\frac{d-2}{2}} \tilde{F} & 2 \mid d \\ 4(-1)^{\frac{d-3}{2}} \tilde{A} \ln(R/\epsilon) + (-1)^{\frac{d-1}{2}} F & 2 \nmid d \end{cases}$$

Want to extract universal contribution: define "boundary entropy" (à la g-function in 2D BCFT [Affleck, Ludwig])

$$\begin{aligned} S_\partial(R) &\equiv S^{(\mathrm{BCFT})}[HB_R^{d-1}] - \frac{1}{2} S^{(\mathrm{CFT})}[B_R^{d-1}] \\ &= \underbrace{\tilde{a}_{d-3}\,(R/\epsilon)^{d-3} + \tilde{a}_{d-5}\,(R/\epsilon)^{d-5} + \ldots}_{\text{scheme-dependent}} + \underbrace{\begin{cases} (-1)^{\frac{d-2}{2}} \tilde{F} & 2 \mid d \\ 4(-1)^{\frac{d-3}{2}} \tilde{A} \ln(R/\epsilon) & 2 \nmid d \end{cases}}_{\text{universal}} \end{aligned}$$

$$\tilde{F}_{\mathrm{UV}} \overset{?}{\geq} \tilde{F}_{\mathrm{IR}} \qquad \tilde{A}_{\mathrm{UV}} \overset{?}{\geq} \tilde{A}_{\mathrm{IR}}$$

$$\left(\ln Z[HS_R^d] - \frac{1}{2} \ln Z[S_R^d]\right)_{\text{univ.}} = S_\partial(R)_{\text{univ.}}$$

Boundary F-Theorems: Results, Conjectures, Questions

Known:

- $d = 2$: "g-theorem" [Affleck, Ludwig; Friedan, Konechny; Casini, Landea, Torroba]
- $d = 3$: "b-theorem" [Jensen, O'Bannon; Casini, Landea, Torroba]

Conjectured:

- $d = 4$: "boundary F-theorem" [Estes, Jensen, O'Bannon, Tsatis, Wrase; Gaiotto]

Questions:

- Is boundary $\tilde{F}$ decreasing under boundary RG flow in arbitrary dimension? [Nozaki, Takayanagi, Ugajin; Estes, Jensen, O'Bannon, Tsatis, Wrase; Kobayashi, Nishioka, Sato, Watanabe; Giombi, Khanchandani]

Investigate in the most tractable theories we can think of...

Half-Supersymmetric B.C.s for $\mathcal{N} = 4$ SYM

- Detailed classification of $OSp(2,2|4)$-preserving boundary conditions of $\mathcal{N} = 4$ SYM [Gaiotto, Witten]
- Holographic duals for vacuum states known explicitly in type IIB

  [D'Hoker, Estes, Gutperle; Aharony, Berdichevsky, Berkooz, Shamir]
- Partition function amenable to supersymmetric localization, can draw on previous results:
  - 4D $\mathcal{N} = 2$ SUSY on sphere [Pestun] and hemisphere

    [Gava, Narain, Muteeb, Giraldo-Rivera; Wang; Komatsu,Wang]
  - 3D $\mathcal{N} = 2$ SUSY on S^3

    [Kapustin, Willet, Yaakov; Benvenuti, Pasquetti; Nishioka, Tachikawa, Yamazaki]

Our Mission

Compute boundary F in $\mathcal{N} = 4$ SYM theory with half-SUSY BCs

We are able to find:

- General BCs: F_∂ from holographic calculation in classical SUGRA
- Dirichlet-type and Neumann-type: Exact F_∂ from localization

Boundary F behaves like a measure of local boundary degrees of freedom:

- Mainly negative for Dirichlet-type and small 't Hooft
- Mainly positive for Neumann-type and small 't Hooft
- Boundary F is unbounded from above

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Half-SUSY Boundary Conditions of $\mathcal{N} = 4$ SYM

Fields of 4D $\mathcal{N} = 4$ multiplet:

- Gauge field A^μ
- Fermion Ψ
- Scalars Φ^i (in fundamental of $SO(6)_R$)

$$PSU(2,2|4) \xrightarrow{\text{SUSY boundary at } x_3 = 0} OSp(2,2|4)$$

$$4\text{D } \mathcal{N} = 4 \xrightarrow{\text{SUSY boundary at } x_3 = 0} 3\text{D } \mathcal{N} = 4$$

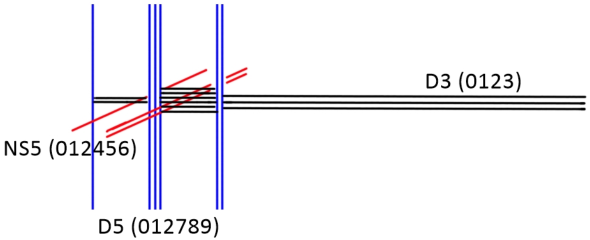
$$SO(6)_R \xrightarrow{\text{SUSY boundary at } x_3 = 0} SO(3) \times SO(3)_R$$

$$\left(A^\mu, \Psi, \Phi^{1,\ldots,6}\right) \xrightarrow{\text{SUSY boundary at } x_3 = 0} \boxed{3\text{D } \mathcal{N} = 4 \text{ vector } \left(A^{0,1,2}, \Psi_+, Y^{1,2,3}\right)}$$

$$\boxed{3\text{D } \mathcal{N} = 4 \text{ hyper } \left(A_3, \Psi_-, X^{1,2,3}\right)}$$

$(X^1, X^2, X^3) = (\Phi^4, \Phi^5, \Phi^6)$ $\quad (Y^1, Y^2, Y^3) = (\Phi^7, \Phi^8, \Phi^9)$ $\quad \Psi_\pm = (1 \pm \Gamma_{3456})\Psi$

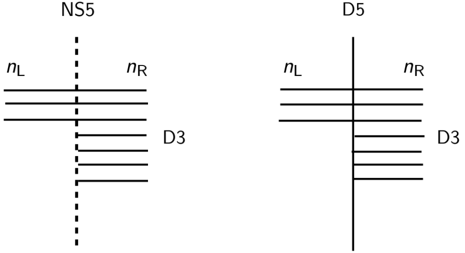
Half-BPS Boundary Conditions of $\mathcal{N} = 4$ SYM



- Theories are low energy description of half-BPS configuration involving D3s ending on D5s/NS5s
- D3-branes semi-infinite in x_3 direction
- D5s/NS5s separated in x_3 direction, span different transverse directions

Half-BPS Boundary Conditions of $\mathcal{N} = 4$ SYM

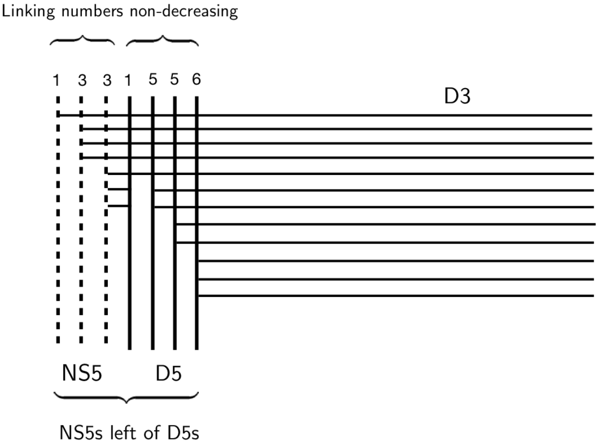
- To completely specify boundary condition, need:
  - D5-brane "linking numbers" $\vec{L} = (L_1, \ldots, L_{N_{D5}})$
  - NS5-brane "linking numbers" $\vec{K} = (K_1, \ldots, K_{N_{NS5}})$



$K = n_R - n_L + N_{D5}^{\text{Left}}$ $\quad$ $L = n_R - n_L + N_{NS5}^{\text{Left}}$

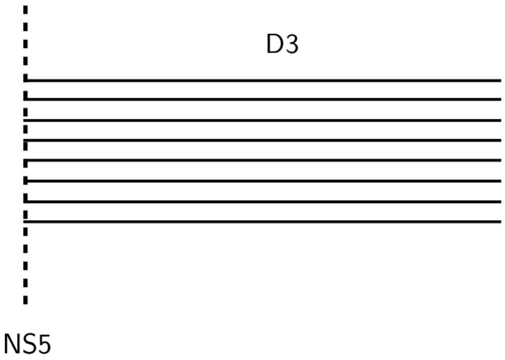
Linking number: Net number of D3-branes ending on given 5-brane from the right, plus number of 5-branes of the opposite type to the left (i.e. smaller x_3).

Example: General Boundary Condition



$$N = 12\,, \qquad \vec{K} = (1,3,3)\,, \qquad \vec{L} = (1,5,5,6)$$

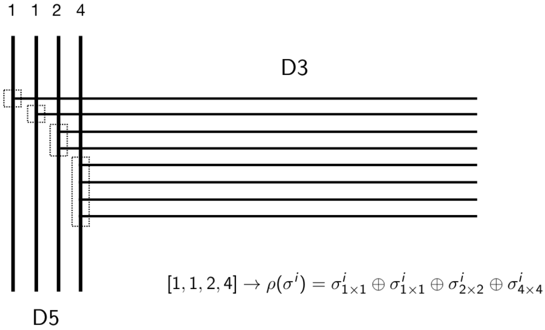
Example: Neumann Boundary Condition



$$\text{Neumann for 3D vector} \longrightarrow F_{3\mu}| = D_3 Y^i| = 0$$

$$\text{Dirichlet for 3D hyper} \longrightarrow X^i| = \Psi_-| = 0$$

Example: Generalized Dirichlet Boundary Condition



$$\text{Dirichlet for 3D vector} \longrightarrow Y^i| = \Psi_+| = 0$$

$$\text{Nahm pole for 3D hyper} \longrightarrow F_{\mu\nu}| = \left[D_3 X^i - \frac{i}{2}\epsilon_{ijk}[X^j, X^k] \right] | = 0$$

- Nahm's equation satisfied by "Nahm pole" $X^i \sim \frac{t^i}{x_3}$, where t^i obey $\mathfrak{su}(2)$
- Irrep dimensions determined by linking numbers

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Supergravity Duals

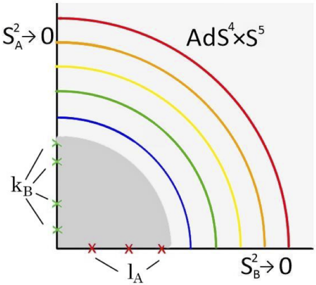
- Vacuum of $OSp(2,2|4)$-invariant $U(N)$ $\mathcal{N}=4$ BCFT should be dual to $OSp(2,2|4)$-invariant solution of type IIB supergravity
- Remarkably, found by D'Hoker/Estes/Gutperle by explicitly solving BPS equations with ansatz

ds^2	$=$	AdS_4	$\times$	S^2_1	$\times$	S^2_2	$\times$	Σ
glob. symm.		$SO(2,3)$	$\times$	$SO(3)$	$\times$	$SO(3)$		
		conformal symm. of boundary		R-symmetry				

- Σ Riemann surface over which metric fibred
- Solutions given in terms of two harmonic functions $h_1, h_2 : \Sigma \to \mathbb{R}$

Supergravity Duals

Non-singular geometry $\rightarrow$ poles of h_1, h_2 on boundary of Σ



SUGRA Data:

Poles of h_1: l_A multiplicity: c_A

Poles of h_2: k_B multiplicity: d_B

Field Theory Data:

Linking numbers: L_A, K_B

Multiplicities: $N_{D5}^{(A)}, N_{NS5}^{(A)}$

$$L_A = \sqrt{g}\, l_A + \frac{2}{\pi} \sum_{B=1}^{n} N_{NS5}^{(B)} \arctan\left(\frac{k_B}{l_A}\right) \qquad N_{D5}^{(A)} = \frac{1}{\sqrt{g}} c_A$$

$$K_B = \frac{1}{\sqrt{g}} k_B + \frac{2}{\pi} \sum_{A=1}^{n} N_{D5}^{(A)} \arctan\left(\frac{k_B}{l_A}\right) \qquad N_{NS5}^{(B)} = \sqrt{g}\, d_B$$

Holographic Computation of Boundary F

- Use RT formula $S[HB^3_R] = \frac{\text{Area(RT surface)}}{4G_N}$
- Extract boundary F using

$$F_\partial = \lim_{\epsilon\to 0}\left(R\frac{d}{dR} - 1\right) S_\partial(R)$$

- Find general expression of the form

$$\begin{aligned} F_\partial(c_A, d_A, k_A, l_A) = & \boxed{\sum_A c_A \mathcal{I}^c(l_A) + \sum_A d_A \mathcal{I}^d(k_A)} \quad \text{linear} \\ & + \boxed{\sum_{A,B} c_A c_B \mathcal{I}^{cc}(l_A, l_B) + \sum_{A,B} d_A d_B \mathcal{I}^{dd}(k_A, k_B) + \sum_{A,B} c_A d_B \mathcal{I}^{cd}(l_A, k_B)} \quad \text{quadratic} \\ & + \boxed{\sum_{A,B,C} c_A c_B d_C \mathcal{I}^{ccd}(l_A, l_B, k_C) + \sum_{A,B,C} d_A d_B c_C \mathcal{I}^{ddc}(k_A, k_B, l_C)} \quad \text{cubic} \\ & + \boxed{\sum_{A,B,C,D} c_A c_B d_C d_D \mathcal{I}^{ccdd}(l_A, l_B, k_C, k_D)} \quad \text{quartic} \end{aligned}$$

- **Take-away:** we can write general F_∂ in terms of SUGRA parameters c_A, d_A, k_A, l_A

Interesting Features: D5-Like and NS5-Like

For BCs with D5-branes only or NS5-branes only:

$$F_\partial^{D5} = \frac{N^2}{4}\left(\frac{3}{2} + \ln\left(\frac{\lambda}{4\pi^2}\right)\right) - \frac{\pi^2 N}{3\lambda}\sum_A L_A^3$$
$$- \frac{1}{16}\sum_{A,B}\left\{(L_A + L_B)^2 \ln\left((L_A + L_B)^2\right) - (L_A - L_B)^2 \ln\left((L_A - L_B)^2\right)\right\}$$
$$F_\partial^{NS5} = \frac{N^2}{4}\left(\frac{3}{2} + \ln\left(\frac{4N^2}{\lambda}\right)\right) - \frac{\lambda}{48N}\sum_A K_A^3$$
$$- \frac{1}{16}\sum_{A,B}\left\{(K_A + K_B)^2 \ln\left((K_A + K_B)^2\right) - (K_A - K_B)^2 \ln\left((K_A - K_B)^2\right)\right\}$$

- Simple λ-dependence
- Related by S-duality $\{K_A\} \leftrightarrow \{L_A\}$, $\frac{\lambda}{4\pi N} \leftrightarrow \frac{4\pi N}{\lambda}$

Interesting Features: D5-Like and NS5-Like

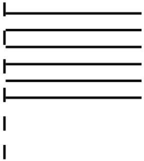
For both D5-brane only and NS5-brane only BCs, one can show that

$$F_\partial^- \leq F_\partial \leq F_\partial^+$$

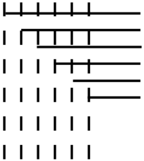
with

$$F_\partial^- = F_\partial \text{ of "minimum entropy" config.}$$
$$F_\partial^+ = F_\partial \text{ of "maximum entropy" config.}$$

"minimum entropy"



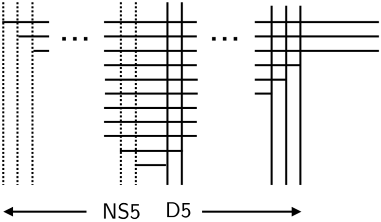
"maximum entropy"



Interesting Features: Unboundedness of F_∂

Consider family of BCs (parametrized by N_{NS5})

$$K = 1 \qquad L = N_{NS5} - 1$$



- Corresponds to coupling in increasingly large 3D quiver for $N_{NS5} \to \infty$
- We find $F_\partial = N_{NS5}^2 \ln N_{NS5} + O(N_{NS5}^2)$
- Expected corrections to classical SUGRA approximation $O(N_5^2)$

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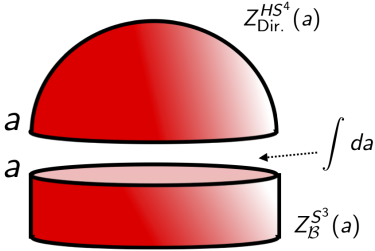
Localization Calculation of F_∂

Apply SUSY "gluing formula" to write partition function for NS5-type boundary condition [Dedushenko]

$$Z_{\mathcal{B}}^{HS^4} = \int da\, \mu(a) \langle \mathcal{B} | a \rangle \langle a | HS^4 \rangle$$
$$= \int da\, \mu(a) Z_{\mathcal{B}}^{S^3}(a) Z_{\text{Dir.}}^{HS^4}(a)$$

- $Z_{\text{Dir.}}^{HS^4}$ for $\mathcal{N} = 2$ SUSY gauge theory from [Gava, Narain, Muteeb, Giraldo-Rivera]
- $Z_{\mathcal{B}}^{S^3}$ for $\mathcal{N} = 2$ SUSY gauge theory from [Kapustin, Willet, Yaakov; Benvenuti, Pasquetti; Nishioka, Tachikawa, Yamazaki]



Result

- We are able to compute the resultant integral for a general NS5-like boundary condition $L = \emptyset, K = (K_1, \ldots, K_{NS5})$
- In the limit
$$K_A \gg 1 \qquad K_{A+1} - K_A \gg 1$$
exact result at leading order is
$$\begin{aligned} F_\partial &= \frac{N^2}{4}\left(\frac{3}{2} + \ln\left(\frac{4N^2}{\lambda}\right)\right) - \frac{\lambda}{48N}\sum_A K_A^3 \\ &\quad - \frac{1}{16}\sum_{A,B}\left\{(K_A + K_B)^2 \ln\left((K_A+K_B)^2\right) - (K_A - K_B)^2 \ln\left((K_A - K_B)^2\right)\right\} \end{aligned}$$
- Reproduces precisely the result from SUGRA (for any value λ)!
- Can obtain exact result for D5-like BCs using Montonen-Olive duality of $\mathcal{N} = 4$ SYM

Distribution of Boundary F

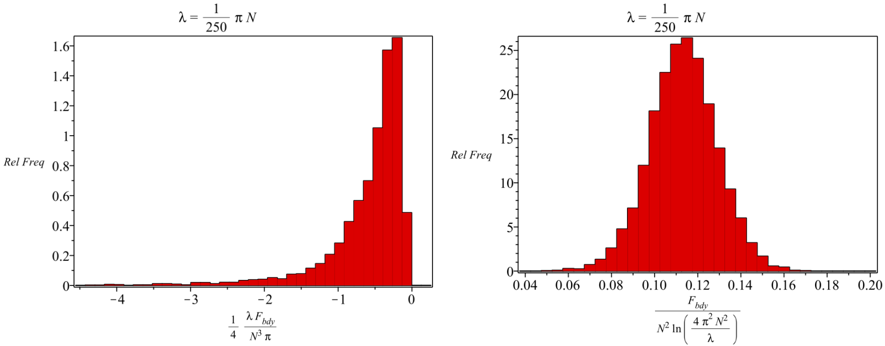


Figure: (Left) Histograms of F_∂ for D5-like BCs; (Right) Histograms of F_∂ for NS5-like BCs. In both cases, we have fixed $N = 100$. We have scaled F_∂ by a positive quantity for convenience.

$$\lambda > 4\pi N \text{ D5-type dist.} \xleftrightarrow{S-duality} \lambda < 4\pi N \text{ NS5-type dist.} \text{ (and vice versa)}$$

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Conclusions and Outlook

We have

- Computed F_∂ holographically in type IIB supergravity for all $OSp(2,2|4)$ boundary conditions of $U(N)$ $\mathcal{N}=4$ SYM
- Computed F_∂ via supersymmetric localization for the "D5-like" and "NS5-like" boundary conditions of $U(N)$ $\mathcal{N}=4$ SYM
- Found that F_∂ behaves as expected for measure of local degrees of freedom at the boundary
    - Unbounded from above
    - Mostly negative/positive for D5-like/NS5-like and small λ
    - Minimized/maximized on "minimum/maximum entropy" BCs for D5-only and NS5-only

Natural follow-up:

- Use results for detailed analysis of boundary RG flows
- Use identical techniques to analyze theories with same symmetry (interface theories)

Thank you!