From spin chains to real-time thermal field theory using tensor networks

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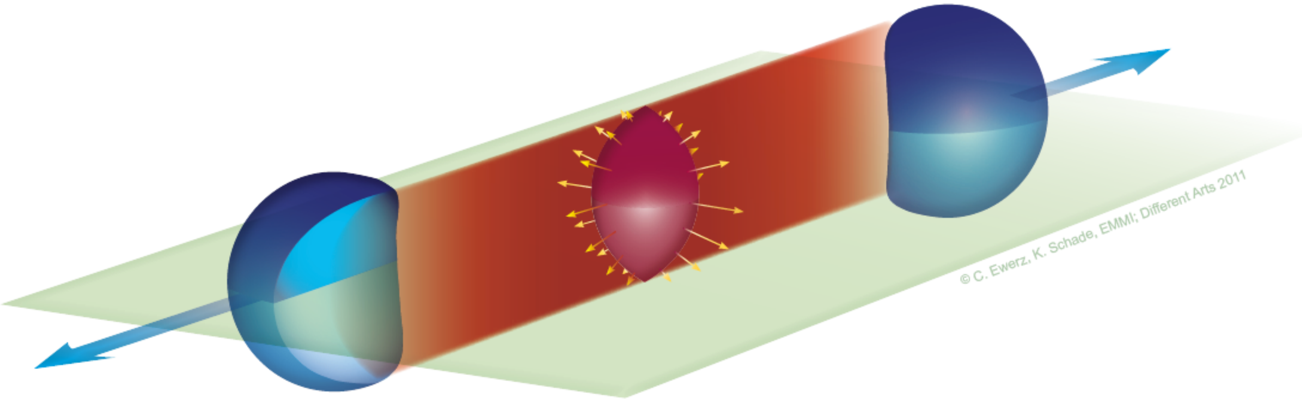
October 2020

Introduction and Motivation

- the understanding of dynamical quantum many-body systems is of central interest in condensed matter and high-energy physics



- hep-th: collective phases of QCD matter are probed in heavy-ion collisions: relaxation from non-equilibrium to QGP
- gauge/gravity duality: strongly-coupled QFTs with many constituents serve as microscopic description of quantum gravity
- cond-mat: integrability in the context of thermalization and relaxation dynamics

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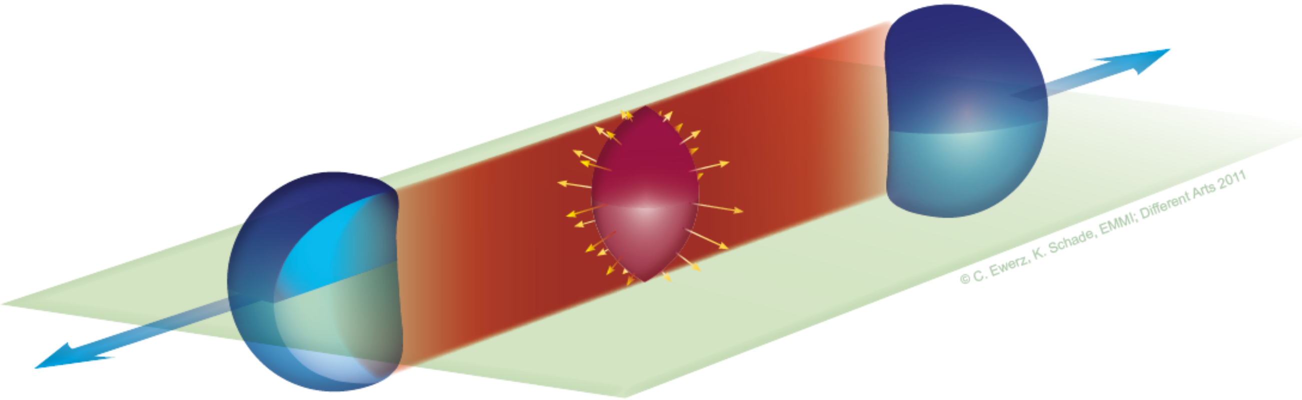

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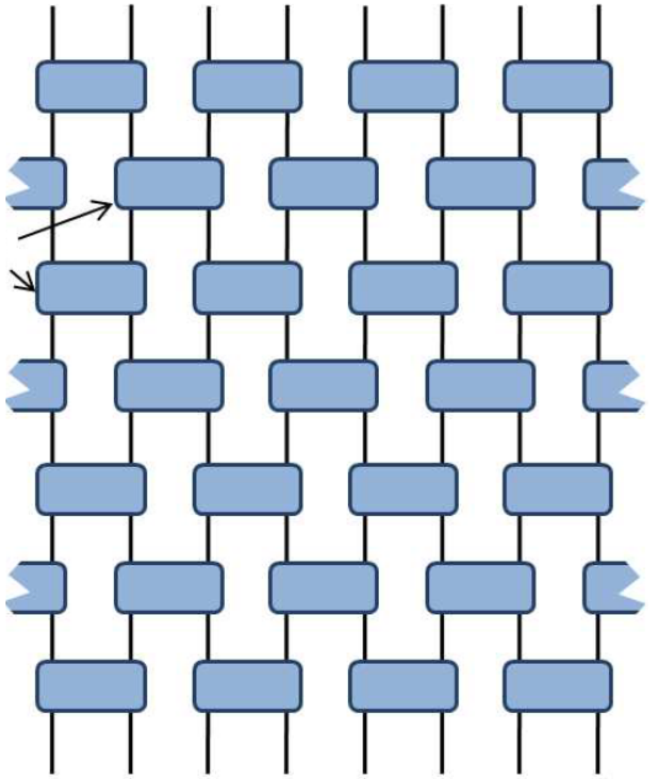
- gauge/gravity duality: strongly-coupled QFTs with many constituents serve as microscopic description of quantum gravity
- cond–mat: integrability in the context of thermalization and relaxation dynamics



- Tensor Networks (TNs) are representations of quantum many-body states in a tensor product basis
- They capture relevant entanglement properties and allow efficient time simulation

⇒ explore thermal quenches of 1D Ising spin chain to extract ab initio real-time QFT dynamics and make nontrivial predictions

Dynamical physical quantities

- dynamics in linear response theory:

$$\delta\langle\mathcal{O}(t,p)\rangle = \int d\omega\, e^{-i\,\omega\,t}\, G_R^{\mathcal{O}}(\omega,p)\, \mathcal{J}(-\omega,-p)$$

local operator

source of Hamiltonian H

retarded 2-point function at non-zero temperature $T = 1/\beta$

$$G_R^{\mathcal{O}}(t,x) = i\,\theta(t)\;\mathrm{Tr}\,\{\rho_\beta[\mathcal{O}(t,x),\mathcal{O}(0,0)]\}$$

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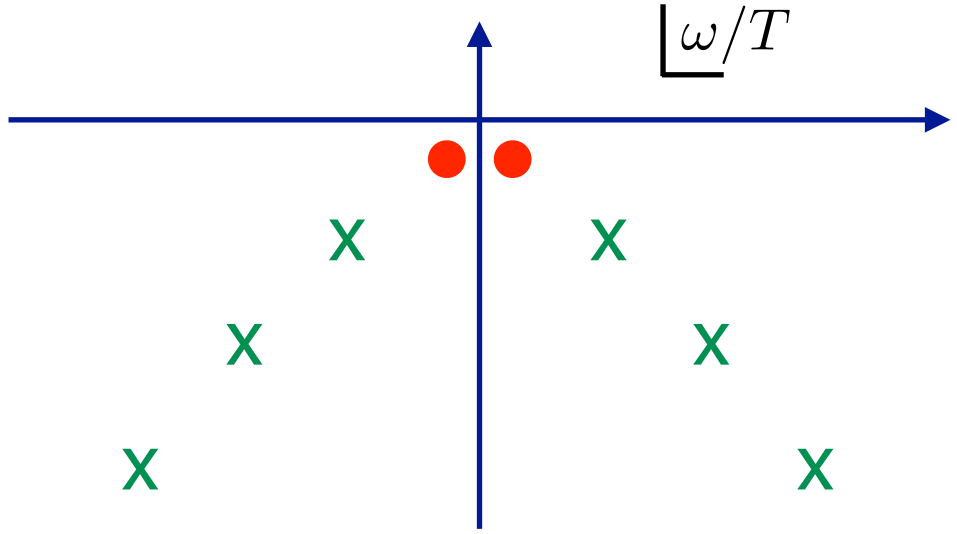
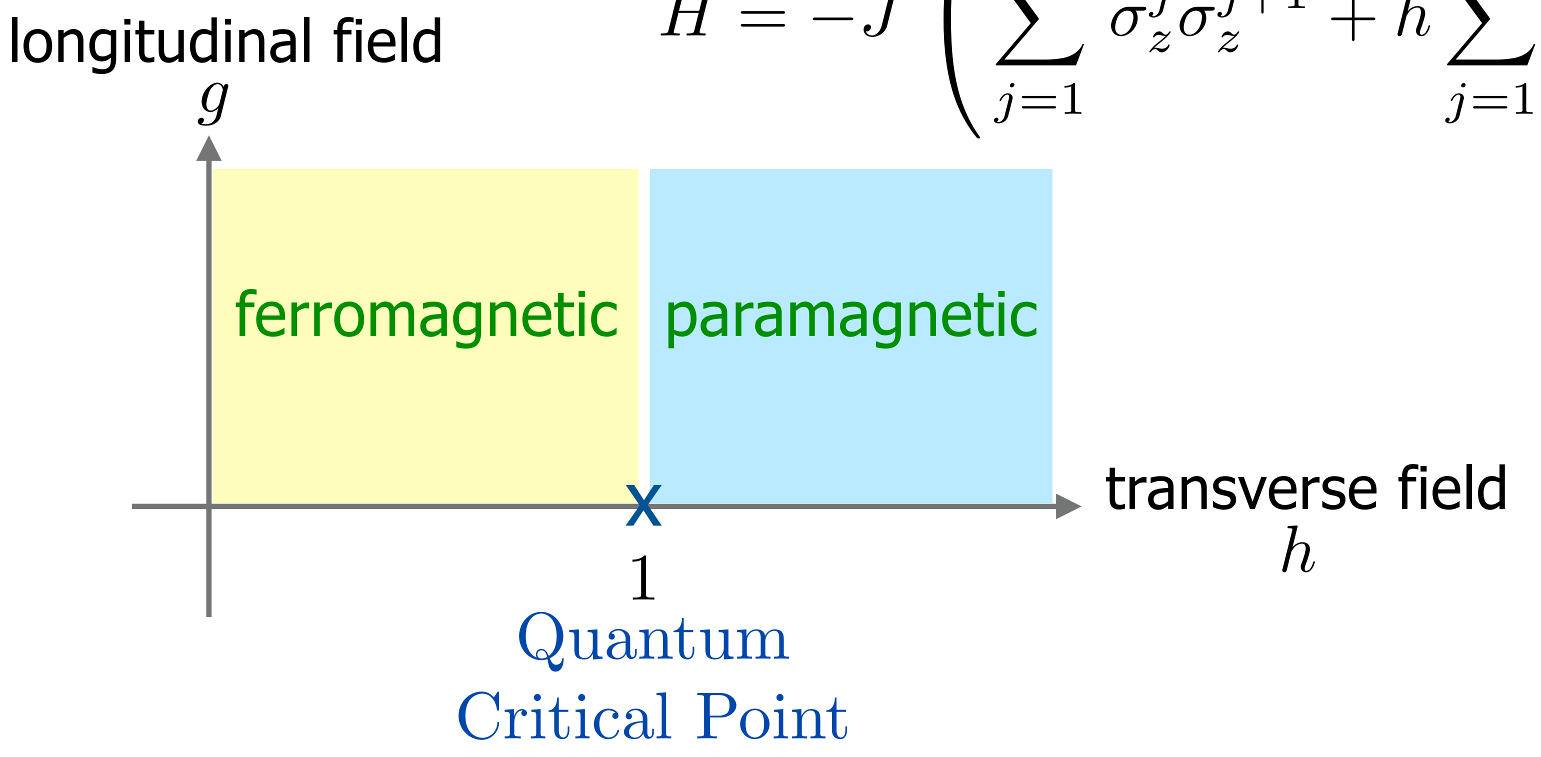
- insights from holography in 4D:



characteristic frequencies at which BHs in AdS space absorb matter (quasinormal modes) set time scale for dissipation in dual QFTs

The quantum Ising model

$$H = -J\left(\sum_{j=1}^{N-1}\sigma_z^j\sigma_z^{j+1} + h\sum_{j=1}^{N}\sigma_x^j + g\sum_{j=1}^{N}\sigma_z^j\right)$$



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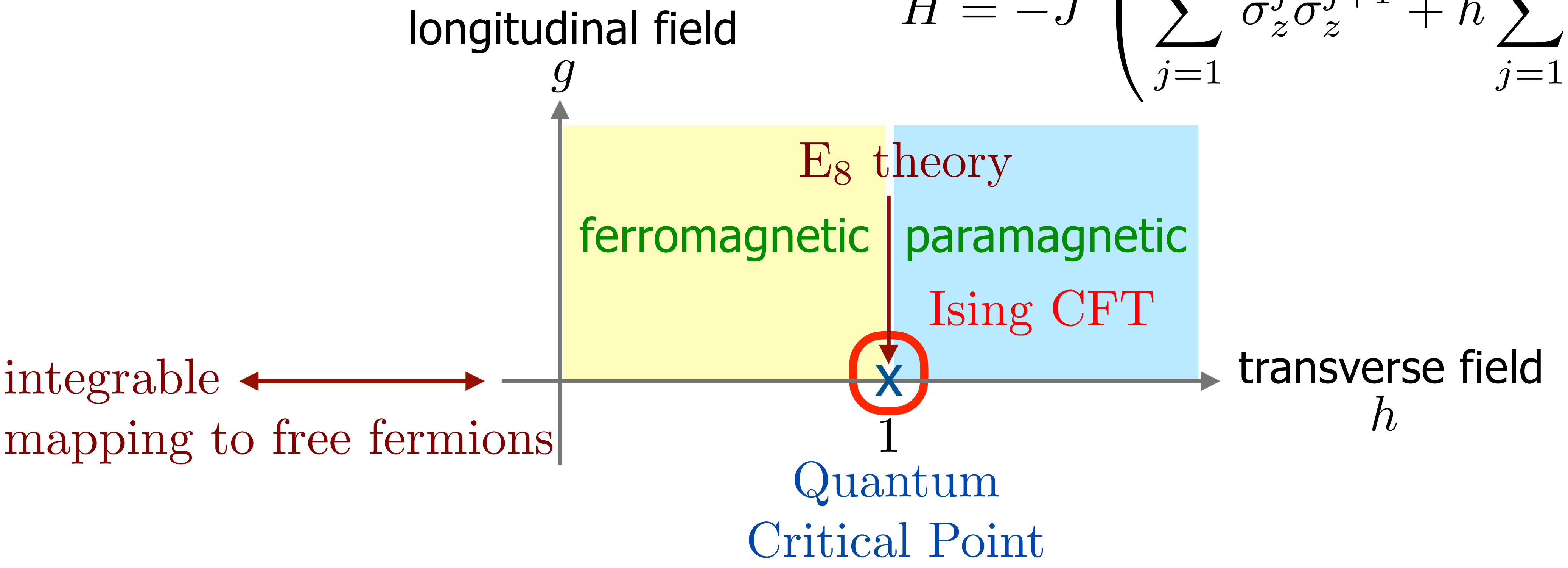

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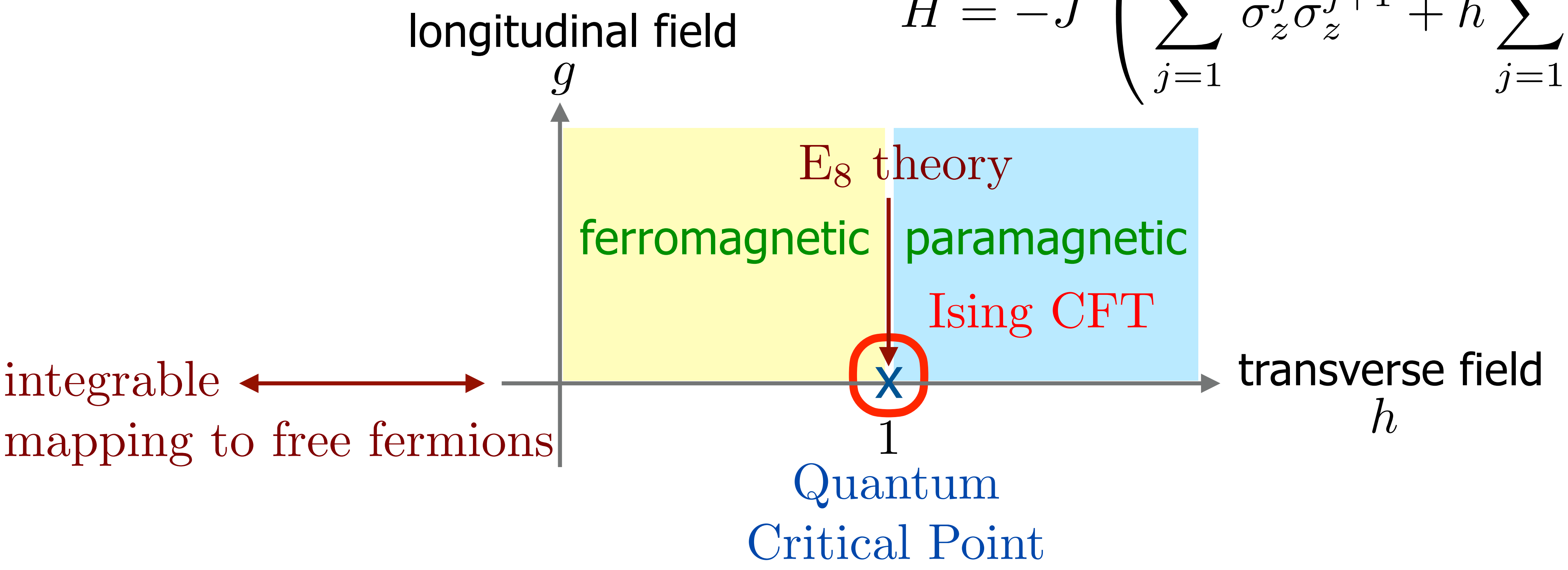


- the full scaling Ising field theory Hamiltonian in presence of transverse and longitudinal perturbations has the form [Rakovszky et al. 2016]:

$$H = \int_{-\infty}^{\infty} dx \left\{ \frac{i}{2\pi}\left[\frac{1}{2}\left(\psi\partial_x\psi - \bar{\psi}\partial_x\bar{\psi}\right) - M_h\bar{\psi}\psi\right] + \mathcal{C}M_g^{15/8}\,\sigma(x)\right\}$$

continuum limit: $a = \frac{2}{J} \to 0,\ L \equiv Na \to \infty,\ \beta J \gg 1$

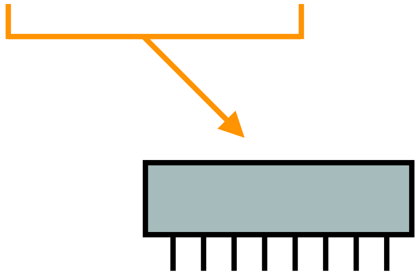
$$M_h = 2J|1-h| \qquad M_g \equiv \mathcal{D}J\,|g|^{8/15}$$

- $M_h = M_g = 0$: free Majorana Ising CFT, $c = \frac{1}{2}$, scalar primary operators $\epsilon = i\bar{\psi}\psi \sim \sigma_x^j,\ \sigma \sim \sigma_z^j$

Tensor Networks *… a classical simulation for a quantum problem*

- the Hilbert space of a generic quantum state is huge:

$$|\Psi\rangle = \sum_{i_1,i_2,\ldots,i_N} \psi_{i_1,i_2,\ldots,i_N} |i_1\rangle\,|i_2\rangle \cdots |i_N\rangle\,, \quad i_n = 1 \ldots d$$



N-legged tensor:
exponentially many coefficients
in N-body Hilbert space: d^N

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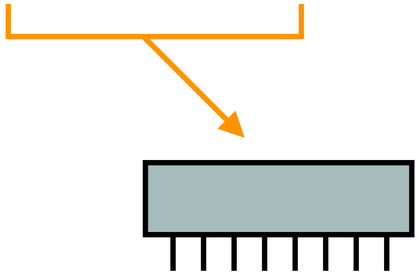

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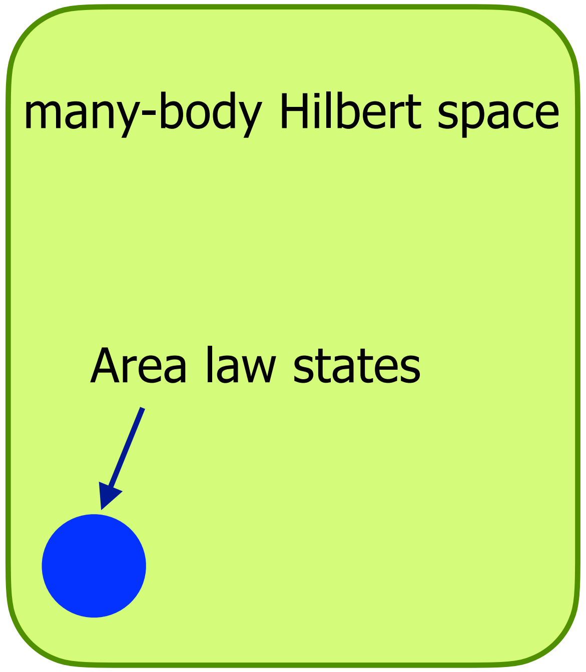


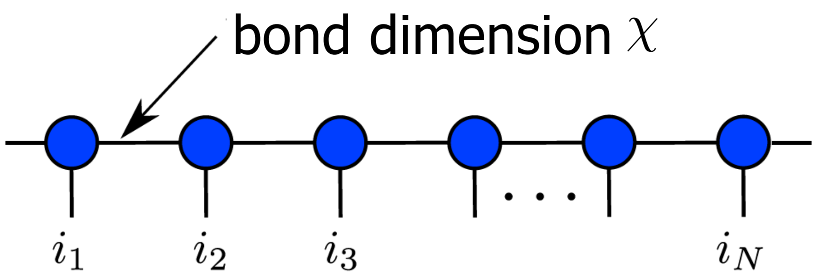
- ground states of local gapped Hamiltonians satisfy Area law for entanglement entropy [Hastings 2007]:

$$S(L) \sim L^{D-1}$$

- Matrix Product States (MPS) as ansätze satisfy this by construction [Schollwöck 2011]:

$$|\Psi\rangle = \sum_{i_1,i_2,\ldots,i_N} A^1_{i_1} A^2_{i_2} \cdots A^N_{i_N} |i_1\rangle |i_2\rangle \cdots |i_N\rangle$$

$$\psi_{i_1,i_2,\ldots,i_N} \sim$$



$\mathcal{O}(Nd\chi^2)$ parameters

efficient

Numerics with MPO + Signal analysis with Prony

- retarded thermal 2-point function:

$$G_R^{\mathcal{O}}(t,x) = i\,\theta(t)\ \mathrm{Tr}\,\{\rho_\beta[\mathcal{O}(t,x),\mathcal{O}(0,0)]\}$$

for operators σ_x or σ_z (global perturbation at zero momentum)
is calculated using the TEBD algorithm (time-evolving block decimation [Vidal 2004])
for matrix product operators (MPO)

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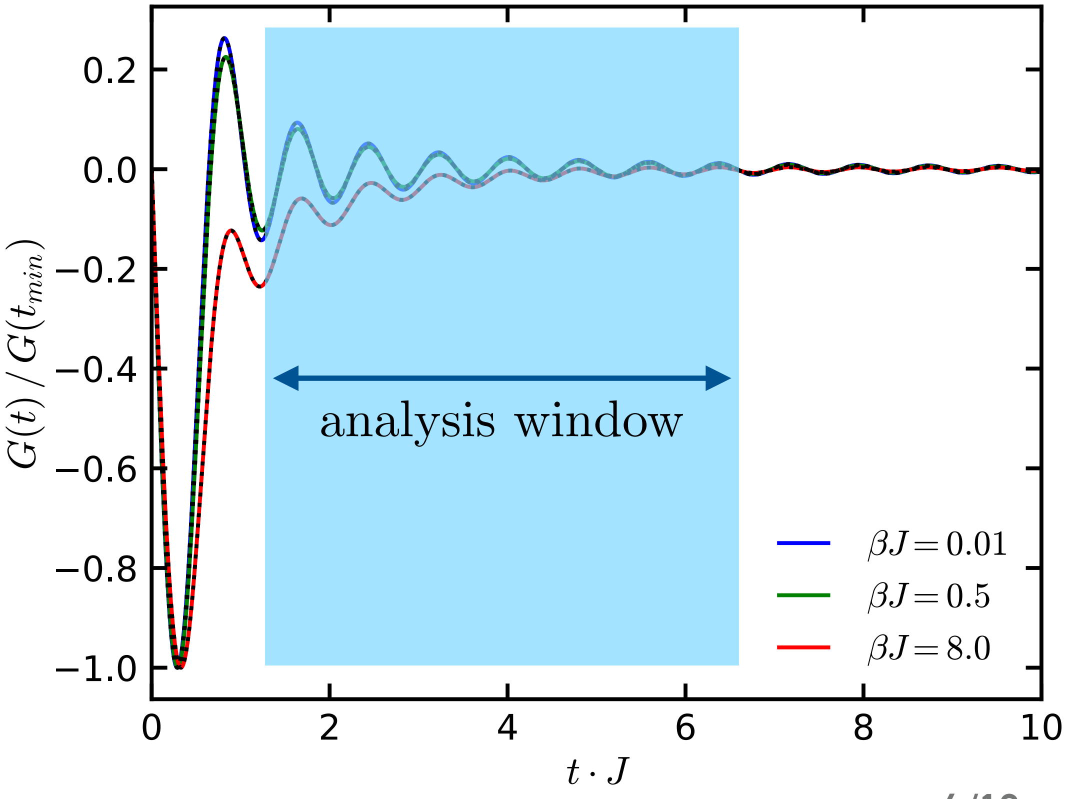
- represent function as sum of complex exponentials:

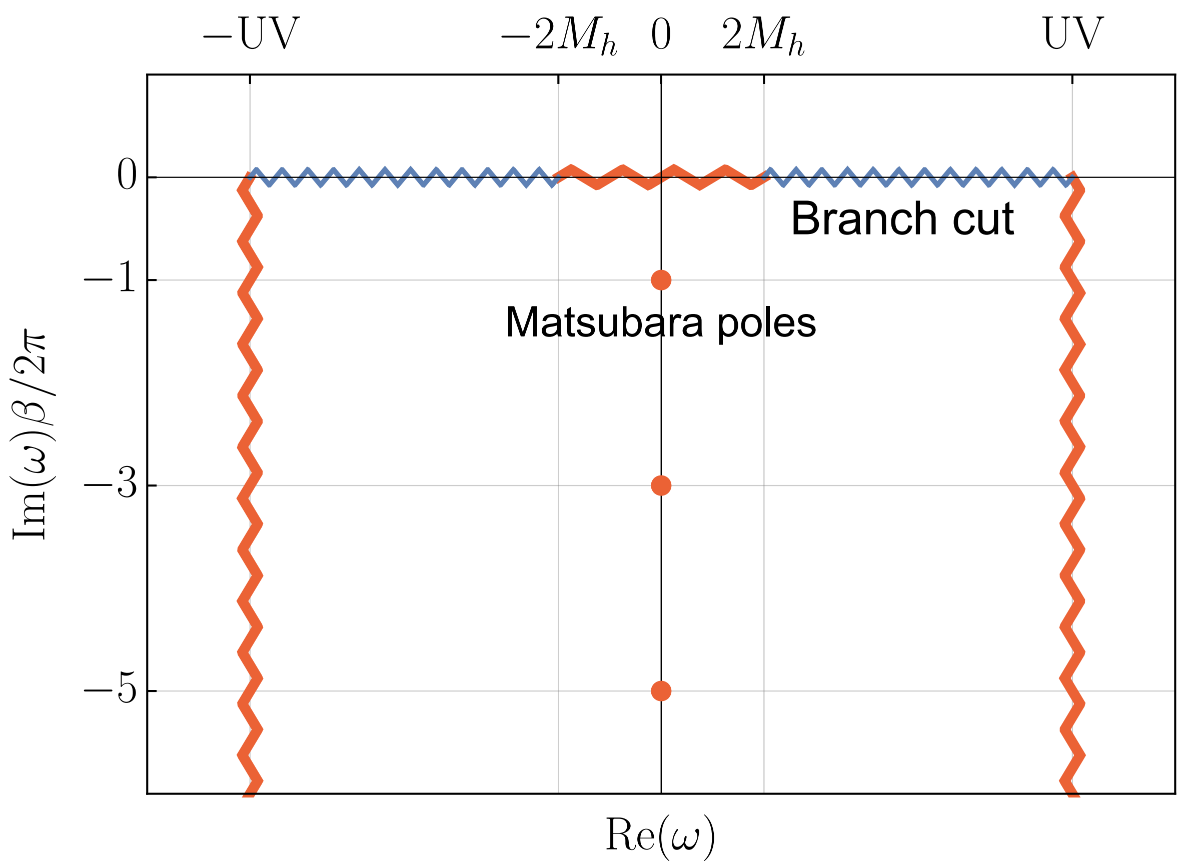
$$G(t) = \sum_{k=1}^{M} c_k e^{-i\omega_k t} \qquad c_k,\ \omega_k \in \mathbb{C}$$

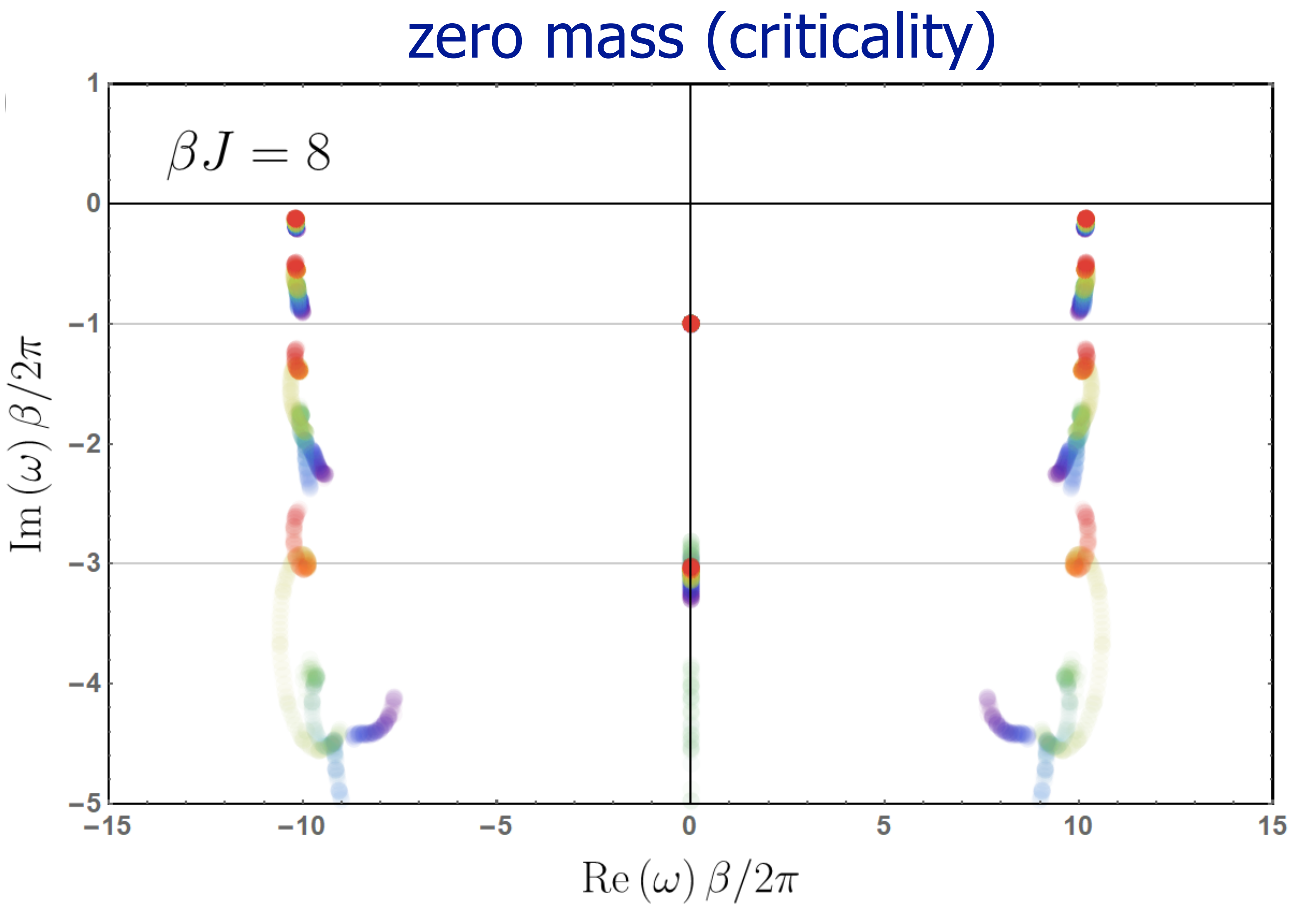
1. Determine ω_k independent of c_k (ESPRIT)
2. Fit c_k by least squares

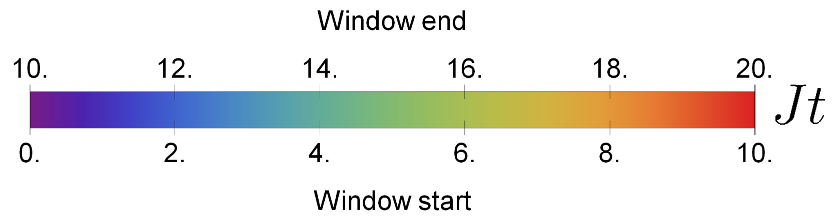
⇒ estimation of stability and uncertainty of poles from parameter variation in Prony and time-shifted analysis window



Retarded thermal correlators in solvable cases




zero mass (criticality)



- (1+1)D CFT:

$$\omega = \pm p - i\,2\pi\,T\,(\Delta + 2\,n)$$

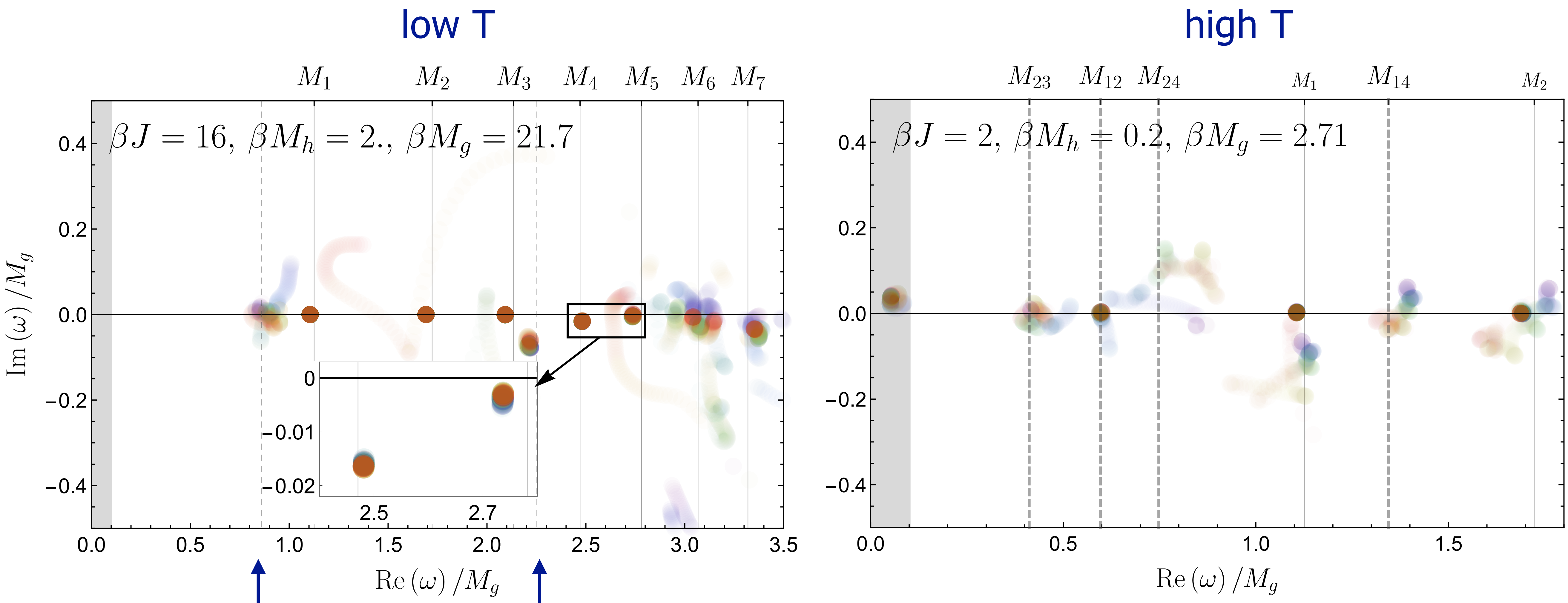
$$\Delta_{\epsilon = i\bar{\psi}\psi} = 1 \qquad \Delta_\sigma = \frac{1}{8}$$

$\Rightarrow$ holographic interpretation as BH quasinormal modes [Sachs et al. 2002]

- two equivalent representations for massive free fermions (transverse Ising model at zero momentum)
- precise numerical extraction of first transient position and its residue in QFT limit

Meson studies

- identification of nontrivial meson / particle masses and their decay rates of perturbed Ising CFT in different vacuum phases
  [Zamolodchikov 2006, 2013; Delfino et al. 2006]


low T

high T

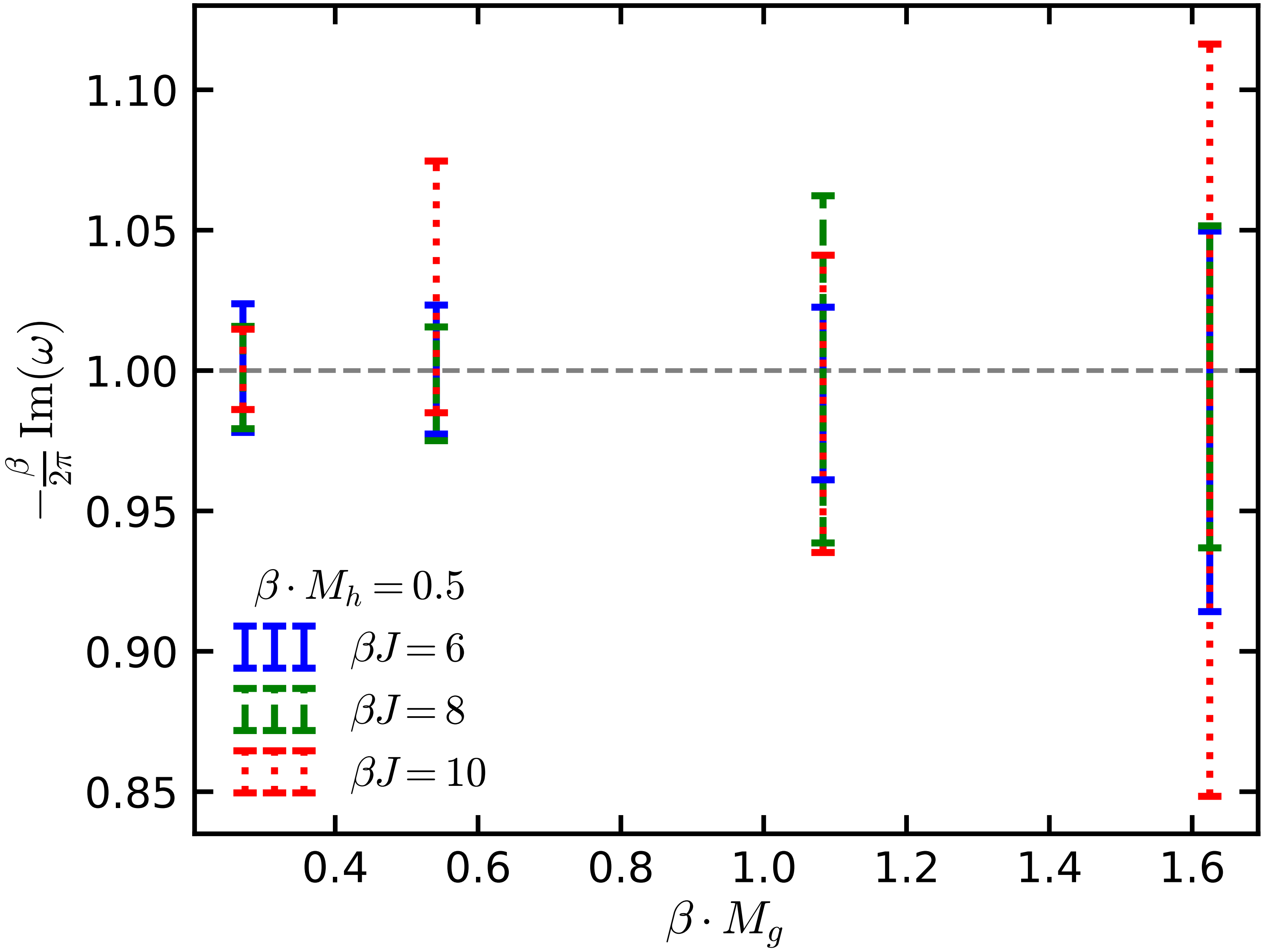
boundary excitation
(finite size effect)

continuum threshold $2M_1$

- temperature dependence of meson states:
  appearance of mass differences in correlator at higher temperature

The non-integrable QFT limit: MPO predictions for the transients

- position of first transient:



⇒ no movement of poles visible within uncertainties in ferromagnetic and paramagnetic phase (zero momentum)

CONCLUSIONS

- Tensor network techniques can be used to extract *ab initio* nontrivial real-time thermal field theory dynamics at intermediate times.
- Prony method can be used to numerically evaluate structure of retarded 2-point function in complex frequency plane

  ⇒ agreement with CFT result / free fermions in integrable regime

  ⇒ meson/particle masses and decay rates match predictions from Ising QFT at low T, predictions for thermal behavior in non-integrable interacting QFT

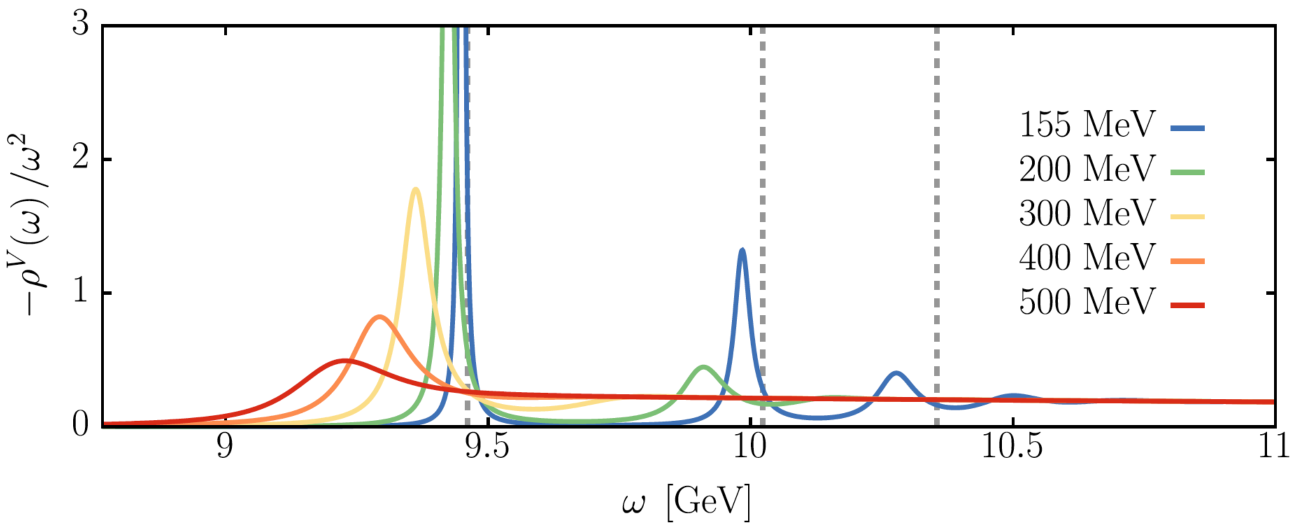
  ⇒ no movement of first decaying thermodynamic pole for non-integrable perturbations

Future directions:

- realizations for experimental quantum simulations?
- higher-dimensional simulations

OUTLOOK: Meson Melting

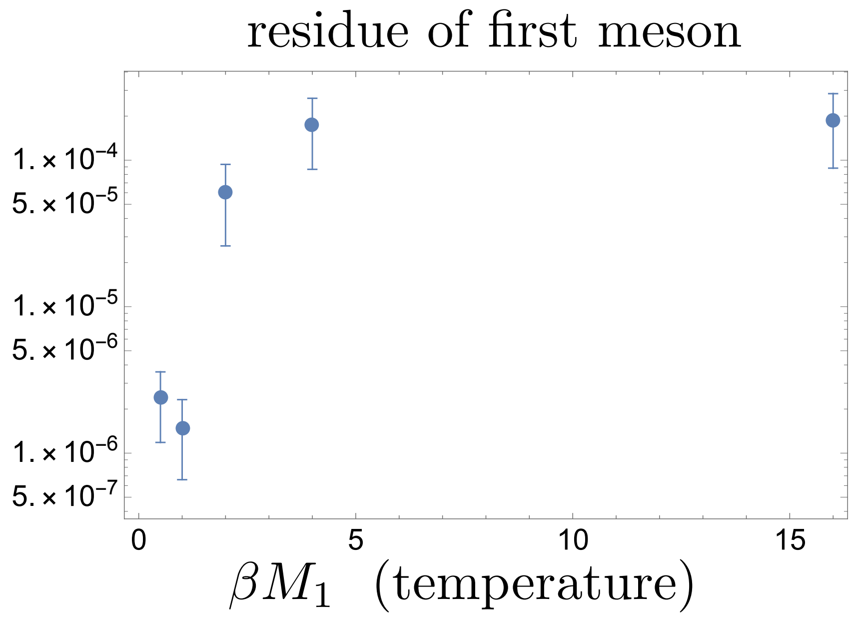
- QCD at high temperatures:



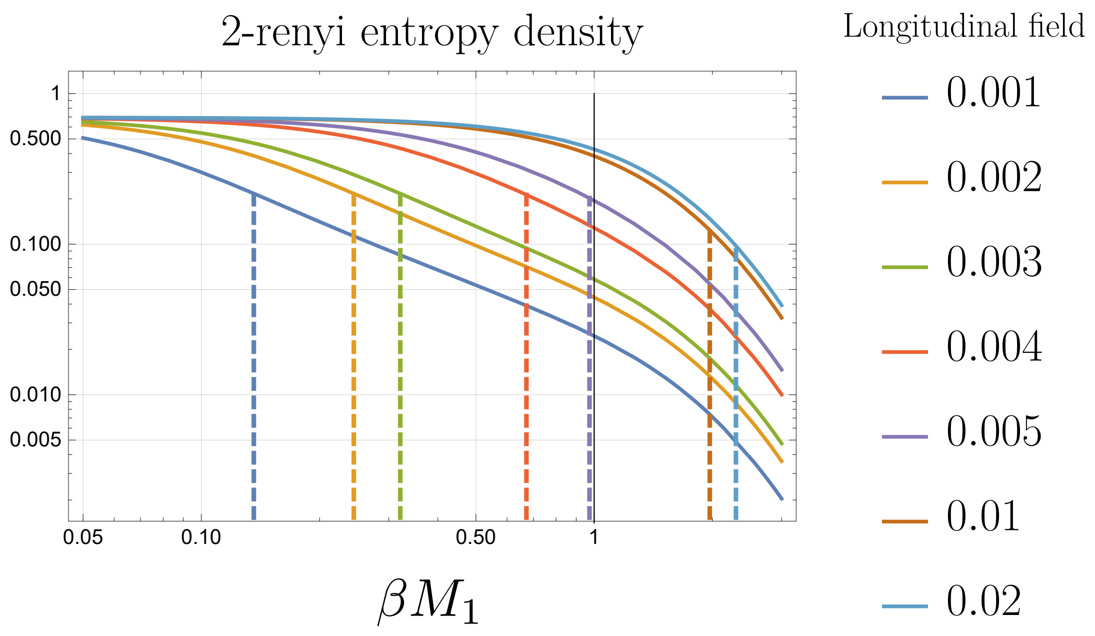
sequential melting of mesons into unbound states in thermal environment is visible as broadening of in-medium spectral functions [Rothkopf 2020]

- MPO simulation of Ising QFT:

non-integrable regime:



integrable E₈ regime:

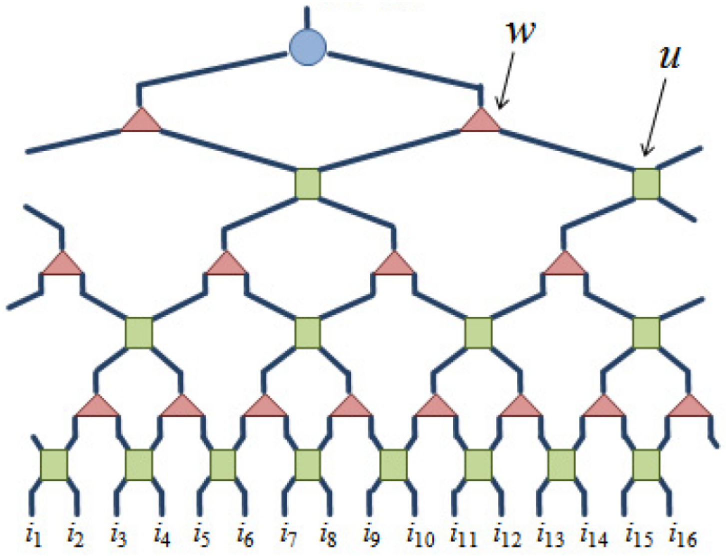


high T ($\beta M_1 < 1$): $s_2 \sim \beta^{-1}$

low T ($\beta M_1 > 1$): $s_2 \sim e^{-\beta M_1}$

OUTLOOK: coarsegrained systems w/ Sukhi Singh

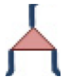
- MERA tensor network implements a RG flow [Vidal 2006]



analytic wavelet tensors in terms of

Pauli matrices [Evenbly, White (2016)]

$$= \frac{\sqrt{3}+2}{4} II + \frac{\sqrt{3}-2}{4} ZZ + \frac{i}{4} XY + \frac{i}{4} YX$$

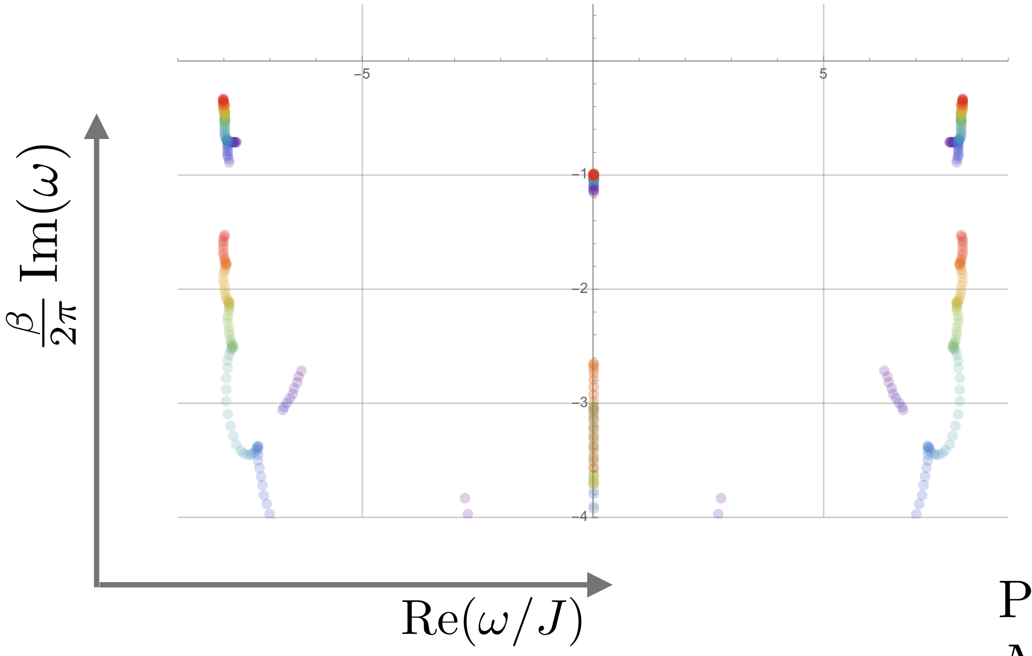


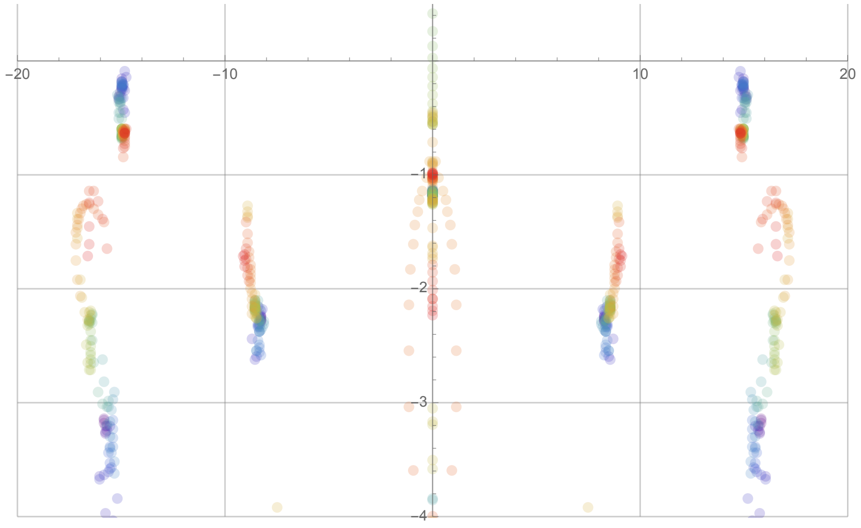
$$= \frac{\sqrt{3}+\sqrt{2}}{4} II + \frac{\sqrt{3}-\sqrt{2}}{4} ZZ + \frac{i(1+\sqrt{2})}{4} XY + \frac{i(1-\sqrt{2})}{4} YX$$

⇒ allow identification of scaling operator ϵ

and scale-invariant densities h, p

- comparison of MPO simulations of retarded correlation functions for ϵ :





Prony analyses for small systems of sizes
$N_{bare} = 40$ (left) or $N_{coarsegrained} = 20$ (right)

BACKUP

- mapping of the integrable transverse field Ising model (g=0) to massive free fermions:

$$H = -J\left(\sum_{j=1}^{N-1}\sigma_z^j\sigma_z^{j+1} + h\sum_{j=1}^{N}\sigma_x^j + g\sum_{j=1}^{N}\sigma_z^j\right)$$

Jordan-Wigner trafo:

$$\sigma_x^j = 1 - 2\,b_j^\dagger\,b_j \quad \text{and} \quad \sigma_z^j = \left(\prod_{l<j}(1-2\,b_l^\dagger\,b_l)\right)(b_j + b_j^\dagger)$$

Bogoliubov trafo: $\gamma_k = u_k c_k - i v_k c_{-k}^\dagger$ in Fourier space: $c_k = \frac{1}{\sqrt{N}}\sum_i c_i e^{ikj}$

$$u_k = \cos(\theta_k/2), \quad v_k = \sin(\theta_k/2), \quad \tan\theta_k = \tfrac{\sin k}{h-\cos k}$$

Hamiltonian: $H = \sum_k \varepsilon_k(\gamma_k^\dagger\gamma_k - 1/2) \qquad \varepsilon_k = 2\sqrt{J^2(1+h^2-2h\cos k)}$

- define two independent Majorana fermion fields:

$$\psi(x=ja) = \sqrt{\pi/a}(b_j^\dagger + b_j), \text{ and } \bar\psi(x=ja) = -i\sqrt{\pi/a}(b_j^\dagger - b_j) \text{ for } a = 2/J$$

$$\{\psi(x),\psi(y)\} = \{\bar\psi(x),\bar\psi(y)\} = 2\pi\,\delta(x-y) \text{ for } a \to 0$$

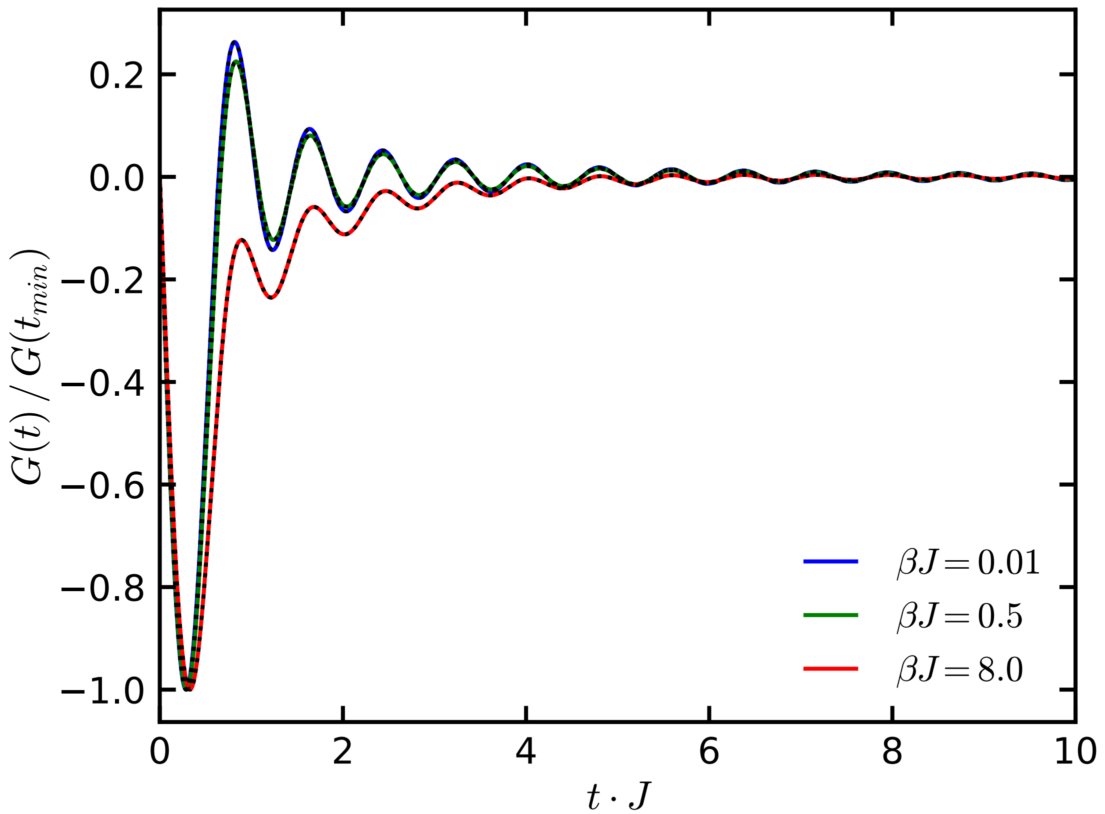
two scalar primary Hermitian operators in c=1/2 Ising CFT:

$$\mathcal{O}_{\Delta=1}(ja) = i\bar\psi(ja)\psi(ja) = -\tfrac{a}{\pi}\,\sigma_x^j \qquad \mathcal{O}_{\Delta=1/8}(ja) = \sigma(ja) \approx \sigma_z^j$$

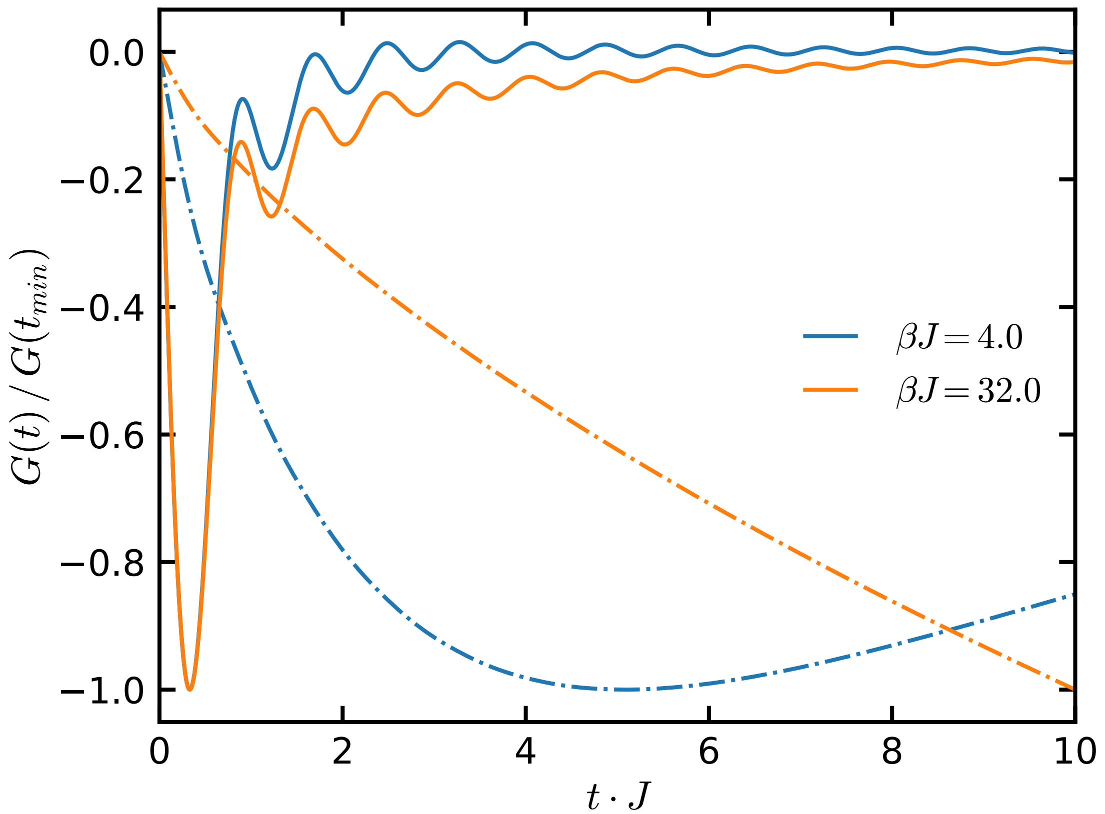
- retarded correlator:

$$G_R^{-\frac{\pi}{a}\,\sigma_x}(t>0,p=0) = 2J\int_{-\pi}^{\pi} dk\,(2n_k-1)\,\sin^2\theta_k\,\sin(2\,\varepsilon_k\,t) \qquad n_k = (1+e^{\beta\,\varepsilon_k})^{-1}$$

- numerical simulations with MPO:



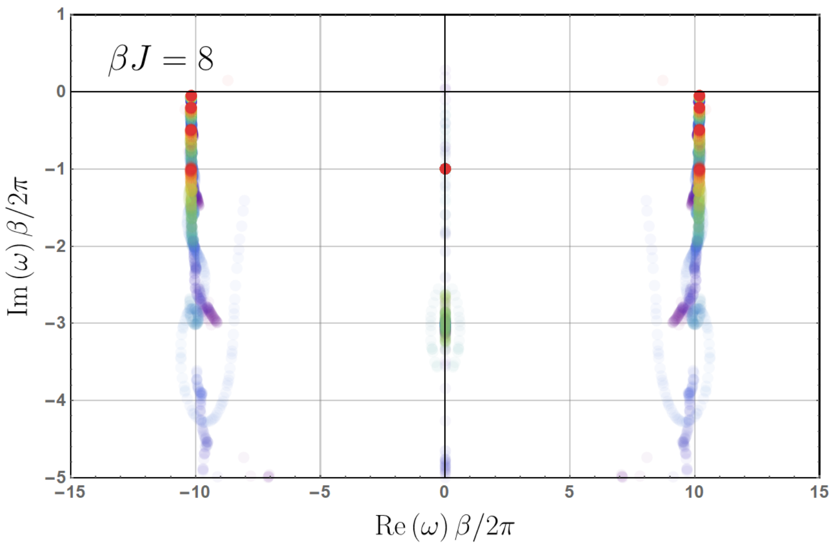
agreement with free fermion mapping at criticality



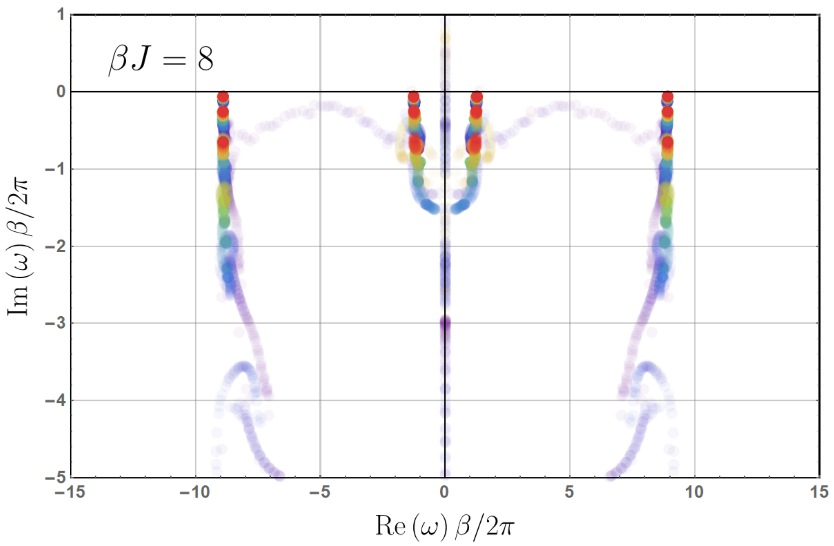
transverse vs. longitudinal response function

- examples of Prony analyses of the retarded transverse 2-point correlator (from numerical evaluation of integral):

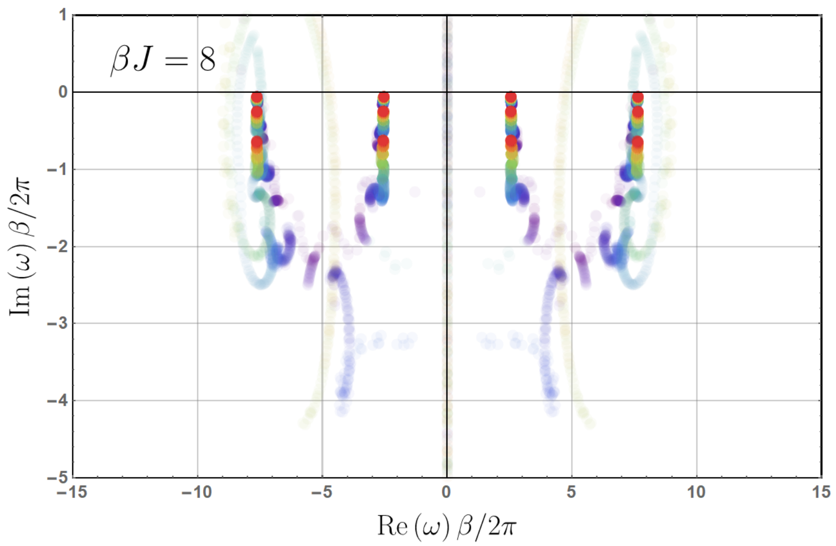
zero (transverse) mass

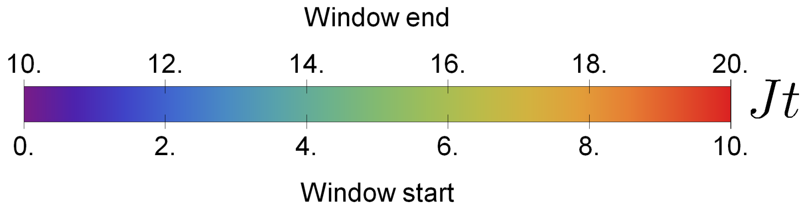


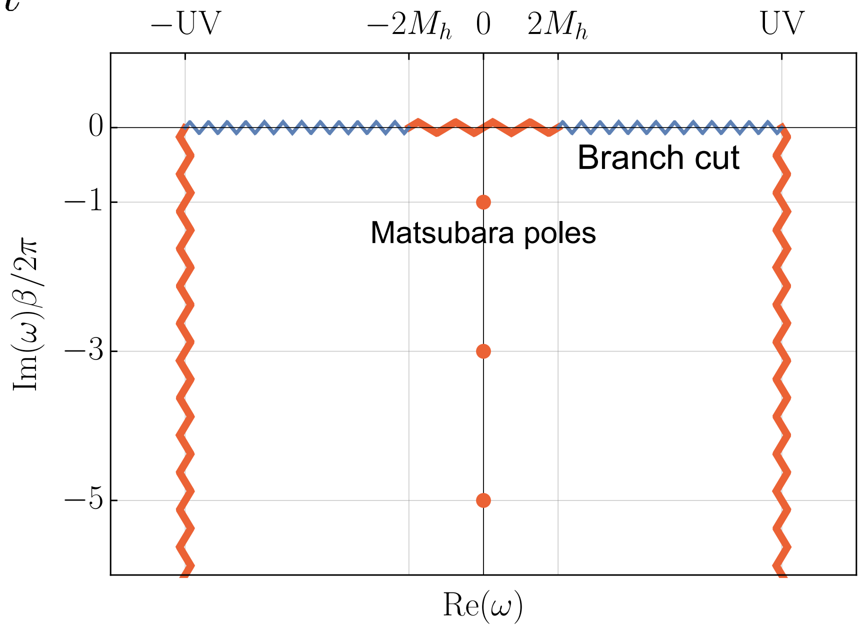
intermediate mass



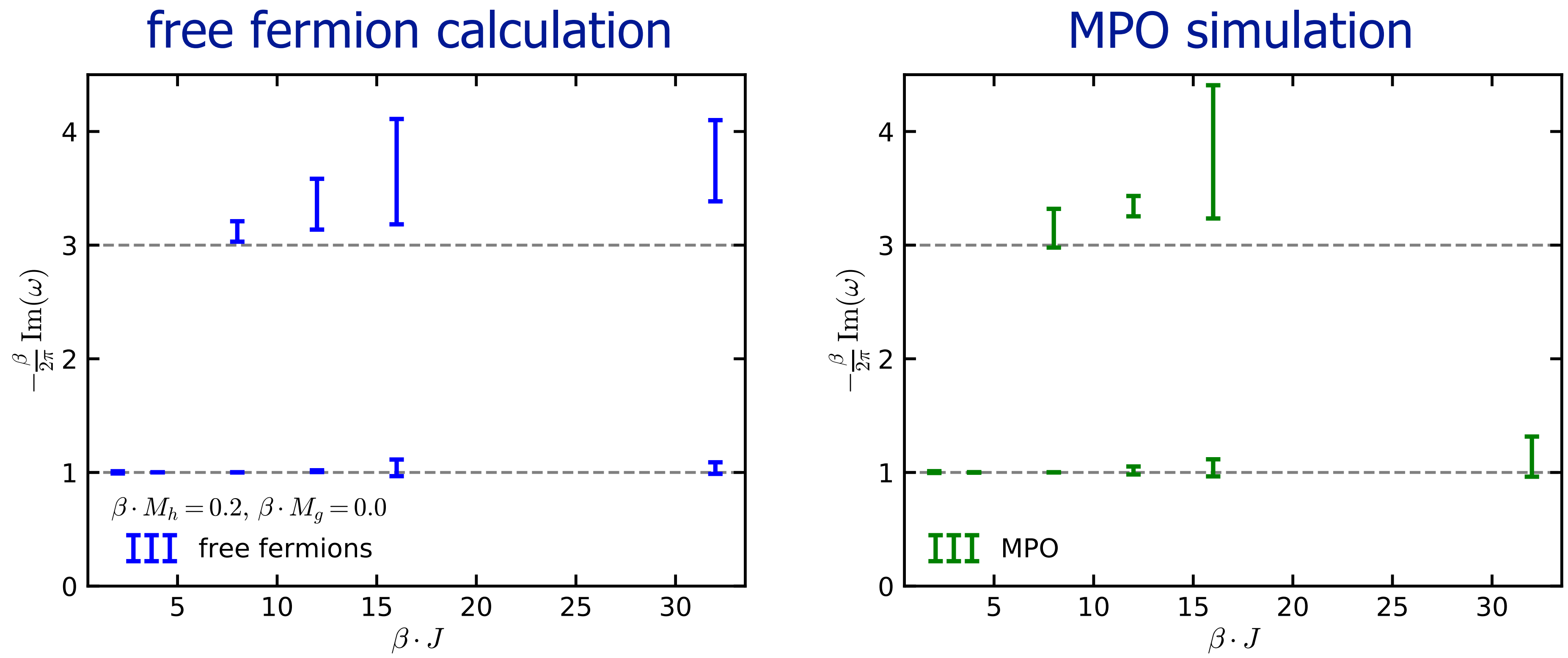
large mass







- extracted decaying thermodynamic poles in integrable QFT limit:



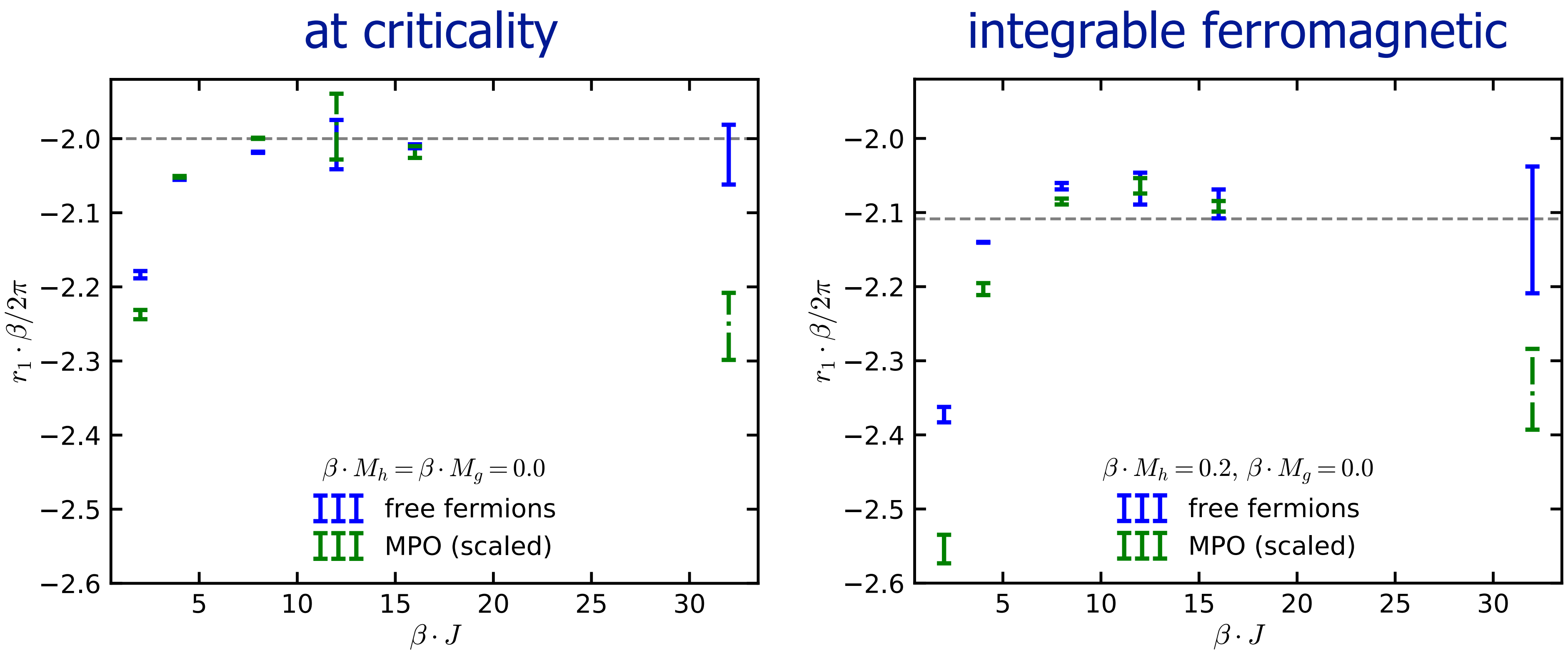
free fermion calculation

MPO simulation

+ good agreement with analytical result for first pole

+ second pole partially identifiable

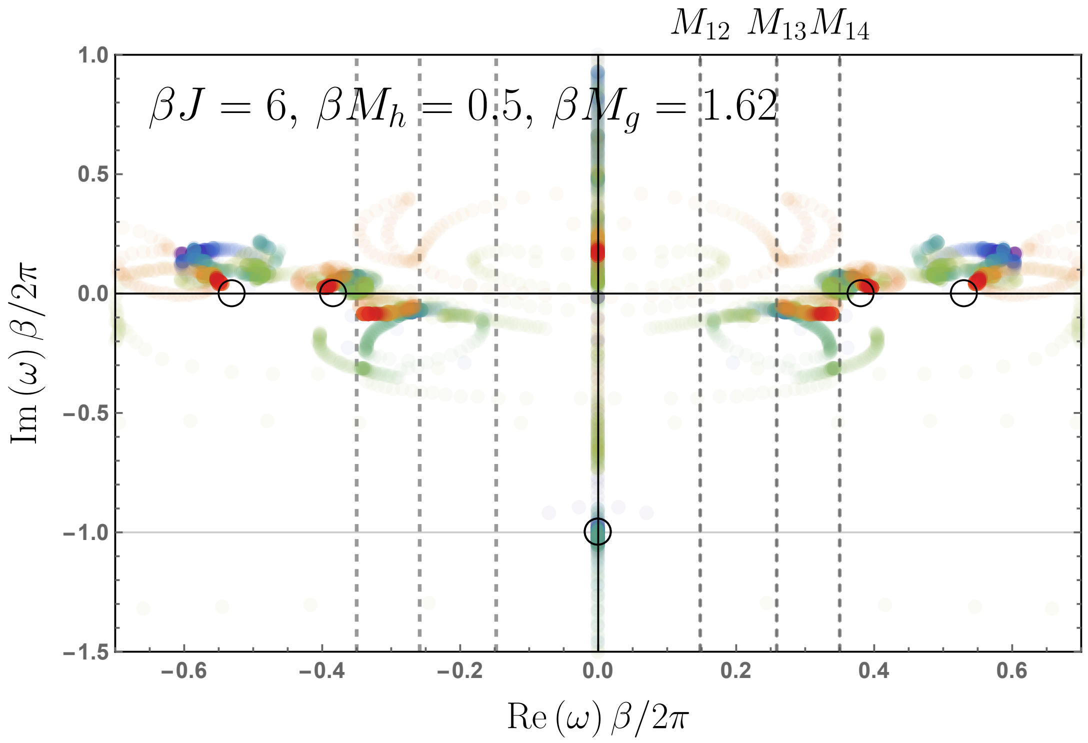
- residues consistent with analytical result in continuum limit:



at criticality

integrable ferromagnetic

- transients and meson states can be observed simultaneously in the QFT regime:

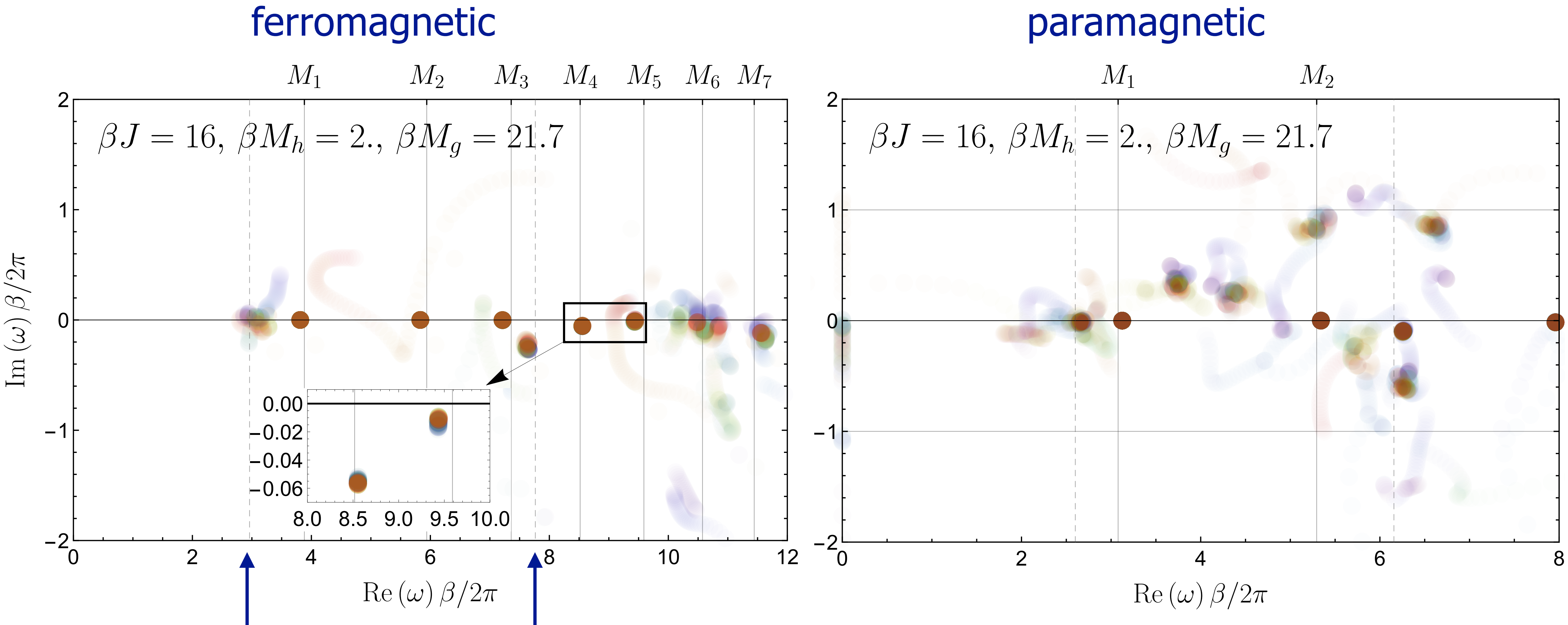


- very precise identification of stable particle masses in integrable E_8 theory:

	m_2/m_1	m_3/m_1	m_4/m_1	m_5/m_1	m_6/m_1	m_7/m_1
MPS+Prony	1.615	1.96	2.41	2.94	3.17	3.5
Analytical	1.618	1.989	2.405	2.956	3.218	3.891

Table 1: Ratios of masses of mesons extracted from MPS+Prony compared to the analytical expectations for the integrable E_8 theory.

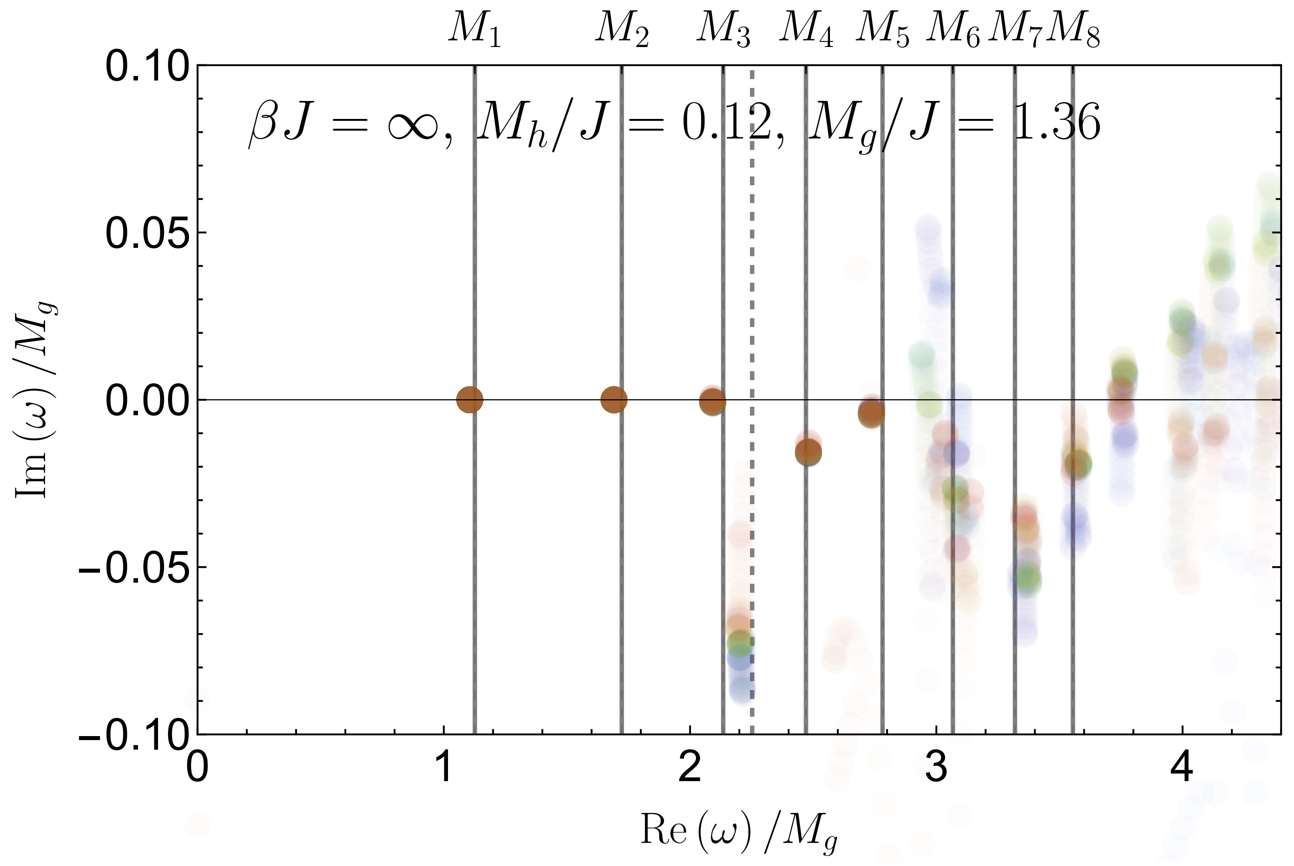
- identification of nontrivial meson / particle masses and their decay rates of perturbed Ising CFT in different vacuum phases
  [Zamolodchikov 2006, 2013; Delfino et al. 2006]



boundary excitation
(finite size effect)

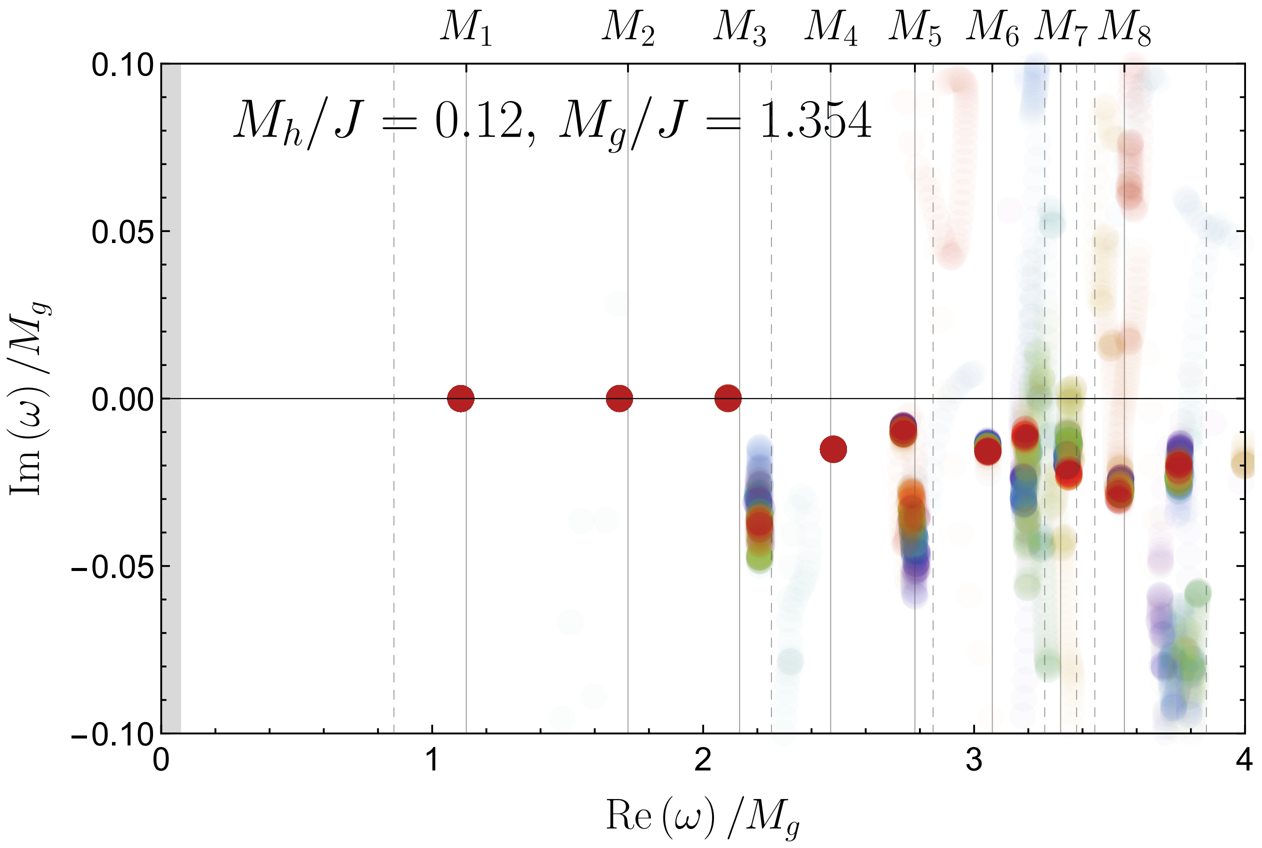
continuum threshold $2M_1$

- ground state quenches:
  vacuum state $|0\rangle$ is found by DMRG



MPO representation $|0\rangle\,\langle 0|$

with bond dimension $\sqrt{\chi}\cdot\sqrt{\chi} = \chi$



MPS representation $|0\rangle$

with bond dimension χ^2

- clear difference to thermal predictions: no mass differences appear