Outer entropy
=
Bartnik-Bray quasilocal mass

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Outline

- This work brings together two notions from different contexts: **Outer Entropy** from holography, and **Bartnik-Bray quasilocal mass** from mathematical relativity, and shows their **equivalence**.

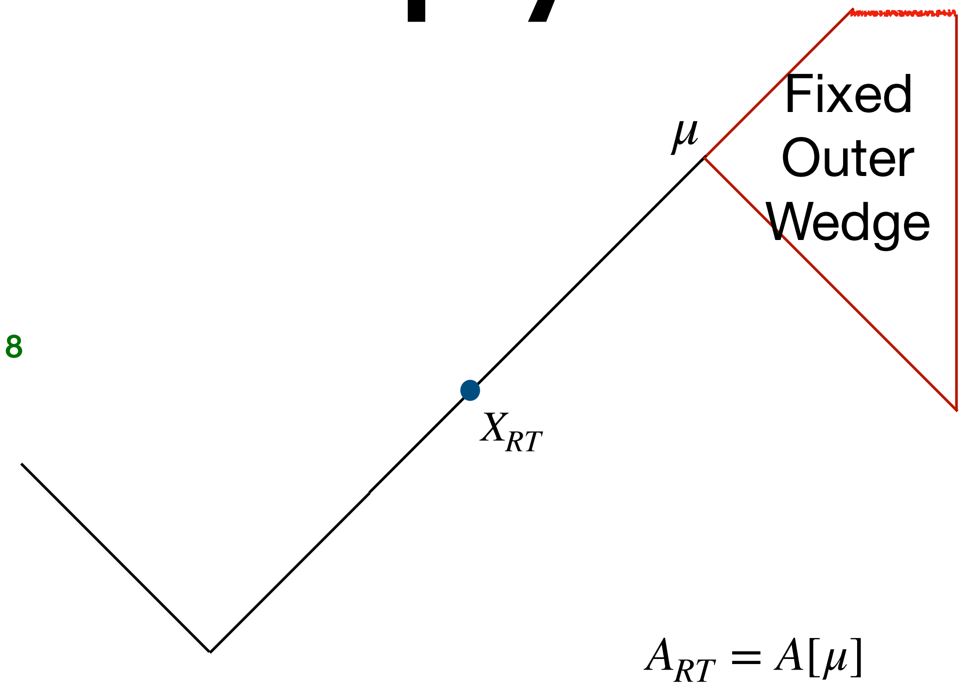
- Motivated by holography, it gives us some insights about an unsolved geometrical problem in geometric analysis/mathematical relativity.

The outer entropy

- The outer entropy is introduced by Engelhardt & Wall (**EW**) as a **coarse-grained** Black Hole entropy associated with an apparent horizon μ.

Engelhardt, Wall, 2018

$$\mathcal{S}(\mu) := \sup_{\rho} S_{\rm vN}(\rho) : D(\overline{\Omega}) \text{ fixed}$$



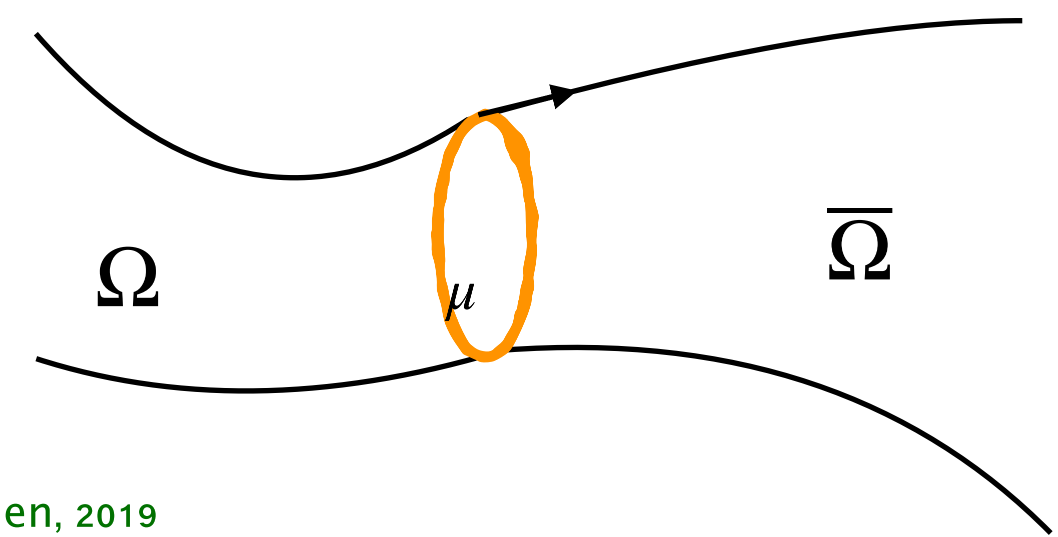
Black line: characteristic initial data

- Motivated by holography, the von Neumann entropy is computed using the **Ryu-Takayanagi** surface X_{RT}, that is the extremal surface with the minimal area homologous to the boundary region.

- EW shows that for an apparent horizon μ, the maximiser always exists. $\mathcal{S}(\mu) = \text{Area}[\mu]/4G\hbar$.

- Statistical interpretation of the BH entropy and area law. Built-in area laws associated with trapping horizons.

- Generalised by Bousso, Nomura & Remmen (BNR) to normal surfaces. They develop a EWBNR algorithm for construct fill-in data.



Nomura, Remmen, 2018
Bousso, Nomura, Remmen, 2019

Outer entropy

The outer entropy of the outer wedge data $(\overline{\Omega}, h_0, K_0)$ *bounded by* $\Sigma = \partial\overline{\Omega}$ *with the asymptotic end* B *is*

maximin — Wall, 2013

$$\boxed{\mathcal{S}(\overline{\Omega}, h_0, K_0) := \sup_{(\Omega,h,K)} \frac{A[HRT(B)]}{4G_N\hbar}} = \sup_{(\Omega,h,K)} \max_{N\subset D(\Omega\cup_\Sigma\overline{\Omega})} \min_{\substack{\sigma\subset N \\ \sigma\in[B]}} \frac{A[\sigma]}{4G_N\hbar}$$

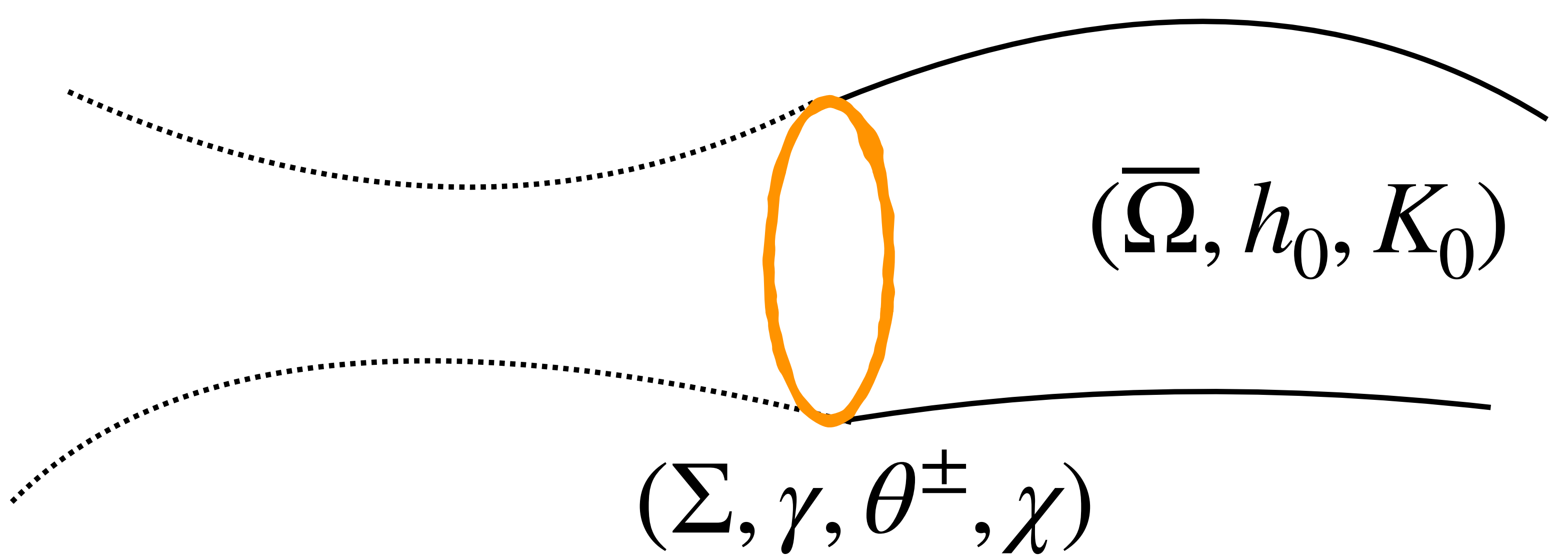
where (Ω, h, K) *is the fill-in data that joins the fixed* $(\overline{\Omega}, h_0, K_0)$ *at* Σ *satisfying DEC and the following constraints:*

$$\gamma|_{\Sigma_{in}} = \gamma|_{\Sigma_{out}}; \quad \theta^\pm|_{\Sigma_{in}} = \theta^\pm|_{\Sigma_{out}}; \quad \chi|_{\Sigma_{in}} = \chi|_{\Sigma_{out}}$$

gluing conditions

where $\chi := K(\cdot, \ell^-)$ *is the twist or anholonomicity 1-form and* ℓ^- *is the ingoing null vector normal to* Σ.

Engelhardt, Wall, 2018



Bartnik mass

- Bartnik had an idea about quasilocal mass back in the 80's. **The Bartnik mass** has been an active research interest in geometric analysis ever since. We seek a minimal mass extension $(\overline{\Omega}, h, K)$ of initial data **without horizons** satisfying DEC.

- Given the positive mass theorem, the Bartnik mass $M_B(\Sigma)$ can be defined as the **infimum** ADM mass over all **horizon-free** extensions of the given surface Σ.

Bartnik, 1989

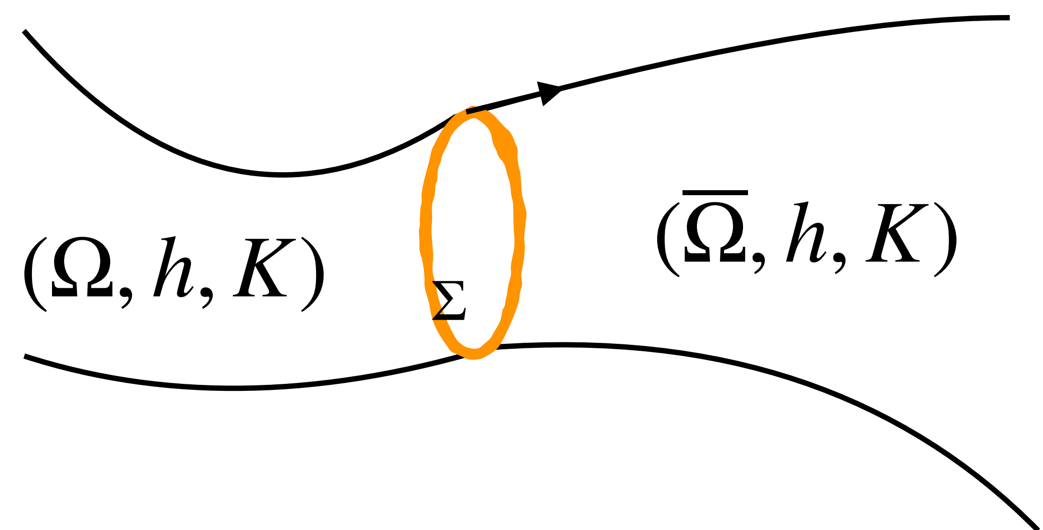
$$M_{\text{outer}}(\Sigma) := \inf_{(\overline{\Omega},h,K)} M(\overline{\Omega}, h, K)\,.$$

- Bray proposed a **dual/inner** version of the Bartnik mass in his seminal paper proving the Riemannian Penrose Inequality (RPI) :

$$M(N, h, K) \geq M_{\text{irr}}(A[\mu]) := \frac{1}{2}\left(\frac{A[\mu]}{\Omega_{n-2}}\right)^{\frac{n-3}{n-2}}.$$

Bray, 2001

$$M_{\text{inner}}(\Sigma) := \sup_{(\Omega,h,K)} \min_{\substack{\sigma \subset \Omega,\\ \sigma \in [\Sigma]}} M_{\text{irr}}(A[\sigma])\,.$$



Equivalence

Our main result is that the outer entropy is **equivalent** to the Bartnik-Bray inner mass:

$$M_{\text{inner}}(\Sigma) = M_{\text{irr}}(4\hbar G_N \mathcal{S}(\Sigma))$$

for an **outer-minimising, mean-convex** Σ.

The proof uses standard focusing arguments.
Check out the details in the paper 2007.00030

A few remarks

- Σ is **outer-minimizing** means that for any Σ' enclosing Σ, $A[\Sigma] \leq A[\Sigma']$.

- Σ is **mean-convex** (normal) means that $\pm\theta^{\pm} \geq 0$.

- Both **(1) outer-minimisation** and **(2) mean-convexity** are “necessary”.

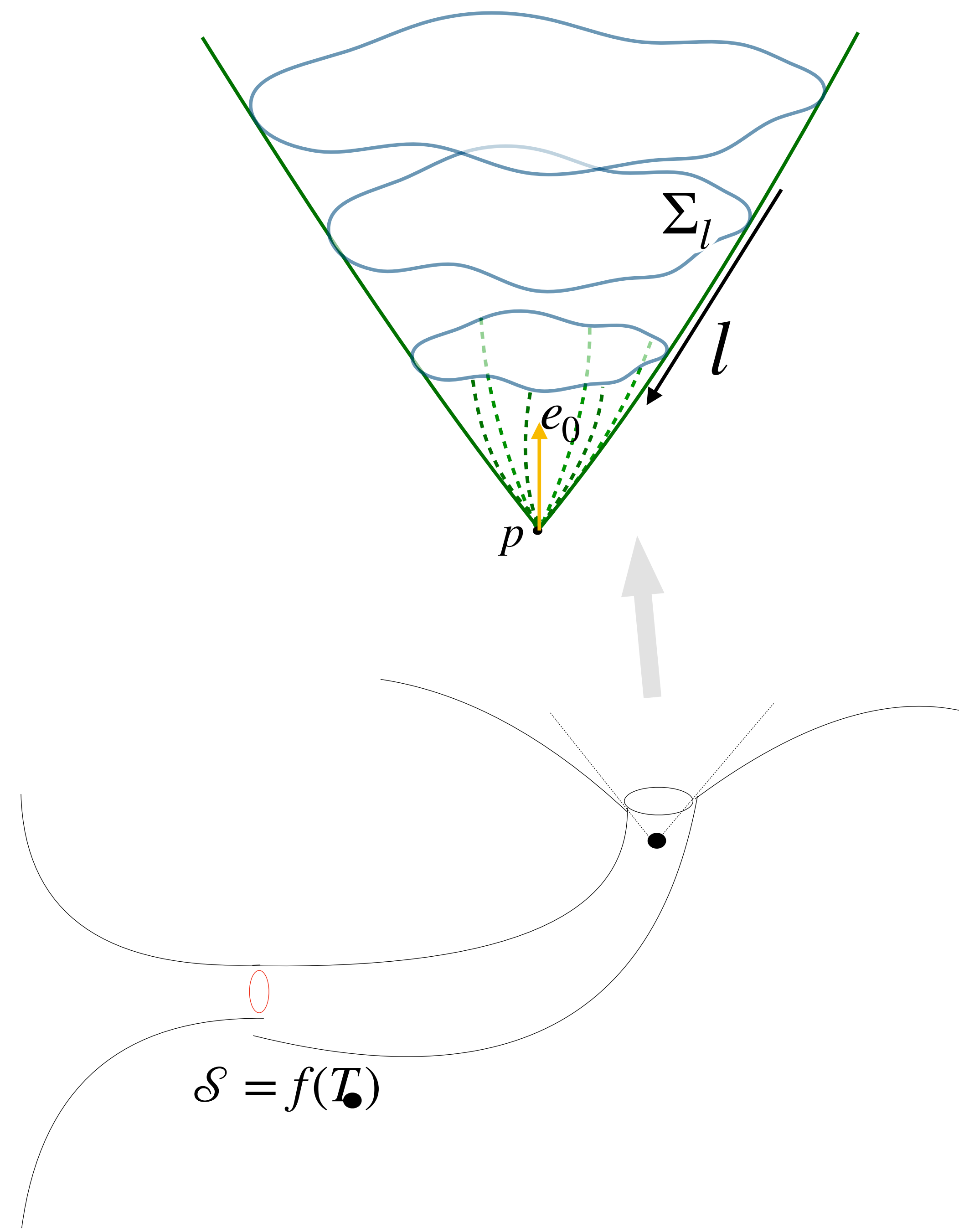
- Bartnik: (1) is used to avoid “bag of gold”-like extensions trivialising the Bartnik mass.

  EW: (1), as part of their “minimar” condition, is used to ensure the HRT surface can be found following their procedure.

- (2) is common in geometric analysis. e.g. Weyl problem, positivity of Brown-York mass, Liu-Yau mass, etc.

A few implications

- The area laws = monotonicity of the quasilocal mass.
- In the **small sphere limit**, any quasilocal mass should reduce to the **stress tensor**, so should the outer entropy. Calculation using the EWBNR algorithm confirms this:

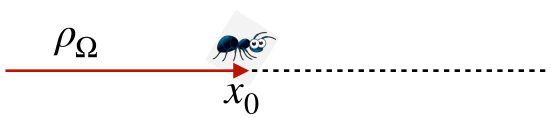


$$\mathcal{S}(\Sigma_l) = \frac{\Omega_{n-2} l^{n-2}}{4 G_N \hbar} \left(\frac{2 l^2 \Omega_{n-2} G_N T(e_0, e_0)|_p}{n-1} \right)^{\frac{n-2}{n-3}}.$$

$$\lim_{l \to 0} l^{-(n-1)} M_{inner}(\Sigma_l) = \frac{\Omega_{n-2} T(e_0, e_0)|_p}{n-1}.$$

Wall's ant conjecture

- A marching ant wonders what's the **minimal energy** given what she has observed so far. We look for the energy-minimising state over the purifications while **holding fixed** the interior marginal state.



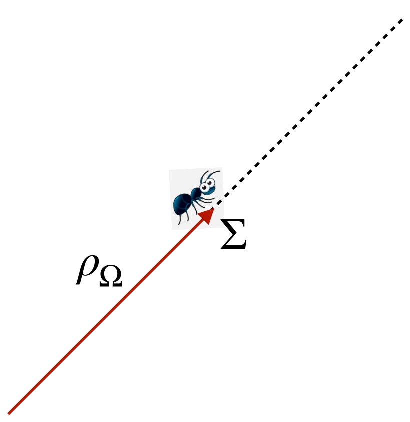
- Wall's conjecture in 1+1 dimensions:

$$\inf_{\rho:\, \mathrm{tr}_{\overline{\Omega}}\rho=\rho_\Omega} \int_{\overline{\Omega}} \langle T \rangle_\rho \, \mathrm{d}x = -\frac{\hbar}{2\pi} \partial_x S_\rho(\Omega)\,|_{x_0}, \quad \overline{\Omega} = (x_0, \infty), \Omega = (-\infty, x_0)\,.$$

Wall, 2017

- **Ceyhan-Faulkner** proved it for a «null» ant on a Rindler Horizon in Minkowski spacetime w.r.t. any **null** variation X on a cut Σ.

$$\inf_{\rho:\, \mathrm{tr}_{\overline{\Omega}}\rho=\rho_\Omega} \int_{\Omega\cup\overline{\Omega}} \langle T \rangle_\rho \, \mathrm{d}x = \frac{\hbar}{2\pi} \mathscr{L}_X D(\rho_\Omega \,||\, \sigma_\Omega)\,|_\Sigma\,.$$



Ceyhan, Faulkner, 2020

- This conjecture concerns the **matter sector**, do we have one for the **gravity sector**?

Gravitational ant conjecture

- The Penrose Inequality implies $\inf_{(\overline{\Omega},h,K)} M(\overline{\Omega},h,K) \geq M_{\rm irr}(4\hbar G_N \mathcal{S}(\Sigma))$.

- We propose the **gravitational analog** of the ant conjecture (GAC):

$$\inf_{(\overline{\Omega},h,K)} M(\overline{\Omega},h,K) = M_{\rm irr}(4\hbar G_N \mathcal{S}(\Sigma)).$$

- In words, it conjectures the Bartnik mass of some closed surface is given by the irreducible mass of the largest black hole that can be fit behind it.

- It's natural that the outer entropy sits at the lower bound. It relates **coarse-grained** entropy with quasilocal mass, whereas Wall's ant conjecture concerns **fine-grained** entropy and stress tensor.

- It is a **purely geometrical** statement. Trivially, the conjecture is true if the Bartnik data can be isometrically embedded into Schwarzschild. There are also some non-trivial examples supporting this claim in the Riemannian setting.

Mantoulidis-Schoen,

Conclusions

- We've shown that the Bartnik-Bray quasilocal mass is equivalent to the Outer Entropy.

- We propose the gravitational ant conjecture:

$$\inf_{(\overline{\Omega},h,K)} M(\overline{\Omega},h,K) = M_{\text{irr}}(4\hbar G_N \mathcal{S}(\Sigma)).$$

- Can we prove a Riemannian version of the conjecture ? It would be an important result in geometric analysis/mathematical relativity.

- Can we understand the microscopic origin of **any** quasilocal mass in **any** candidate theory of quantum gravity?

Thank You !

Equivalence

- We want to prove $M_{\text{inner}}(\Sigma) = M_{\text{irr}}(4\hbar G_N \mathcal{S}(\Sigma))$ for an **outer-minimising mean-convex** Σ.
- Firstly, the gluing conditions in both problems are essentially the same. They are related by a basis change in the normal bundle. In particular, $(\theta^+, \theta^-) \to (\text{tr}_\Sigma K, H)$.
- We need the following lemma to "**quasilocalise**" the outer entropy:

  Lemma 1: For an outer-minimising surface Σ, the HRT surface for the outer entropy, if it exists, always lies inside the inner wedge.

  Nomura, Remmen, 2018

- The outer entropy is more restrictive than the inner mass:

  Lemma 2: For an outer-minimising surface Σ, the supremum areas for the Bartnik-Bray inner mass and the HRT surface satisfy $A_{inner} \leq A_{HRT}$.

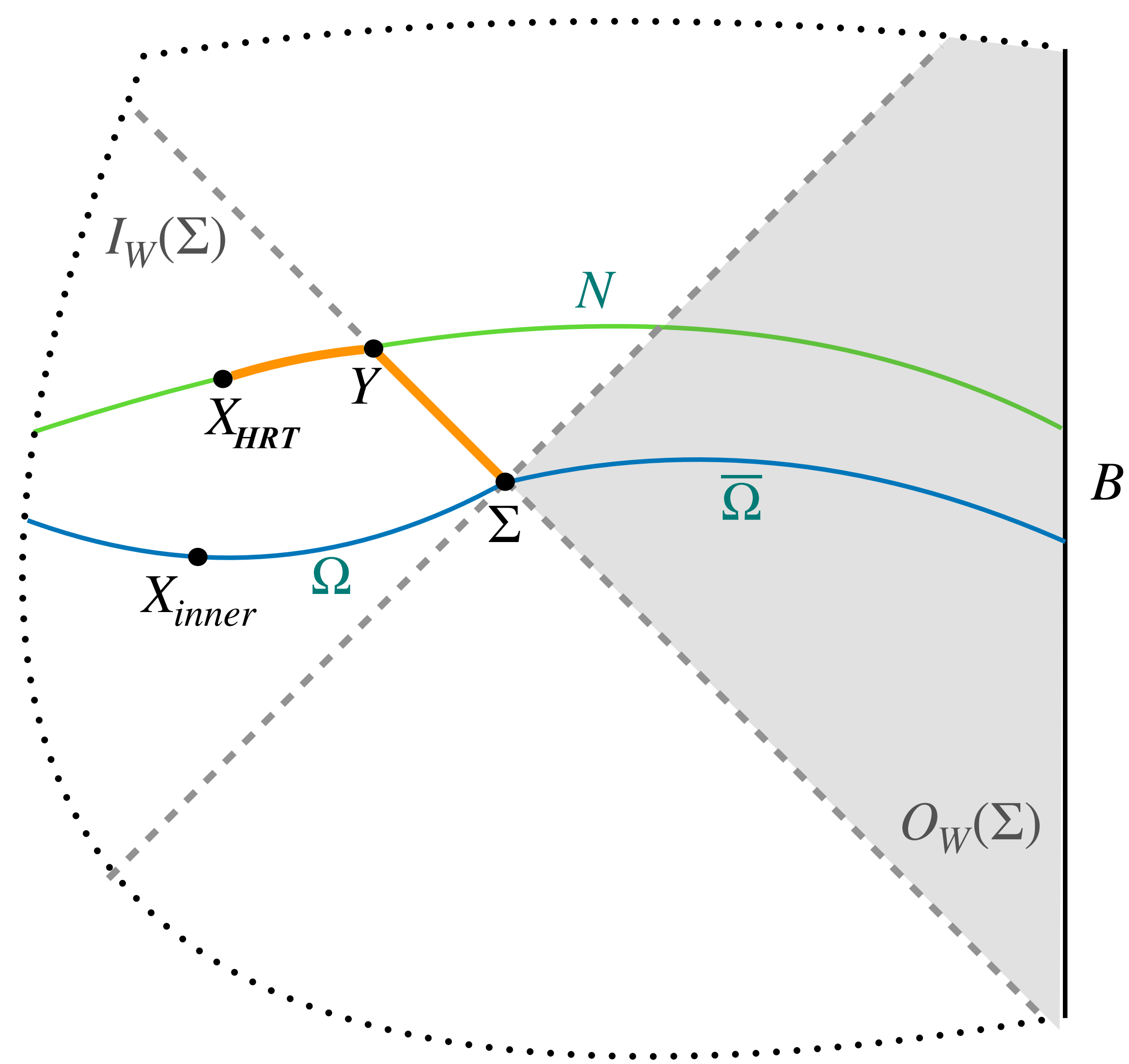
- We also want $A_{inner} \leq A_{HRT} \leq A[\Sigma]$. For this, we need mean-convexity.

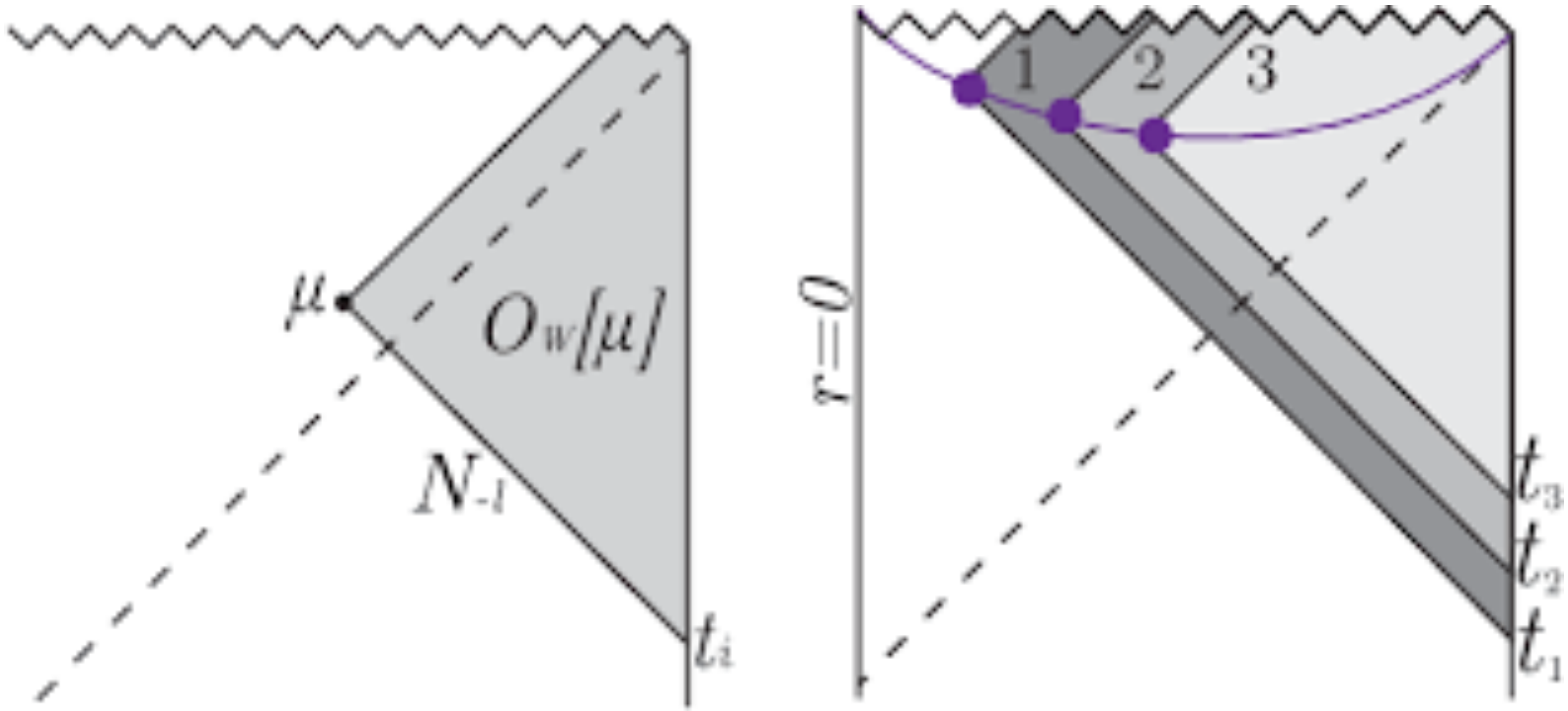
  Lemma 3: For a mean-convex surface Σ, the HRT surface for the outer entropy, if it exists, has area $A_{HRT} \leq A[\Sigma]$.

  Engelhardt, Wall, 2018
  Nomura, Remmen, 2018

Equivalence

- Suppose X_{HRT}, X_{inner} **both exist** but differ, X_{HRT} is extremal on some slice N in the interior (Lemma 1), so $A[X] \leq A[Y]$.
- The standard focusing argument, mean-convexity and DEC gives $A[X_{HRT}] \leq A[Y] \leq A[\Sigma]$, so X_{HRT} is minimal on slice $X_{HRT} - Y - \Sigma$.
- By definition of inner mass, $A[X_{inner}] \geq A[X_{HRT}]$. Lemma 2 also implies that $A[X_{inner}] \leq A[X_{HRT}]$, so they are equal.
- Consider now the case that **neither optimiser exists**. Suppose the supremum areas satisfy $A_{HRT} > A_{inner}$, then $A_{HRT} - \epsilon > A_{inner}$ for some ϵ. Area $A_{HRT} - \epsilon$ is realised at some fill-in, so $A_{inner} \geq A_{HRT} - \epsilon$. $\Longrightarrow$ Contradiction.
- Similarly, one can show that X_{HRT} & X_{inner} both exist or neither exists.





(a) (b)

Secretly an Semidefinite program?

Is the holographic dual an SDP ? SDP is a very$^{\otimes N}$important tool in QI.

$$\inf_{(\overline{\Omega},h,K)} M(\overline{\Omega},h,K) \geq M_{\mathrm{irr}}(4\hbar G_N \mathcal{S}(\Sigma))$$

Primal **Dual**

$$\inf_{(\overline{\Omega},h,K)} M(\overline{\Omega},h,K) = M_{\mathrm{irr}}(4\hbar G_N \mathcal{S}(\Sigma))$$

The gravitational ant conjecture implies the Penrose Inequality.

PI is Weak Duality:

Primal $\geq$ Dual

GAC is Strong Duality:

Primal $=$ Dual