Complexity of Purification in free CFT_2's

arXiv:2009.11881 [hep-th]

Hugo Camargo

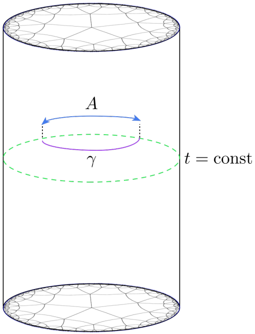
Gravity, Quantum Fields and Information (GQFI)
Max Planck Institute for Gravitational Physics - AEI

hugo.camargo@aei.mpg.de

HoloTube Jr. - October 30, 2020

Holography

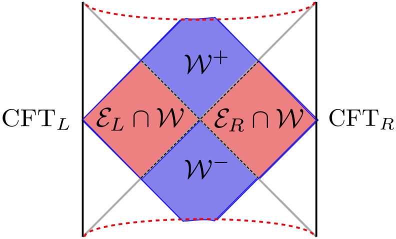
In holography, the emergence of spacetime is encoded in the quantum data at the boundary CFT:



$$S[A] = \min_{\partial\gamma=\partial \mathrm{A}} \frac{\text{Area}(\gamma)}{4G_N} \quad (\text{Leading order in } G_N) \tag{1}$$

Holographic Subregion Complexity

- Holographic EE is insufficient to determine the entire bulk geometry!
- Holographic "complexity" $\mathcal{C}$ encodes additional information about states to which EE is insensitive to.
- Is there a holographic "complexity" measure associated with a single CFT subregion?



- Three subregion-complexity proposals with particular *divergent structure*: subregion-$\mathcal{C}_V$,-$\mathcal{C}_A$ and -$\mathcal{C}_{V\,2.0}$.

Complexity of *Pure* States (*à la* Nielsen)

Complexity $\mathcal{C}$ tells us how difficult it is to produce/reach a certain *target* state $|\psi_{\mathrm{T}}\rangle$.

- Defined via quantum circuits:

$$|\psi_{\mathrm{T}}\rangle = \hat{U}\,|\psi_{\mathrm{R}}\rangle \tag{2}$$

  where $|\psi_{\mathrm{R}}\rangle$ is some (unentangled) *reference* state (e.g. $|00\ldots\rangle$), and where $\hat{U}$ is a unitary operator that can be written as:

$$\hat{U} = \overleftarrow{\mathcal{P}}\left\{ e^{-i\int_0^1 \mathrm{d}\tau \sum_{\mathrm{I}} Y^{\mathrm{I}}(\tau)\hat{\mathcal{O}}_{\mathrm{I}}} \right\} \tag{3}$$

- The Hermitian operators $\{\hat{\mathcal{O}}_{\mathrm{I}}\}$ (*gate generators*) form an operator algebra $\mathcal{A}$, while the parameters $\{Y^{\mathrm{I}}(\tau)\}$ are tangent vectors to $\hat{U}$.

Computational (circuit) Complexity

To compute complexity, we define a *circuit depth* $\mathcal{D}$:

$$\mathcal{D}(\hat{U}) = \int_0^1 d\tau \mathcal{F}(Y^{\mathrm{I}}(\tau)) \tag{4}$$

where the *cost function* $\mathcal{F}(Y^{\mathrm{I}}(\tau))$ is a local functional of the position and the tangent vectors.

- Extremising $\mathcal{D}$ for a given choice of $\mathcal{F}$ yields the computational (circuit) *complexity* $\mathcal{C}$ of $\hat{U}$.
- Different choices for $\mathcal{F}$ give different measures for the circuit complexity $\mathcal{C}$.
- Example: L^2-norm: $\mathcal{F}_2 = \sqrt{\sum_{\mathrm{IJ}} \eta_{\mathrm{IJ}} Y^{\mathrm{I}} Y^{\mathrm{J}}}$. Minimizing $\mathcal{D} \implies$ finding geodesics in a Riemannian space and $\mathcal{C} =$ length of geodesic.

Gaussian States in free QFTs

Gaussian states, with vanishing one-point functions, are completely characterized by their two-point functions, which make up their *covariance matrix*:

$$G_{\pm}^{ab} = \langle\psi|\xi^a\xi^b \pm \xi^b\xi^a|\psi\rangle, \tag{5}$$

where $\xi^a \equiv \{x^1, p^1, \ldots, x^N, p^N\}$ are the dimensionless phase-space operators for N dof. Natural symmetry group: $\mathrm{Sp}(2N, \mathbb{R})$.

- For 1 Harmonic oscillator (N=1):

$$\psi(x) \sim \exp\left\{-\frac{1}{2}(a + \mathrm{i}b)x^2\right\} \implies G = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ -\frac{b}{a} & \frac{a^2+b^2}{a} \end{pmatrix} \tag{6}$$

- Complexity $\mathcal{C}$ can be then recast purely as a function of the *spectrum* of G! $\implies$ Info. about full state $|\psi\rangle$.

Complexity of *Mixed* States: CoP

Is there a measure of complexity for mixed states? Yes! Complexity of Purification (CoP).

- Given a mixed state ρ_{A} in some Hilbert Space $\mathcal{H}_A$, we define a new Hilbert space:

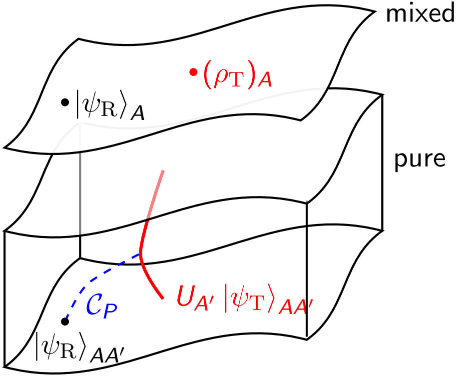
$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A'} \,, \tag{7}$$

  with ancillary system A'. There exist many purifications $|\psi_{\mathrm{T}}\rangle \in \mathcal{H}$ such that $\rho_{\mathrm{A}} = \mathrm{Tr}_{\mathcal{H}_{A'}}(|\psi_{\mathrm{T}}\rangle\langle\psi_{\mathrm{T}}|)$.
- CoP is defined as the minimum of complexity $\mathcal{C}$ with respect to a reference state $|\psi_{\mathrm{R}}\rangle$ and to all possible purifications $|\psi_{\mathrm{T}}\rangle$:

$$\mathcal{C}_P[\rho_{\mathrm{A}}] = \min_{|\psi\rangle\in\mathcal{H}} \mathcal{C}(|\psi_{\mathrm{R}}\rangle, |\psi_{\mathrm{T}}\rangle) \,. \tag{8}$$

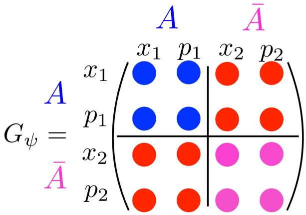
Geometric Meaning of CoP

The manifold (red line) of all possible purifications $|\psi_{\mathrm{T}}\rangle_{AA'}$ related by (e.g. Gaussian) unitaries $U_{A'}$. $\mathcal{C}_P$ is given by the geodesic distance (blue) between the purified reference state $|\psi_{\mathrm{R}}\rangle_{AA'}$ and $U_{A'}\,|\psi_{\mathrm{T}}\rangle_{AA'}$.



CoP for 2-Modes (arXiv:1807.07075 [hep-th])

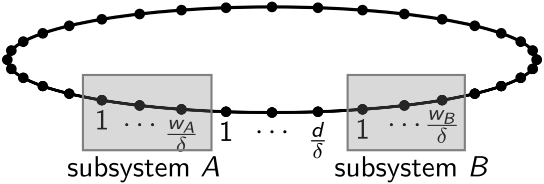
What's the CoP for a system of 2 harmonic oscillators? Look at the covariance matrix:



- EE is only sensitive to G_A (i.e. determined by spectrum of ρ_A).
- *Pure State* Complexity: Sensitive to full $G_{|\psi\rangle}$.
- *Mixed State* Complexity: Fix G_A (given by ρ_A) and consider $G_{|\psi'\rangle}$. Then $\mathcal{C}_P[\rho_A] := \min_{|\psi'\rangle} \mathcal{C}(|\psi'\rangle)$.
- $\mathcal{C}$ satisfies: $\mathcal{C}_P[\rho_A] + \mathcal{C}_P[\rho_{\bar{A}}] \geq \mathcal{C}(|\psi\rangle)$.

Gaussian Purifications of Vacuum Subregions of CFT$_{(1+1)}$'s

Focus on 2 CFT's: the Klein-Gordon field in the massless limit ($c = 1$) and of the critical transverse-field Ising model ($c = 1/2$) on a circle.



The vacuum states $|0\rangle$ of these theories are both Gaussian!

$$G_{ij}^{ab} = \sum_{\kappa \in \mathcal{K}^+} c_\kappa(i-j) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad c_\kappa(j) = |\cos\tfrac{\kappa}{2}|\cos(\kappa j) - \frac{\sin\kappa\sin(\kappa j)}{2|\cos\frac{\kappa}{2}|} \tag{9}$$

$$G_{ij}^{ab} = \sum_{k=1}^{N} e^{\mathrm{i}\frac{2\pi k}{N}(i-j)} \begin{pmatrix} \omega_k & 0 \\ 0 & \frac{1}{\omega_k} \end{pmatrix} \quad , \quad \omega_k = \sqrt{m^2 + \frac{4}{\delta^2}\sin^2\frac{\pi k}{N}} \tag{10}$$

L^2 Cost Function and Single Interval Behaviour

We consider a L^2 cost function (based on geodesic distance):

$$\mathcal{C}(|G_{\rm T}\rangle\,,|G_{\rm R}\rangle) = \sqrt{\frac{|\,\mathrm{tr}\,(\log\,(-G_{\rm T}\,G_{\rm R}))^2\,|}{8}}. \tag{11}$$

- $|G_{\rm R}\rangle$ = spatially unentangled subregion, and $|G_{\rm T}\rangle$= vacuum subregion.

We minimize $\mathcal{C}$ over the most general Gaussian *minimal* purifications. For a single interval of size w/δ:

- For the bosonic CFT (with reference scale μ):

$$\mathcal{C}_P = \left(f_2(\mu\,\delta)\frac{w}{\delta} + f_1\left(\frac{m}{\mu},\mu\,\delta\right)\log\frac{w}{\delta} + f_0\left(\frac{m}{\mu},\mu\,\delta\right)\right)^{\frac{1}{2}} \tag{12}$$

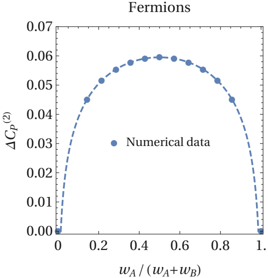
- For the Ising CFT:

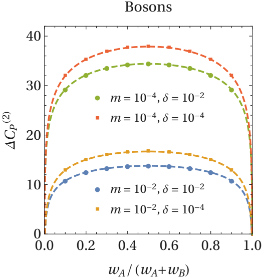
$$\mathcal{C}_P = \left(e_2\,\frac{w}{\delta} + e_1\log\frac{w}{\delta} + e_0\right)^{\frac{1}{2}} \tag{13}$$

Adjacent-Interval Behaviour

For the adjacent interval case, we consider a regularization akin to MI called *mutual complexity* (of purification):

$$\Delta \mathcal{C}_P^{(2)} \equiv \mathcal{C}_P(A)^2 + \mathcal{C}_P(B)^2 - \mathcal{C}_P(A \cup B)^2 \tag{14}$$





$$\Delta \mathcal{C}_P^{(2)} = e_1 \log \frac{w_A w_B}{(w_A + w_B)\,\delta} + e_0$$

$$\Delta \mathcal{C}_P^{(2)} = f_1\left(\frac{m}{\mu}, \mu\,\delta\right) \log \frac{w_A w_B}{(w_A + w_B)\delta} + f_0\left(\frac{m}{\mu}, \mu\,\delta\right) \tag{15}$$

$\mathcal{C}_P$ vs Holographic Subregion Complexity?

The divergent structure of $\mathcal{C}_P$ is comparable to the divergent structure of the holographic subregion complexity proposals. Particularly with the subregion-$\mathcal{C}_{V2.0}$ and subregion-$\mathcal{C}_A$ proposals.

- Single Interval:

$$\mathcal{C} \propto \begin{cases} \frac{w}{\delta} - 2\log\left(\frac{w}{\delta}\right) - \frac{\pi^2}{4} & (\mathrm{s}{-}\mathcal{C}_{\mathrm{V}\,2.0}) \\ \log\left(\frac{\ell_{CT}}{\mathcal{L}}\right)\frac{w}{2\delta} - \log\left(2\frac{\ell_{CT}}{\mathcal{L}}\right)\log\left(\frac{w}{\delta}\right) + \frac{\pi^2}{8} & (\mathrm{s}{-}\mathcal{C}_{\mathrm{A}}) \end{cases} \tag{16}$$

- Adjacent Intervals: (using $\Delta\mathcal{C} \equiv \mathcal{C}_P(A) + \mathcal{C}_P(B) - \mathcal{C}_P(A \cup B)$)

$$\Delta\mathcal{C} \propto \begin{cases} \log\frac{w_A\, w_B}{(w_A+w_B)\delta} + \frac{\pi^2}{8} & (\mathrm{s}{-}\mathcal{C}_{\mathrm{V}\,2.0}) \\ log\left(2\frac{\ell_{CT}}{\mathcal{L}}\right)\log\frac{w_A\, w_B}{(w_A+w_B)\delta} - \frac{\pi^2}{8} & (\mathrm{s} - \mathcal{C}_{\mathrm{A}}) \end{cases} \tag{17}$$

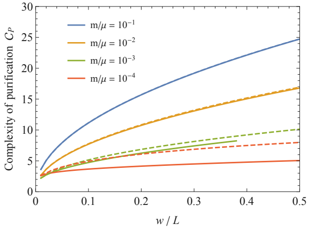
Summary and Outlook

- Complexity $\mathcal{C}$ captures more info. about a quantum state than EE,
- Vacuum/Ground States in free QFTs are Gaussian $\implies$ Full description in terms of Covariance Matrices and $\mathrm{Sp}(2N, \mathbb{R})$.
- The most general Gaussian purifications capture the universal properties of $\mathcal{C}_P$ for the single and adjacent interval cases.
- Numerical tests show $\mathcal{C}$ satisfies: $\mathcal{C}_P[\rho_A] + \mathcal{C}_P[\rho_{\bar{A}}] \geq \mathcal{C}(|\psi\rangle)$,
- Complexity of purification $\mathcal{C}_P$ of vacuum subregions has comparable divergence structure to holographic subregion proposals.
- Beyond Gaussianity: Non-Gaussian states $\implies$ Towards complexity $\mathcal{C}$ in *interacting* field theories.
- Is it $\mathcal{C}$=Action, $\mathcal{C}$=Volume$_{1,2}$, $\mathcal{C}=\cdots$?

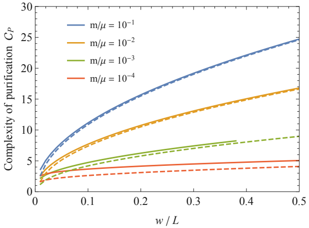
Comparison with Mode-by-mode Purifications and Fisher-Rao Distance Function

There are two other approaches for complexity of mixed states.

- Mode-by-mode Purifications = Split the problem of finding $\mathcal{C}_P$ for a system with N_A modes into N_A problems for a single mode.
- Fisher-Rao Distance = A natural distance function in the manifold of covariance matrices.



(a) "$\mathcal{C}_P$ vs Single-Mode Purifications"



(b) "$\mathcal{C}_P$ vs Fisher-Rao Distance"

Non-Unitary Time-Evolution (arXiv:1904.02713 [hep-th])

Consider the thermal state $\rho_\beta = e^{-\beta H}$ in a CFT_{1+1}. We can view ρ_β as the Euclidean time-evolution operator $\mathcal{V}$ that acts on a state $|\phi(\tau = 0, x)\rangle$:

$$\mathcal{V} = \overleftarrow{\mathcal{P}} \left\{ \exp\left(- \int_0^\beta \mathrm{d}\tau H \right) \right\} \tag{18}$$

Can we interpret the Euclidean path integral $\langle \phi(\beta, x) | \mathcal{V} | \phi(0, x) \rangle |_{e^{2\omega}\delta_{\mu\nu}}$ on a Weyl-rescaled geometry as a circuit?

- Yes! The price to pay: it is built from Hermitian and skew-Hermitian gates: $\mathcal{V} = \overleftarrow{\mathcal{P}} \exp(\int dtdy\, [a(t,y)h(y) + ib(t,y)p(y)])$
- Cost function? An expansion of DBI-inspired one yields the *Liouville action*:

$$\mathcal{D}_L \sim \int d\tau dx \left\{ \frac{e^{2\omega}}{\epsilon^2} + \frac{1}{2}\eta(\dot{\omega}^2 + \omega'^2) + \ldots \right\} \tag{19}$$