# How **right** was Landau?

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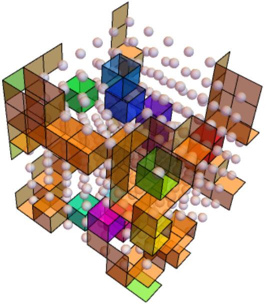
based on work with
Nabil Iqbal (Durham) 2003.04349 and in progress



2020-09

# Plan

- The generalized Landau paradigm.
- Consequences of emergent (one-form) symmetries.
- Mean string field theory.
- 3d Ising model as a string theory.
  - 3d ‘planar’ Ising model and simulation results.
  - Speculations about the worldsheet.
  - A mystery about the vanilla 3d Ising model.

# Motivation

# Landau was even more right than we thought.

**Landau paradigm part 1:**

Phases of matter are classified by how they represent their symmetries.

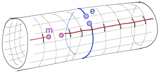
(Phases of matter are classified by the symmetries they break.)

Gapless excitations or degeneracy (in a phase) are Goldstone modes for spontaneously broken symmetries.

**Some apparent exceptions:**

- topological order [Wegner, Wen]

*e.g.* deconfined phase of $\mathbb{Z}_2$ lattice gauge theory, fractional quantum Hall states.




- other deconfined states of gauge theory (*e.g.* Coulomb phase of E&M).
- (Landau) Fermi liquid.
- topological insulator and integer quantum Hall states.
- CFTs with no (symmetric) relevant operators.

But...

# Higher-form symmetries. [Nussinov-Ortiz, Willett et al, Hofman-Iqbal, Lake...]

($D \equiv$ number of spacetime dimensions.)

## 0-form symmetry:

$\partial^\mu j_\mu = 0$ (*i.e.* $d \star j = 0$)
$\implies Q = \int_{\Sigma_{D-1}} \star j$ is independent of time-slice $\Sigma$,

*i.e.* **is topological.** [Thorngren]

Charged objects are local operators
$\delta \mathcal{O}(x) = \mathbf{i}[Q, \mathcal{O}(x)] = \mathbf{i} q \mathcal{O}(x)$.

Finite transformation:
$U_{g=e^{\mathbf{i}\alpha}} = e^{\mathbf{i}\alpha Q} = e^{\mathbf{i}\alpha \int_{\Sigma_{D-1}} \star j}$.

Charged particle worldlines can't end (except on charged operators).

Discrete ($\mathbb{Z}_k$) version: particles can disappear in groups of $k$.

## 1-form symmetry:

$J_{\mu\nu} = -J_{\nu\mu}$ with $\partial^\mu J_{\mu\nu} = 0$
(*i.e.* $d \star J = 0$)

$\implies Q_\Sigma = \int_{\Sigma_{D-2}} \star J$ depends only on the topological class of $\Sigma$.

Charged objects are loop operators:
$\delta W(C) = \mathbf{i}[Q_\Sigma, W(C)] = \mathbf{i} q \# (\Sigma, C)\, W(C)$
*e.g.* in free Maxwell theory:
$J^M = F, W^M(C) = e^{\mathbf{i} \oint_C A}$ and
$J^E = \star F, W^E(C) = e^{\mathbf{i} \oint_C \tilde{A}}$ ($dA \equiv \star d\tilde{A}$).

Finite transformation:
$U_{g=e^{\mathbf{i}\alpha}}(\Sigma_{D-p-1}) = e^{\mathbf{i}\alpha Q_\Sigma} = e^{\mathbf{i}\alpha \int_{\Sigma_{D-p-1}} \star J}$.

Charged string worldsheets can't end (except on charged operators).

Discrete ($\mathbb{Z}_k$) version: strings can disappear or end in groups of $k$.

# Higher-form symmetries.

[Nussinov-Ortiz, Willett et al, Hofman-Iqbal, Lake]

## 0-form symmetry:

Unbroken phase: correlations of charged operators are short-ranged, decay when the charged object ($S^0 =$ two points) grows.

$\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \rangle \sim e^{-m|x|}$

($|x| = \text{Area}(S^0(x))$.)

Broken phase for 0-form sym:

$\langle \mathcal{O}(x)^\dagger \mathcal{O}(0) \rangle = \langle \mathcal{O}^\dagger \rangle \langle \mathcal{O} \rangle + \ldots$ independent of size of $S^0$.

## 1-form symmetry:

Unbroken phase: correlations of charged operators are short-ranged, decay when the charged object grows.

$\langle W(C) \rangle \sim e^{-T_{p+1}\text{Area}(C)}$

For E&M, area law for $\langle W^E(C) \rangle$ is the superconducting phase.

Broken phase for 1-form sym:

$\langle W(C) \rangle = e^{-T_p \text{Perimeter}(C)} + \ldots$

(set to 1 by counterterms local to $C$: large loop has a vev)

# Landau was even more right than we thought.

- topological order $\stackrel{?}{=}$ SSB of *discrete* higher-form symmetry.

TO $\equiv$ degenerate groundstates which are *locally indistinguishable*.

SSB of $(q > 1)$-form discrete symmetry implies topological order, since the algebra of loop (or surface) operators must be realized on the vacuum.

- eg 1 ($\mathbb{Z}_k$ gauge theory/toric code): in $D$ spacetime dimensions with $\mathbb{Z}_k^{(1)} \times \mathbb{Z}_k^{(D-2)}$ 1-form and $(D-2)$-form symmetries, represented by $U^m(C_1), V^n(M_{D-2}), m, n = 1..k$

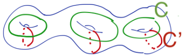
$$U^m(C)V^n(M) = e^{\frac{2\pi \mathbf{i} mn\#(C,M)}{k}} V^n(M)U^m(C). \quad (\#(C,M) \equiv \text{intersection } \#)$$

This is the algebra of electric and magnetic flux surfaces in $\mathbb{Z}_k$ gauge theory. Simple realization is $BF$ theory:

$$S = \frac{k}{2\pi}\int_D B_{D-2} \wedge dA, \qquad U^m(C) = e^{\mathbf{i}m\int_C A}, \quad V^n(M) = e^{\mathbf{i}n\int_M B_{D-2}}$$

- eg 2 (Laughlin FQHE): in $D = 2+1$, $\mathbb{Z}_k^{(1)}$ 1-form symmetry with an 't Hooft anomaly

$$U^m(C)U^n(C') = e^{\frac{2\pi \mathbf{i} mn\#(C,C')}{k}} U^n(C')U^m(C).$$




(the flux carries charge) gives $k$ groundstates on $T^2$.

(Whether the most general topologically ordered state can be understood in this way is an open question [Wen 18].)

# Landau was even more right than we thought.

[Willett et al, Hofman-Iqbal, Lake]

- The gaplessness of the photon can be understood as required by spontaneously broken $\mathsf{U}(1)$ 1-form symmetry.

If we couple to a bg field $\Delta L = j_\mu \mathcal{A}^\mu$,

$$\mathcal{L}_{\text{eff}} = \frac{\kappa}{2}\left(\underbrace{d\phi}_{\text{Goldstone}} + \mathcal{A}\right)^2 .$$

The goldstone transforms nonlinearly $\phi \to \phi + \lambda, \mathcal{A} \to \mathcal{A} - d\lambda$. This is a global symmetry if $d\lambda = 0$.

(By $(\text{form})^2$ I mean $(\text{form}) \wedge \star(\text{form})$.)

If we couple to a bg field $\Delta L = J_{\mu\nu}\mathcal{B}^{\mu\nu}$,

$$\mathcal{L}_{\text{eff}} = \frac{g^2}{4}\left(\underbrace{d\tilde{A}}_{\text{Goldstone}} + \mathcal{B}\right)^2 .$$

The goldstone transforms nonlinearly $\tilde{A} \to \tilde{A} + \lambda, \mathcal{B} \to \mathcal{B} - d\lambda$. This is a global symmetry if $d\lambda = 0$.

Maxwell term for $A$.

# Consequences of emergent 1-form symmetries

# Consequences of emergent symmetries

In particle physics, we are used to the idea of emergent (aka accidental) symmetries, which are explicitly broken by irrelevant operators:

$$S = S_{\text{symmetric}} + \int d^D x \frac{g}{\Lambda^{\Delta - D}} \left( \mathcal{O}_{\Delta,\text{charged}}(x) + h.c. \right).$$

Approximate symmetries can be spontaneously broken and lead to pseudo-Goldstone bosons, with $m_\pi^2 \sim \frac{g}{\Lambda^{\Delta-D}} f_\pi$, only massless when $\Lambda \to \infty$.

Q: The existence of magnetic monopoles with $m = M_{\text{monopole}}$ explicitly breaks the 1-form symmetry of electrodynamics:

$$\partial^\mu J^E_{\mu\nu} = j_\nu^{\text{monopole}}$$

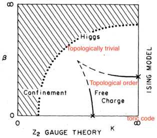
If the photon is a Goldstone for this symmetry, does this mean the photon gets a mass?

Cheap answer #1: No, by dimensional analysis (take $m_e \to \infty$). $m_\gamma \to 0$ when $M_{\text{monopole}} \to \infty$.

Cheap answer #2: The operators which are charged under a 1-form symmetry are loop operators – they are not local. We can't add non-local operators to the action at all.

# Consequences of emergent 1-form symmetries

- Discrete analog: In the toric code, the discrete 1-form symmetries are exact, but in the rest of the deconfined (broken) phase, they are *emergent* (and still spontaneously broken). A rigorous proof of this [Hastings-Wen 04] constructs the string operators by quasi-adiabatic continuation.




- To remove the distinction between 0-form and 1-form symmetries, compactify the theory on a circle of radius $R$. Then the 3+1d gauge field decomposes into a 2+1d gauge field $a$ ($d\sigma = \star_3 da$) and a scalar $\phi = \oint A$.

Worldlines of electrically-charged particles wrapping the circle produce

$$\Delta S_{\text{eff}} \propto e^{iq\oint A} e^{-m_e R} + \cdots = e^{-m_e R} \cos\phi + \cdots \quad \Longrightarrow \quad m_\sigma = 0, m_\phi \sim e^{-m_e R}.$$

Magnetically-charged particles in 3+1d become pointlike sources of magnetic flux – monopole instantons.

$$\overset{\text{[Polyakov 74]}}{\Longrightarrow} \Delta S_{\text{eff}} \sim e^{-m_m R} \cos\sigma \quad \Longrightarrow \quad m_\sigma \sim e^{-m_m R}.$$

When $R \to \infty$, the photon mass goes to zero.

# Consequences of emergent symmetries at $T > 0$

•We could interpret the circle as the thermal circle, $R = \beta$.

Known forms of topological order in $d \leq 3 + 1$ have the property that at any $T > 0$ they are smoothly connected to $T = \infty$ (a trivial product state).

The argument above is perfectly consistent with this fact: if the 1-form symmetry is emergent, then as soon as $T > 0$, a mass *is* generated for the photon, and the state is smoothly connected to $T = \infty$.

Note: $\exists$ stable TO at $T > 0$: 2-form toric code in 4+1d.

[Landahl-Dennis-Kitaev-Preskill, 01]

$\mathsf{U}(1)$ version: 2-form gauge field is massless even at finite temperature $0 < T < T_c$.

Why: a theory with a 2-form symmetry on a circle still has a 1-form symmetry.

(The phase transition at $T_c$ where the topological order is finally destroyed by proliferation of strings now seems even more interesting.)

# Consequences of emergent symmetries, continued

The appeal to locality of the action is not completely satisfying. On the lattice, the action is a sum of loops: $S = \sum_{\text{small loops},C} \prod_{\ell \in C} u_\ell + \cdots$.

The question is: as we coarse-grain, do the loops become larger or smaller?

**0-form sym:** $S = S_{\text{CFT}} + \frac{g}{\Lambda^{D-\Delta}} \int_x (\mathcal{O}_\Delta(x) + h.c.) \equiv S_{\text{CFT}} + \delta S$.

$$Z = \langle e^{-\delta S} \rangle = 1 - \left( \frac{g}{\Lambda^{D-\Delta}} \right)^2 \int_x \int_y \langle \mathcal{O}_\Delta(x) \mathcal{O}_\Delta(y)^\dagger \rangle + \cdots$$

$$= 1 - \left( \frac{g}{\Lambda^{D-\Delta}} \right)^2 \left( L^{2(D-\Delta)} - a^{2(D-\Delta)} \right) + \cdots$$

IR divergence if $\Delta < D$. Relevant op changes the IR behavior.

**1-form sym:** Consider $\delta S = g \sum_C (W[C] + h.c.)$.

$Z = \langle e^{-\delta S} \rangle = 1 - g \sum_C \langle W[C] \rangle + h.c.$.

Assumptions: (1) regulate $\sum_C$ with cubic lattice

(2) strict area law (broken phase): $\langle W[C] \rangle = t^{\ell[C]}$.

(3) Approximate loops as independent. Result (use transfer matrix):

$\langle \sum_C W[C] \rangle = L^D \frac{1}{2} \sum_{\ell=1}^{\infty} \langle 0 | \frac{t^\ell T^\ell}{\ell} | 0 \rangle = -\frac{L^D}{2} \int đ^D q \log \left( 1 - 2t \sum_{\mu=1}^{D} \cos a q_\mu \right)$

For $t < t_c \equiv \frac{1}{2D}$, the sum over loops is finite. Near the critical point, this is

$\langle \sum_C W[C] \rangle \sim -\frac{L^D}{2} \int đ^D q \log \left( q^2 + m^2 \right), \quad m^2 = \frac{1 - 2tD}{2ta^2}$.

This is just the worldline description of a Higgs particle. If it condenses, the photon is massive, otherwise not.

# Method of the missing box

| | | |
|---|---|---|
| 0-form symmetry | : | mean field theory |
| | :: | |
| 1-form symmetry | : | ? |

# Mean String Field Theory

[work in progress]

Recall: order parameter for $\mathsf{U}(1)$ 0-form symmetry-breaking, $\phi(x) \mapsto e^{\mathbf{i}\alpha}\phi(x)$. Demand locality in space, derivative expansion:

$$S_{\text{Landau-Ginzburg-Wilson}}[\phi] = \int d^D x \left( r|\phi|^2 + u|\phi|^4 + \cdots + |\partial\phi|^2 + \cdots \right) \ .$$

One way to think about this action is as a variational statement: $|\text{groundstate}\rangle \stackrel{?}{=} \otimes_x |\phi(x)\rangle$.

---

Order parameter for $\mathsf{U}(1)$ 1-form symmetry-breaking, $\Psi[C] \mapsto \Psi[C]e^{\mathbf{i}\int_C \Lambda}$. Demand locality in loop space, area derivative expansion:

$$S_{\text{LGW}}[\Psi] = \int [dC] \left( v\left(|\Psi[C]|^2\right) + \left| \frac{\delta}{\delta C_{\mu\nu}} \Psi[C] \right|^2 + \cdots \right) \ , \ \ v(x) \equiv rx + ux^2 + \cdots \ .$$

$\frac{\delta}{\delta C^{\mu\nu}}$: area derivative [Migdal, Polyakov]

variational statement: $|\text{groundstate}\rangle \stackrel{?}{=} \sum_C \Psi[C]|C\rangle$.

Plausible goal: develop a crude picture of the phase diagram (and transitions) for systems with 1-form symmetries.

# Mean String Field Theory

[work in progress]

$$\text{eom:} \quad 0 = \frac{\delta S}{\delta \Psi[C]^\dagger} = \oint ds \frac{\delta}{\delta C^{\mu\nu}(s)} \frac{\delta}{\delta C_{\mu\nu}(s)} \Psi[C] - r\Psi[C] + \cdots$$

**Unbroken phase:** (confinement) $r > 0 \implies \Psi[C] \sim 0.$

$$\text{Ansatz:} \quad \Psi[C] = e^{-\beta A[C]}, \quad A[C] = \min_{\Sigma, \partial\Sigma = C} \text{Area}(\Sigma)$$

For large $\beta$, $\beta = \sqrt{r/2}$. Area law.
**Broken phase:** $\Psi[C] \sim 1$ "string condensed phase" [Levin, Wen]
Component field ansatz:

$$\Psi[C] = \exp\left(\oint_C ds \left(-\mu(x(s)) + \mathbf{i} A_\mu(x(s))\dot{x}^\mu(s) + h_{\mu\nu}(x(s))\dot{x}^\mu \dot{x}^\nu + \cdots\right)\right) \quad .$$

Assume $\langle \mu(x) \rangle \equiv \mu$, worldline tension, plug back into action:

$$S[\Psi] = \int [dC] \left( e^{-2\mu\ell[C]} \oint ds F_{\mu\nu}(x(s)) F^{\mu\nu}(x(s)) - v\left(e^{-2\mu\ell[C]}\right)\right)$$

$$\overset{\mu \gg 1}{\simeq} \frac{1}{g^2} \int d^D x F_{\mu\nu}(x) F^{\mu\nu}(x), \qquad \frac{1}{g^2} = \sqrt{\frac{|r|}{2u}}.$$

Check: Following the logic for $\mathbb{Z}_N$ gives the $(BF)$ EFT for the toric code.

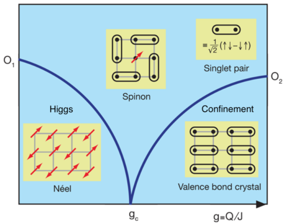
# Generalized Landau paradigm, part 2

# ‘Beyond-Landau’ critical points?

**Landau paradigm part 2**:
At a critical point, the critical dofs are the fluctuations of the order parameter.

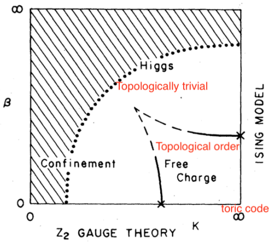
**Apparent exceptions**:

- Direct transitions between states which break *different symmetries* (deconfined quantum critical points), *e.g.* Neel to VBS in $D = 2 + 1$.



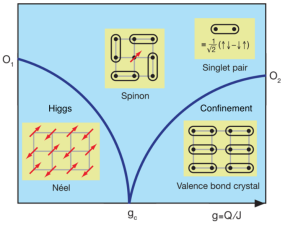

[Image: Alan Stonebraker]

- Transitions out of deconfined phases, such as topologically-ordered states (no local order parameter).



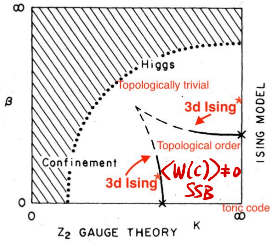

[Image: Fradkin-Shenker]

# 'Beyond-Landau' critical points?

- 


  Can be understood as a consequence of symmetries with mixed 't Hooft anomalies [Metlitski-Thorngren 18]

  $\Longrightarrow$ WZW terms coupling the order parameters on both sides. (Not today's focus.)

- 


  Can we understand the critical theory in terms of fluctuations of the string order parameter $W(C)$? But by Wegner's duality, this theory (up to global data) is in the same universality class as the 3d Ising model.

This suggests that the near-critical 3d Ising model should have a description as a string theory.

# 3d Ising model as a string theory

# 3d Ising model as a string theory.

This is something which has been suggested before, from other points of view. [Fradkin-Srednicki-Susskind 80, Polyakov 81, Dotsenko, Itzykson 82, Casher-Foerster-Windey 85, Kavalov, Sedrakyan, Distler 92, Caselle-Gliozzi-Vinti, Magnea 94]

Reasons to hope for progress here:

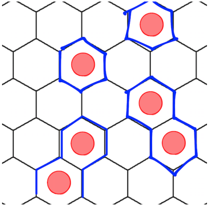
- We're going to propose a modification to the Ising model, which we think may have a better string theory description.
- We've learned a lot about non-perturbative string theory since 1994!

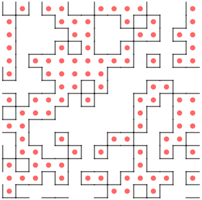
# Fermions from 2d Ising model. [Jordan-Wigner, Lieb-Mattis, ..., Polyakov]

$$Z_\triangle(\beta) = \sum_\sigma e^{-\beta \sum_{\langle ij \rangle}(1-\sigma_i\sigma_j)}$$

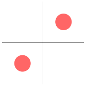
$$= 2\sum_\gamma e^{-2\beta L[\gamma]}$$

(● = spin up)



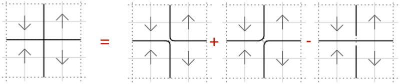


On the square lattice, this can happen:



(This is an avoidable, non-universal technicality, but its resolution is instructive.)

Resolution:



$$Z_\square(\beta) = 2\sum_\gamma (-1)^{n[\gamma]} e^{-2\beta L[\gamma]} \qquad n[\gamma] \equiv \#\text{ of self-intersections}$$

# Fermions from 2d Ising model.

[Jordan-Wigner, Lieb-Mattis, ..., Polyakov]

$$Z_\square(\beta) = 2\sum_\gamma (-1)^{n[\gamma]} e^{-2\beta L[\gamma]} = 2\exp\left(\underbrace{\sum_{\gamma,\text{connected}} (-1)^{n[\gamma]} e^{-2\beta L[\gamma]}}_{\text{worldline sum for real fermion}}\right)$$

$n[\gamma] \equiv \#$ of self-intersections

w/ PBC: only even winding configs $w_{x,y}[\gamma] \in 2\mathbb{Z}$ correspond to spins

$$Z_{T^2} = \sum_\gamma \frac{1}{2}\left(1 + (-1)^{w_x(\gamma)}\right)\frac{1}{2}\left(1 + (-1)^{w_y(\gamma)}\right)(-1)^{n[\gamma]} e^{-2\beta L[\gamma]}$$

$$= Z_{++} + Z_{+-} + Z_{-+} + Z_{--}.$$

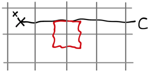
This sum over spin structures says $(-1)^F$ is gauged.

# Fermions from 2d Ising model. [Jordan-Wigner, Lieb-Mattis, ..., Polyakov]

More explicitly, we can make fermion operators:

Disorder operator: $\mu(x) \equiv \prod_{\langle ij \rangle \perp C_x} e^{-2\beta\sigma_i\sigma_j}$.

$x \in$ dual lattice. (Flip sign of $\beta$ along links crossed by $C$.)

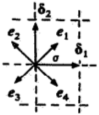



$\mu$ is independent of local changes in $C$ by $\sigma_i \to -\sigma_i$ symmetry. $C$ is a branch cut for $\sigma_i$.

Duality interchanges $\mu \leftrightarrow \sigma$.

The self dual object $\psi_a(x) \equiv \sigma(x)\mu(x + e_a)$ is a fermion

$$R_{2\pi}\left(\psi(x)\right) = \psi_{a+4}(x) = -\psi_a(x)$$

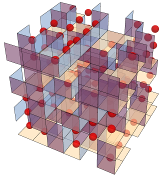



and satisfies

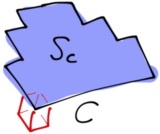
$$\langle \psi_a(x) \rangle = \cosh(2\beta)\langle \psi_{a+1}(x) \rangle - \sinh(2\beta)\langle \psi_{a+2}(x + \delta_{a+1}) \rangle$$

In the continuum limit, this is the Dirac equation, with $m \propto \beta - \beta_c$.

# Fermionic strings from 3d Ising model.



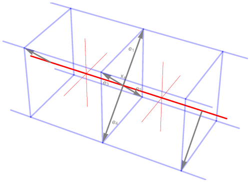
[Polyakov, 80s]




Disorder operator:

$\mu(C) \equiv \prod_{\langle ij\rangle \perp S_C, \partial S_C = C} e^{-2\beta\sigma_i\sigma_j}$.

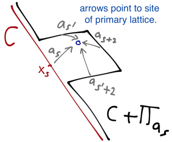
$\mu$ is independent of local changes in $S_C$ by $\sigma_i \to -\sigma_i$ symmetry. $S_C$ is a branch cut for $\sigma_i$.



$\Psi_{a_1 \cdots a_L}(C) \equiv \mu(C) \prod_{s=1}^{L} \sigma(x_s + e_{a_s})$

($x_s$ = center of link $s$) satisfies

$\Psi_{a_1 \cdots a_L}(C) = \cosh(2\beta)\Psi_{a_1 \cdots a_s+1, a_{s+1}, \cdots a_L}(C)$
$- \sinh(2\beta)\Psi_{a_1 \cdots a_{s-1}, a'_s, a_s+2, a'_s+2, a_{s+1}, \cdots a_L}(C + \Pi_{a_s})$




Links like free Dirac particles, connected by unbreakability of domain wall.

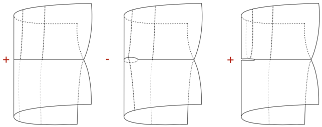
This description is shared by the RNS superstring.

$\psi^\mu \left(\dot{x}_\mu - x'_\mu\right) |\text{phys}\rangle = 0.$

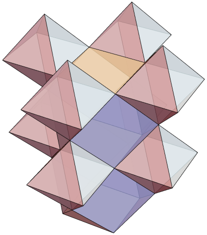
# Strong coupling problem.

Distler (1992) argued that the analog of self-intersection number term in the 3d case is the Euler character

$$Z_{3d}(\beta) = 2\sum_{\Sigma}(-1)^{\chi[\Sigma]}e^{-2\beta\text{Area}[\Sigma]}$$




Just as in the 2d case, we can avoid this issue by working on a lattice where each edge touches only 3 faces, such as this one: (corner-sharing octahedra)



But this highlights the fact that $|g_s| = 1$.

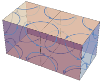
# Appeal to universality.

Q: can we modify the Ising model so that the dual string theory is weakly coupled?

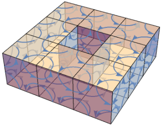
(*i.e.* decrease the weight of domain walls with higher genus in the sum)

$$Z_{3d}(\beta, g_s) = 2 \sum_{\Sigma} (g_s)^{\chi[\Sigma]} e^{-2\beta \mathrm{Area}[\Sigma]}$$

$\chi = 2$:



$\chi = 0$:



Possible outcomes, assuming there is still a continuous transition (there is):

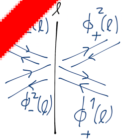
(1) Finite $g_s < 1$ leads to a new universality class, where spherical domain walls dominate.

(2) This changes $T_c$, but stays in the same 3d Ising universality class.

# The planar 3d Ising model

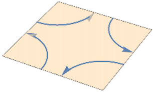
# How not to change $g_s$

First idea: On each link of dual lattice (= face of the primal lattice), place four $N \times N$-matrix-valued real variables $\phi^{1,2}_{\pm}$, associated with the four faces incident on the link:




$$\Delta S[\phi, z] = \sum_{\langle ij\rangle}(1+\sigma_i\sigma_j)\mathbb{1}\sum_{\alpha,\ell\in\langle ij\rangle} \phi^2_\alpha(\ell) + \sum_{\langle ij\rangle}(1-\sigma_i\sigma_j) g \mathrm{tr}\phi^4 + \sum_{\ell,\alpha,\beta} \phi_\alpha(\ell)\phi_\beta(\ell)$$

The $\mathrm{tr}\phi^4$ interaction connects the indices of the matrices on the links bounded by the plaquette like this:



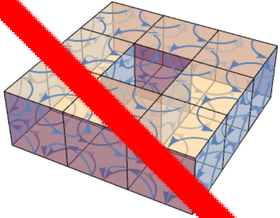
It costs a factor of $g \sim \frac{1}{N}$.

Configurations where the indices are not contracted contribute zero because of the angular integral over the $\phi$s.

The contribution of a spin configuration acquires a factor of

$$g^{\#\text{ of faces}} N^{\#\text{ of index loops}} = \lambda^{\#\text{ of faces}} N^{2-2g}$$

with $\lambda \equiv gN$.



But this model is difficult to simulate and has an extra $\mathsf{O}(N)$ symmetry.

# The planar 3d Ising model.

But there's a much easier way to change the relative weighting of the domain walls depending on their topology: just modify the Boltzmann weights:

$$Z = \sum_s g_s^{-\chi(s)} W_0(s)$$

where $W_0(s) = e^{-\beta \sum_{\langle ij \rangle} Z_i Z_j}$ is the usual Ising model Boltzmann weight, $\chi(s) \equiv F(s) - E(s) + V(s)$.

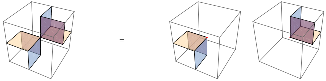
$F, E, V = \#$ of faces, edges and vertices of the dual lattice participating in a domain wall.

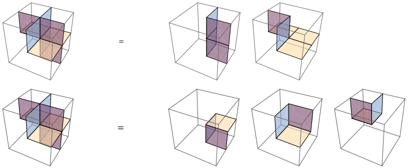
A local Hamiltonian!

This statement requires some refinement.

# Ambiguity & Resolution.

In how many DWs does a vertex participate?

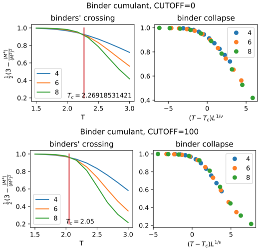



**One possibility**: add an energetic penalty to exclude the (9) ambiguous configurations.

Doing the analogous thing to the 2d Ising model ($\Delta E(\;) = \mathrm{CUTOFF}$) does not change the critical behavior (it merely moves $T_c$, but $\nu = 1$ still).

Bad for the MC acceptance rate.

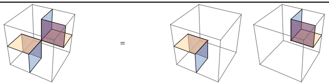



Alternative: decide on a decomposition into elementary constituents.

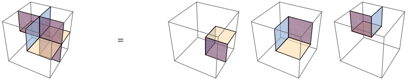
There are $2^8$ possible configs $\alpha$ of the 8 spins adjacent to a vertex of $\hat{\Gamma}$

$\longrightarrow$ Binary vectors, $p_\alpha \in \mathbb{Z}_2^{12}$.

Order them by # of faces = Hamming weight (0 to 12). Choose a basis of lowest weight.

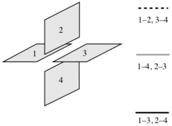


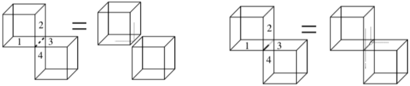
$$p_{77} = p_2 + p_9$$



$$p_{126} = p_6 + p_7 + p_8$$

# Ambiguity & Resolution.

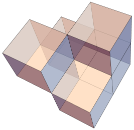


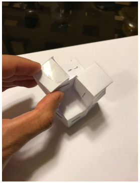



[images: Distler]

**But**: not all vertex resolutions are mutually compatible.

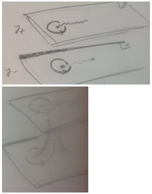
*e.g.* These two touching $S^2$s would be assigned $\chi = 5$:





(1) For each vertex of $\hat{\Gamma}$, record face connections implied by the vertex decomposition.

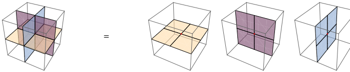
(2) For each edge, check for compatibility between these face connections. If not, that edge carries a $4\pi$ branch point, $\Delta\chi = -1$.



=

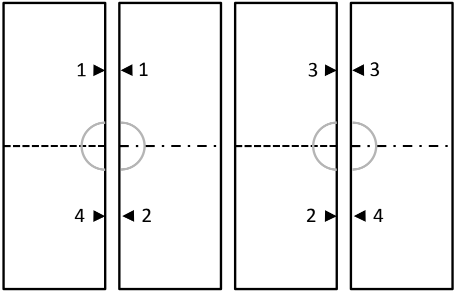
Note: This prescription allows unoriented configurations. (An *unoriented* immersed surface must have an odd number of triple points: $\chi = \#$ of triple points, mod 2. [Banchoff, 74])

a triple point:



# Arts & crafts.

Make your own $4\pi$ branch point!




Cut out along solid black lines.
Fold - - - - - - towards you and — · — · — away from you.
Tape together two parallel edges 1 – 1, 2 – 2, etc.

# Expectations.

$$H = 2\beta A + \phi\chi$$

As $\beta \to \pm\infty$, min/max $A$.

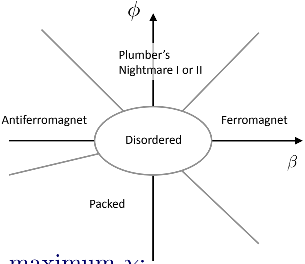
As $\phi \to \pm\infty$, min/max $\chi$.

Assuming $2 \times 2 \times 2$ unit cell, identify winning configurations, and order parameters that distinguish their patterns of symmetry-breaking.

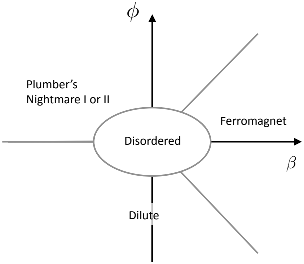
Mean field phase diagrams:

Similar analysis in continuum: [Huse-Leibler 88]

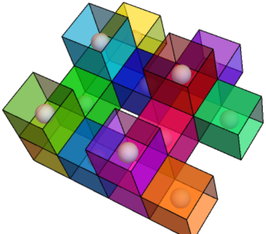
Branch point



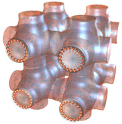

No-touching




'Packed phase' has the maximum $\chi$:



The 'Plumber's nightmare' has the minimum $\chi$:



[fig: Sethna]

# Numerical implementation: cluster updates.

**Critical slow-down**: Near a critical point, correlation lengths grow, and for local Monte Carlo dynamics, so do correlation times.
Remedy: non-local MC dynamics [Sweeny, Wolff, Swendsen-Wang 80s]: propose moves which update an order-1 fraction of spins at once.

Happily, because our modification of the Ising interactions depends on the domain wall configuration, we can adapt these methods to our model.

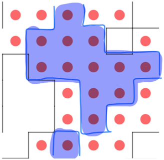
Detailed balance

$$\pi(a)\mathcal{A}\left(a \to b\right) P\left(a \to b\right) \overset{!}{=} \pi(b)\mathcal{A}\left(b \to a\right) P\left(b \to a\right)$$

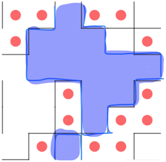
($\pi$ = Boltzmann wt, $\mathcal{A}$ = construction prob,
$P$ = acceptance prob) determines

$$P\left(a \to b\right) = \min\left(1, g_s^{\Delta\chi}\right).$$

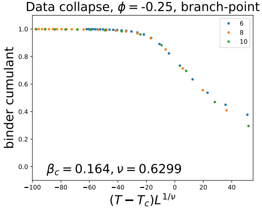
$a$:

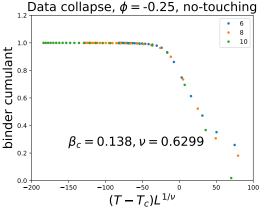


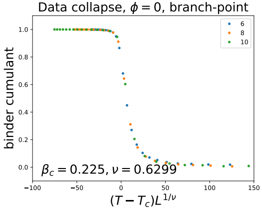
$b$:

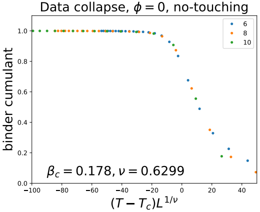


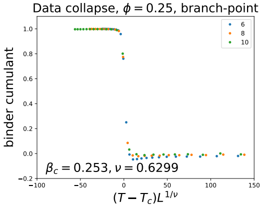
# Simulation results.

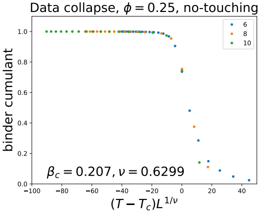


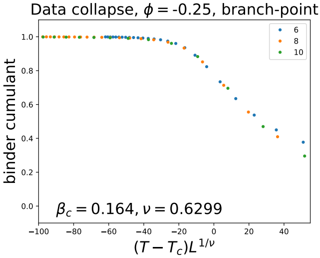



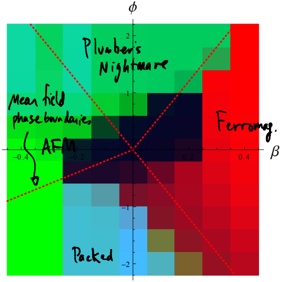



# Simulation results.




We can infer the correlation-length critical exponent $\nu$ from the collapse of the Binder cumulant. We find that the 3d Ising value $\nu = 0.6299$ gives the best data collapse for all values of $g_s$ (option (2) above).

Why: Our perturbation can be decomposed into a sum of symmetric, local scaling operators. The Ising fixed point has only one symmetric relevant local operator.

Red $\propto \sum_i s_i$ (ferro).

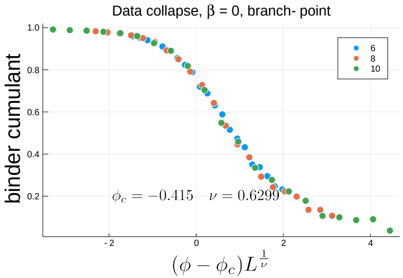
Green $\propto \sum_i (-1)^i s_i$ (AFM).

Blue $\propto \sum_i v_i s_i$ (an order parameter chosen to detect the symmetry-breaking pattern of the packed phase).

$T_c$ varies with $g_s$.

# Simulation results: how weak can we make the bare string coupling?

If we set $\beta = 0$, (with the branch-point method) the Ising transition occurs at $g_s = e^{\phi} = .66$. That is, this is where the Ising $T_c$ has moved off to $T = \infty$.




Comment on universality: the 3d Ising fixed point has a fixed point value of $g_s$, which we cannot change (and do not know yet).

We are merely trying to make the dual string theory weakly coupled *on the way* to the fixed point.

# Speculations about the worldsheet

# Comments on worldsheet theory.

**Important Q**: how does the Ising $\mathbb{Z}_2$ act in the string theory??

**Hint 1**: The string worldsheet is a branch cut for the spin.

**Hint 2**: $2+1$d Ising gauge theory has fermionic excitations – the boundstate of $e$ (end of string) and $m$ (vison) is a fermion.

RNS superstring spectra:

$$\begin{array}{c} \text{RR} \\ \text{NS-R R-NS} \\ \text{NS-NS} \end{array} \overset{\text{mod }\Omega}{\rightarrow} \begin{array}{c} \text{RR} \\ \text{NS-R} \\ \text{NS-NS} \end{array} \overset{?}{=} \begin{array}{c} e\text{-particle?} \\ \text{dyon} \\ \text{glueballs} \end{array}$$

has a spacetime fermion number symmetry.

Orbifolding by $(-1)^{F_s}$ $\overset{\text{mod }(-1)^{F_s}}{\rightarrow} \begin{array}{c} \text{RR}_L \oplus \text{RR}_R \\ -- \\ \text{NS-NS} \end{array} \overset{?}{=} \begin{array}{c} \text{spin} \oplus \text{neutral} \\ -- \\ \text{neutral} \end{array}$

This (unoriented) type 0 theory has two RR sectors, labelled by the chirality operator $\Gamma$.

Conjecture: $\Gamma$ is the Ising $\mathbb{Z}_2$.

# Comments on worldsheet theory.

It is tempting to interpret this as a holographic duality.

Then at the fixed point, the bulk spacetime should be something like $AdS_4$.

**Problem 1:** The bosonic non-linear sigma model with target space $AdS_4$ is not a CFT.

At least at large radius. [Friess-Gubser 05] found evidence for a small-radius fixed point.

Or maybe some of those RR fluxes can hold up the spacetime as in more familiar examples.

**Problem 2:** Adding one extra dimension $\varphi$ doesn't solve the problem of making a critical string theory.

A spacelike linear dilaton (in the radial direction, $\Phi = Q\varphi$)

could be used to cancel the Weyl anomaly.

But linear dilaton and target-space conformal symmetry (required near the critical point) are not compatible:

At the critical point, we expect $ds^2 = ds^2_{AdS} = d\varphi^2 + e^{-2\varphi} d\vec{x}^2$.

If under a spacetime scale transformation $\varphi \to \varphi + \lambda$,

$S_{\text{worldsheet}} \ni \int Q\varphi \frac{R}{2\pi} \to S_{\text{worldsheet}} + Q\lambda\chi$.

# Comments on worldsheet theory.

Possible resolutions:

- [Gursoy 2011]: warped AdS spacetimes can still have conformal invariance.

- [Hellerman-Maeda-Maltz-Swanson 14]: 'composite linear dilaton'. add

$S_{\text{worldsheet}} \ni \int Q\varphi \frac{R}{2\pi}$ where $\varphi = \frac{1}{\Delta} \ln \mathcal{O}_\Delta$ is a composite operator which shifts under a worldsheet scale transformation.

We could choose $\mathcal{O}_2 = e^{-2\varphi}\partial_\alpha X^\mu \partial^\alpha X_\mu + \partial_\alpha \varphi \partial^\alpha \varphi$, the $AdS_4$ kinetic term, which is invariant under *target-space* scale transformations

$X^\mu \to e^\lambda X^\mu, \varphi \to \varphi + \lambda$.

And $\varphi = \frac{1}{\Delta} \ln \mathcal{O}_\Delta = -\varphi + \log|\partial X| + \log|\partial\varphi|$.

What is $\log|\partial X|$?

# Effective string theory.

[Polchinski-Strominger 91, ... Hellerman-Swanson et al]

A less ambitious but more concrete connection with string theory governs the fluctuations of a large flat domain wall.

Worldsheet $X(\sigma,\tau)$ coordinate fields arise as Goldstones for breaking of translations by the wall.

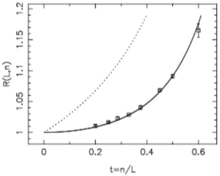
'Large and flat' means $X(\sigma,\tau) = \sigma +$ fluctuations, so $\partial X \neq 0$, and $\log(\partial X)^2$ makes sense.

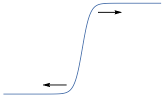
[Caselle-Fiore-Gliozzi-Hasenbusch-Provero 96]

Effective string theory prediction for

$$R(L,n) \equiv \frac{\langle W(L+n,L-n)\rangle}{\langle W(L,L)\rangle} e^{-n^2\sigma} = \sqrt{\frac{\eta(\mathbf{i})\sqrt{1-t}}{\eta\left(\mathbf{i}\,\frac{1+t}{1-t}\right)}},$$

$t \equiv n/L$ matches lattice simulation (at $\beta < \beta_c$ ):



[Kuti 05] find (at $\beta > \beta_c$) a gapped breathing mode on the worldsheet.

Closer to the critical point, we can expect this mode to become gapless: a goldstone for breaking of scale transformations by the profile of the wall. This should be the bulk radial coordinate, $\varphi$.

# Comments and puzzles about such a duality.

- Probably we shouldn't be too dogmatic about the idea that the right strings to think about are literally the domain walls between the regions of up and down spins.

  [Caselle-Gliozzi-Magnea 94] argue that the effective string (in the ordered phase) is actually a coarse-grained object like a "wall full of handles," with a genus-dependent tension.

  Another reasonable choice might be the boundaries of the (Fortuin-Kasteleyn) clusters involved in the cluster updates. These are designed to percolate *at* $T_c$ [Coniglio-Klein 80].

  We haven't studied their statistical topology yet.

- An unoriented string theory without space-filling D-branes?

  Actually these seem to exist [Kaidi, Parra Martinez, Tachikawa 19].

- Large-$N$ puzzle: String theory in flat space has Hagedorn growth of single-string states at high energy. In AdS/CFT, this is matched by the large-$N$ growth of the number of words $\mathrm{tr}\,(XYXXY\cdots)$. But our weak-coupling limit did not involve large-$N$!

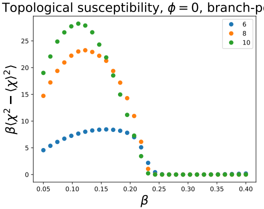
  Perhaps large curvature removes the Hagedorn spectrum.

# A mystery about the 3d Ising model

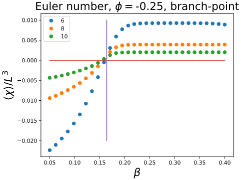
# Simulation results: a mystery

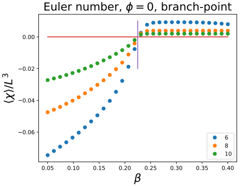
We measured the average euler character *per cluster*, $\langle \chi \rangle$.
$\langle \chi \rangle \overset{T \to 0}{\to} 2$,
$\langle \chi \rangle \overset{T \to \infty}{\to} -\infty$.
The number of clusters is a non-local (but computable [Hoshen-Kopelman 76, Sweeny 83]) observable.

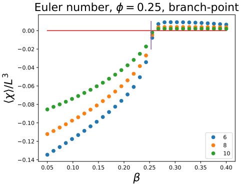
The fluctuations seem small for $T \gtrsim T_c$:



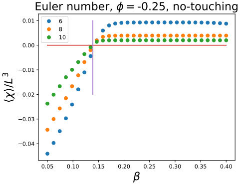

Branch point:

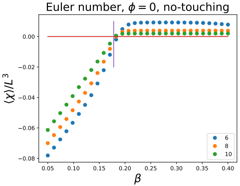


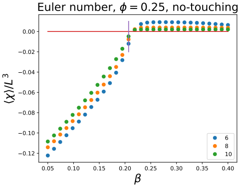




No-touching:



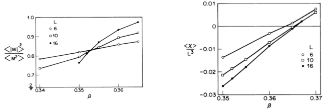



# Is the 3d Ising CFT full of donuts?

After we submitted our paper, we learned from David Huse that the smallness of $\chi$ (not $\chi$ per cluster) near $T_c$ was observed by [Karowski-Thun 92] in a model like our no-touching model.

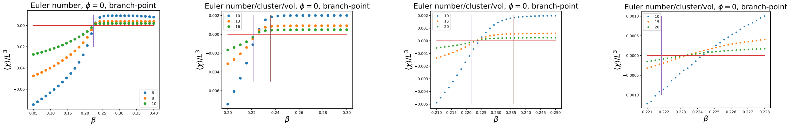
[Huse 93] looked more closely:




There isn't a local scaling variable in the 3d Ising critical theory with small dimension, so this would have been a contradiction.

However, $\langle \chi \rangle$ per cluster is not a local observable. And $N_{\text{clusters}} \sim L^3$.

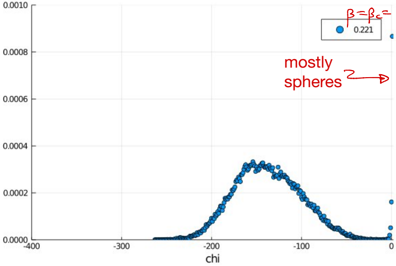
Zooming in near $T_c$, we find:



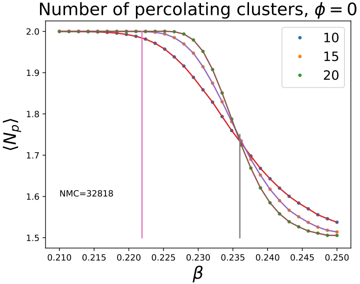
# Is the 3d Ising CFT full of donuts?

No!

Fraction of clusters vs $\chi$:




The other vertical line is $\beta_p$, where the clusters *percolate*:




**Qs:**. Why is the value of $\beta = \beta_\chi$ where $\langle \chi \rangle(\beta_\chi, L) = 0$ so close to $\beta_c$?

Why is the dependence on $L$ of $\chi$/cluster so weak?

Is this a universal phenomenon (associated with the Ising fixed point, or some other one)?

Against: It is not clear that $\chi$/cluster exhibits any singular behavior.

For: It happens both for the no-touching and for the branch point regularizations.

# Final comment.

Landau was more right than we thought.
This seems to be a fruitful principle.

# The end.

Thanks for listening.