Hadronic contributions to the muon $g-2$

Gregorio Herdoíza

Universidad Autónoma de Madrid

DFT, UAM & IFT, UAM-CSIC

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New Physics

Standard Model (SM) : successful description of electromagnetic, weak & strong interactions

However, many questions remain unanswered within the SM:

- large number of fundamental parameters :
  masses and mixing parameters in flavour sector
- matter-dominated Universe ; CP violation
- Dark Matter
- ...

Direct searches of New Physics (NP) : direct production/detection of new particles

- new states at high-p_T at the LHC $\rightsquigarrow$ explore larger $\Lambda_{\rm NP}$
- increased luminosity $\rightsquigarrow$ explore smaller coupling $c_{\rm NP}$
- direct detection of Dark Matter particles
- ...

Indirect searches of New Physics

- lower energies (wrt. "energy frontier")
- comparison between experiment and SM prediction
- precision/rare processes : allowed/suppressed in the SM
- flavour violating/diagonal processes
- experimental efforts :

Belle-II, LHCb, BES-III, NA62, neutrino expts., . . . unitarity triangle analysis

muon g-2, EDM. . . electroweak precision tests

SM prediction : requires good control of QED, weak and QCD contributions

increased precision $\rightsquigarrow$ face new challenges

$$(g-2)_\mu$$

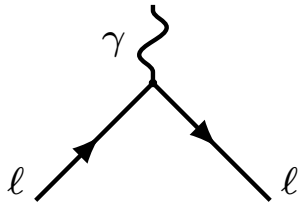
$$a_\mu = \frac{g_\mu - 2}{2}$$

$(g-2)_\ell$

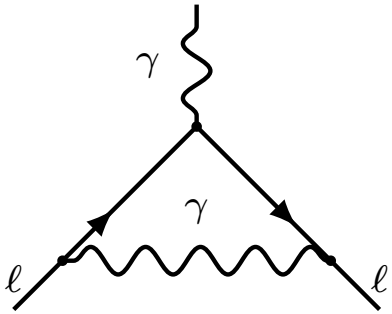
$\ell = e, \mu, \tau$

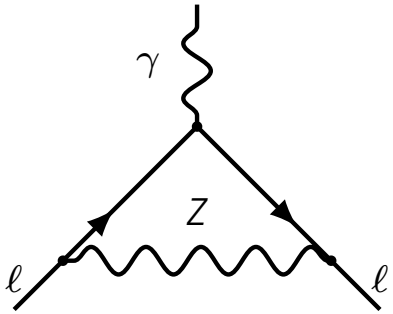
$$\vec{\mu} = g_\ell \frac{e\hbar}{2m_\ell c} \vec{s}$$

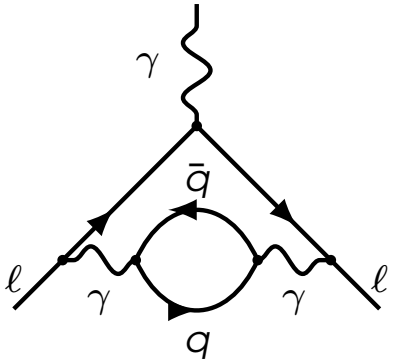
- gyromagnetic ratio: g_ℓ connects magnetic moment $\vec{\mu}$ to the spin $\vec{s}$
- anomalous magnetic moment of the muon: $a_\mu \equiv \frac{g_\mu - 2}{2}$
- elementary fermion: Dirac theory $g_\ell = 2 \rightsquigarrow a_\ell = 0$



- Schwinger [1948]: anomaly $a_\ell = \frac{\alpha}{2\pi} \approx 0.00116 \approx 11\,614\,097.3 \times 10^{-10}$
- SM contributions: QED, weak, hadronic







LO Hadronic Vacuum Polarization (HVP, LO):

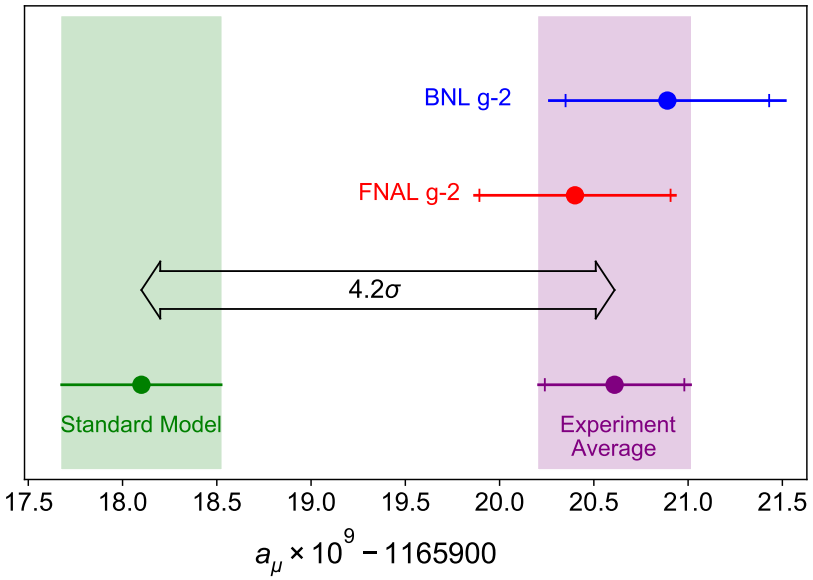
- dispersion relation + $\sigma(e^+e^- \to \text{hadrons})$
- lattice

- loop induced: New Physics $O(m_\ell^2/\Lambda_{\rm NP}^2)$
- $a_\mu^{\rm exp}$: precession of muon spin around the momentum in a homogeneous magnetic field

$(g-2)_\mu$ experiments: CERN, BNL, Fermilab

$(g-2)_\mu$

$$a_\mu \equiv \frac{g_\mu - 2}{2}$$



$$a_\mu^{\text{exp; FNAL'21}} = (11\,659\,204.0 \pm 5.4) \cdot 10^{-10} \qquad [0.46\,\text{ppm}] \leftarrow \{0.43_{\text{stat.}},\ 0.16_{\text{syst.}},\ 0.03_{\text{fund.prms}}\}\ \text{ppm}$$

$$a_\mu^{\text{exp; BNL'06+FNAL'21}} = (11\,659\,206.1 \pm 4.1) \cdot 10^{-10} \qquad [0.35\,\text{ppm}]$$

$$a_\mu^{\text{SM; WP'20}} = (11\,659\,181.0 \pm 4.3) \cdot 10^{-10} \qquad [0.37\,\text{ppm}] \leftarrow \{0.34_{\text{LO hadronic}}, \ldots\}\ \text{ppm}$$

- difference experiment − theory [WP '20]:

$$\Delta a_\mu^{\text{exp}-\text{WP'20}} = (25.1 \pm 5.9) \cdot 10^{-10} \qquad [4.2\sigma]$$

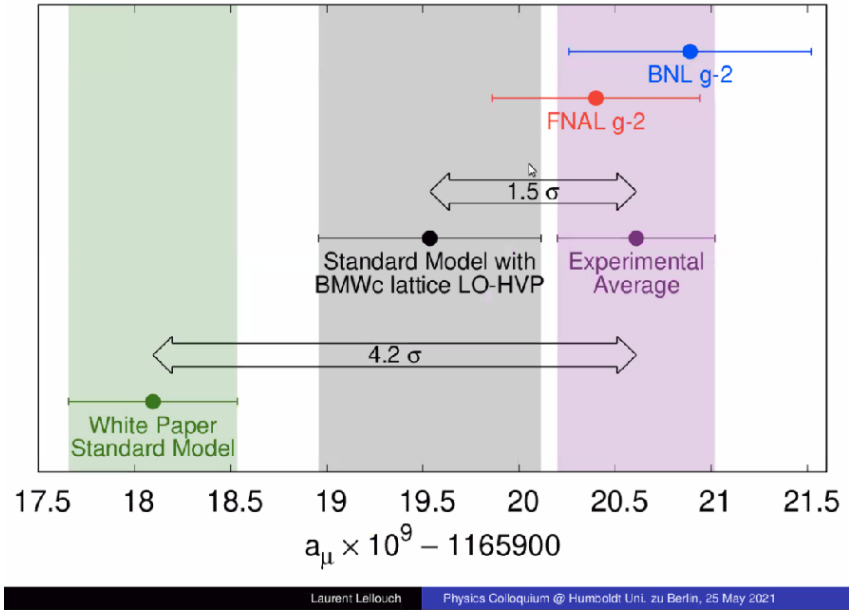
[FNAL '21]: Muon g-2 collab., 2104.03281 [hep-ex], Phys.Rev.Lett. 126 (2021) 14, 141801, Apr. 7, 2021.

[BNL '06]: Muon g-2 E821 collab., hep-ex/0602035, Phys.Rev.D 73 (2006) 072003, Apr. 7, 2006.

[WP '20]: Muon g-2 theory initiative White Paper, 2006.04822 [hep-ph], Phys.Rept. 887 (2020) 1-166, Dec. 3, 2020.

$(g-2)_\mu$

$$a_\mu \equiv \frac{g_\mu - 2}{2}$$



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$$a_\mu^{\text{SM; BMW'21}} = (11\,659\,195.4 \pm 5.7) \cdot 10^{-10} \qquad [0.49\,\text{ppm}] \leftarrow \{0.47_{\text{LO hadronic}},\ \ldots\}\ \text{ppm}$$

- difference experiment – theory [WP '20]:

$$\Delta a_\mu^{\text{exp}-\text{WP'20}} = (25.1 \pm 5.9) \cdot 10^{-10} \qquad [4.2\sigma]$$

- difference experiment – theory [BMW '21]:

$$\Delta a_\mu^{\text{exp}-\text{BMW'21}} = (10.7 \pm 7.0) \cdot 10^{-10} \qquad [1.5\sigma]$$

[FNAL '21]: Muon *g*-2 collab., 2104.03281 [hep-ex], Phys.Rev.Lett. 126 (2021) 14, 141801, Apr. 7, 2021.

[BNL '06]: Muon *g*-2 E821 collab., hep-ex/0602035, Phys.Rev.D 73 (2006) 072003, Apr. 7, 2006.

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[BMW '21]: Budapest Marseille Wuppertal Lattice QCD collab., 2002.12347 (hep-lat), Nature 593 (2021) 7857, 51-55, Apr. 7, 2021.

$(g-2)_\mu$: Standard Model [WP '20]

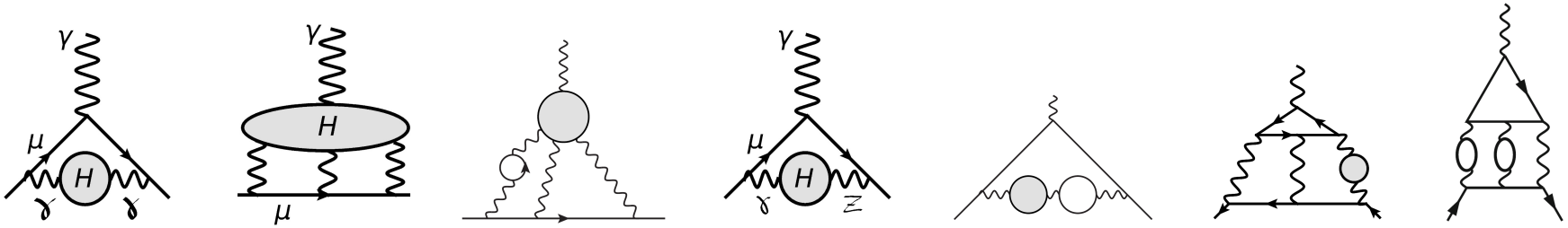
$$\Delta a_\mu^{\text{exp}-\text{WP'20}} = (25.1 \pm 5.9) \cdot 10^{-10} \quad [4.2\sigma]$$

$$a_\mu^{\text{exp; BNL'06+FNAL'21}} = (11\,659\,206.1 \pm 4.1) \cdot 10^{-10} \quad [0.35\,\text{ppm}]$$

$$a_\mu^{\text{SM; WP'20}} = (11\,659\,181.0 \pm 4.3) \cdot 10^{-10} \quad [0.37\,\text{ppm}]$$

- uncertainties:

1. Hadronic VP LO $O(\alpha^2)$: $4.0 \cdot 10^{-10}$
2. Hadronic light-by-light $O(\alpha^3)$: $1.7 \cdot 10^{-10}$
3. Hadronic light-by-light NLO $O(\alpha^4)$: $0.1 \cdot 10^{-10}$
4. Weak hadronic loops & $O(\alpha^3)$ nonleading logs.: $0.10 \cdot 10^{-10}$
5. Hadronic VP NLO $O(\alpha^3)$: $0.07 \cdot 10^{-10}$
6. Hadronic VP NNLO $O(\alpha^4)$: $0.01 \cdot 10^{-10}$
7. QED up to $O(\alpha^5)$: $0.0104 \cdot 10^{-10}$



$(g-2)_\mu$: Standard Model [WP '20]

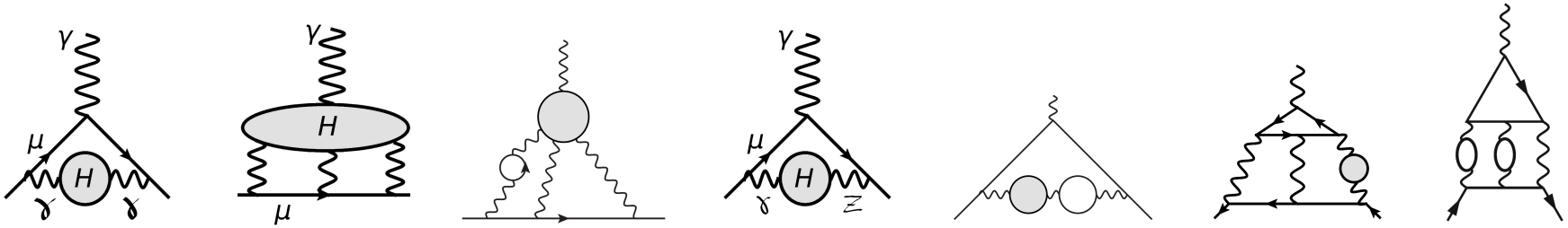
$$\Delta a_\mu^{\text{exp}-\text{WP'20}} = (25.1 \pm 5.9) \cdot 10^{-10} \quad [4.2\sigma]$$

$$a_\mu^{\text{exp; BNL'06+FNAL'21}} = (11\,659\,206.1 \pm 4.1) \cdot 10^{-10} \quad [0.35\,\text{ppm}]$$

$$a_\mu^{\text{SM; WP'20}} = (11\,659\,181.0 \pm 4.3) \cdot 10^{-10} \quad [0.37\,\text{ppm}]$$

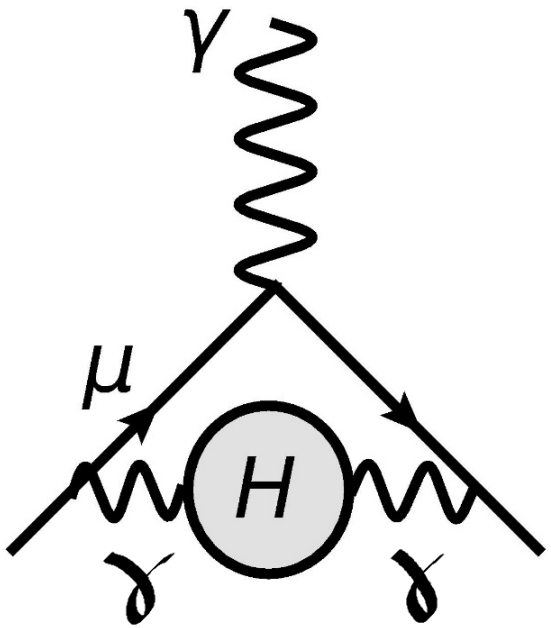
- uncertainties:

1.	Hadronic VP LO $O(\alpha^2)$:	$4.0 \cdot 10^{-10}$	693.1	$[10^{-10}]$
2.	Hadronic light-by-light $O(\alpha^3)$:	$1.7 \cdot 10^{-10}$	9.0	
3.	Hadronic light-by-light NLO $O(\alpha^4)$:	$0.1 \cdot 10^{-10}$	0.2	
4.	Weak hadronic loops & $O(\alpha^3)$ nonleading logs.:	$0.10 \cdot 10^{-10}$	15.36	
5.	Hadronic VP NLO $O(\alpha^3)$:	$0.07 \cdot 10^{-10}$	−9.83	
6.	Hadronic VP NNLO $O(\alpha^4)$:	$0.01 \cdot 10^{-10}$	1.24	
7.	QED up to $O(\alpha^5)$:	$0.0104 \cdot 10^{-10}$	11 658 471.8931	

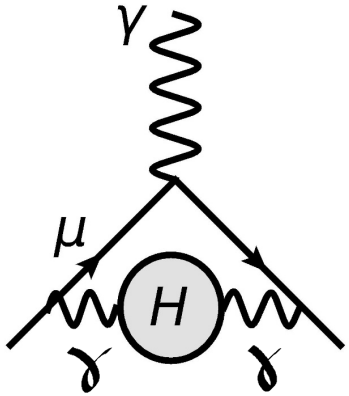


Leading Order Hadronic Vacuum Polarization

$$a_\mu^{\mathrm{HVP,LO}}$$



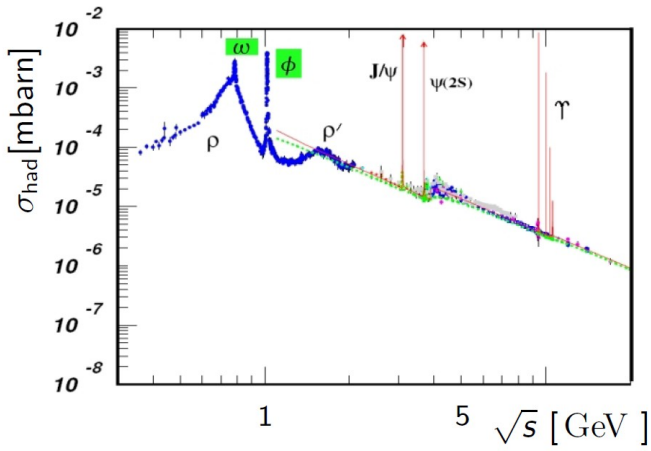
$a_\mu^{\mathrm{HVP,LO}}$



- data-driven approach : combine analyticity and unitarity to experimental data

  dispersion relation + optical theorem + $\sigma_{\mathrm{had}}^{(0)}\,(e^+e^- \to \gamma^* \to \text{hadrons}(+\gamma))$

$$a_\mu^{\mathrm{HVP,LO}} = \frac{m_\mu^2}{12\pi^3} \int_{M_{\pi^0}^2}^{\infty} ds\, \frac{\sigma_{\mathrm{had}}^{(0)}(s)\,\hat{K}(s)}{s}$$



$\hat{K}(s)$ known function

low-energy regions contribute significantly

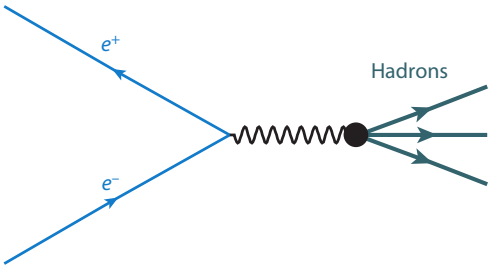
combination of more than 100 data sets + perturbative QCD

theoretical prediction that relies on experimental data

- lattice QCD+QED : regularization of QCD + QED on a Euclidean lattice

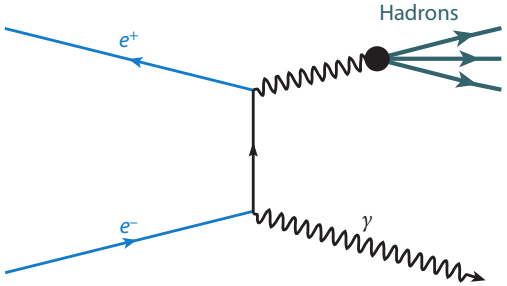
$a_\mu^{\mathrm{HVP,LO}}$: data-driven approach

Direct scan with tunable e^+e^- beams



Radiative return with Initial State Radiation (ISR)

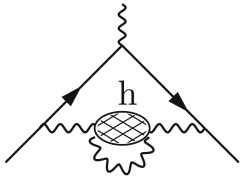
scan at fixed CM energy



$\sigma_{\mathrm{had}}^{(0)}\left(e^+e^- \to \gamma^* \to \text{hadrons}(+\gamma)\right)$:

- inclusive wrt. Final State Radiation (FSR) of photons

  For consistency, not included in higher order HVP



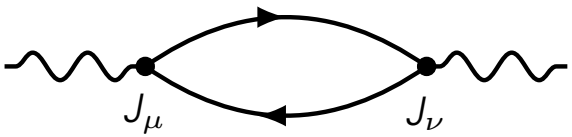
- bare cross-section: exclude VP effects ("undressing", if effects included in data)

  physical ("full") photon propagator contains any number of insertions of the VP function

$\rightsquigarrow$ running of the QED coupling

running of the QED coupling

- vacuum polarization function (VPF)



$$\Pi_{\mu\nu}(Q) = \int \mathrm{d}^4x\, e^{iQx}\, \langle J_\mu(x)\, J_\nu(0)\rangle \;=\; (Q_\mu Q_\nu - Q^2\, \delta_{\mu\nu})\, \Pi(Q^2)$$

$$J_\mu(x) \;=\; \sum_{f=1}^{N_\mathrm{f}} Q_f\, \overline{\psi}_f(x)\gamma_\mu \psi_f(x)$$

quarks: $Q_f \in \{-1/3, 2/3\}$

- $\Pi(Q^2)$: photon vacuum polarisation function (VPF)

$$\Delta\alpha_{\mathrm{QED}}(q^2) \;=\; 4\pi\alpha\, \Big(\Pi(Q^2) - \Pi(0)\Big) \;=\; 4\pi\alpha\, \hat{\Pi}(Q^2)$$

$$Q^2 = -q^2$$

- vacuum polarisation: charge screening $\rightsquigarrow$ running of QED coupling

$$\alpha(q^2) = \frac{\alpha}{1 - \Delta\alpha_{\mathrm{QED}}(q^2)}$$

$$\Delta\alpha_{\mathrm{QED}} = \Delta\alpha_{\mathrm{lep}} + \Delta\alpha^{(5)}_{\mathrm{had}} + \Delta\alpha_{\mathrm{top}} + \ldots$$

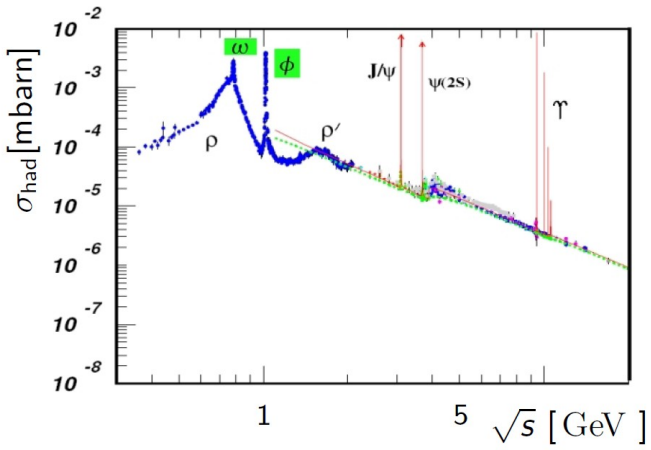
$\Delta\alpha^{(5)}_{\rm had}$

- data-driven approach : combine analyticity and unitarity to experimental data

  dispersion relation + optical theorem + $\sigma^{(0)}_{\rm had}\,(e^+e^- \to \gamma^* \to \text{hadrons}(+\gamma))$

$$\Delta\alpha^{(5)}_{\rm had}(q^2) = -\frac{q^2}{4\pi^2\alpha}\,\mathrm{P}\int_{M^2_{\pi^0}}^{\infty} ds\,\frac{\sigma^{(0)}_{\rm had}(s)}{(s-q^2)}$$

$$\sigma_{\rm had}(s) = \frac{\sigma^{(0)}_{\rm had}(s)}{|1-\hat{\Pi}|^2} \quad \rightsquigarrow \quad \text{iterative procedure}$$

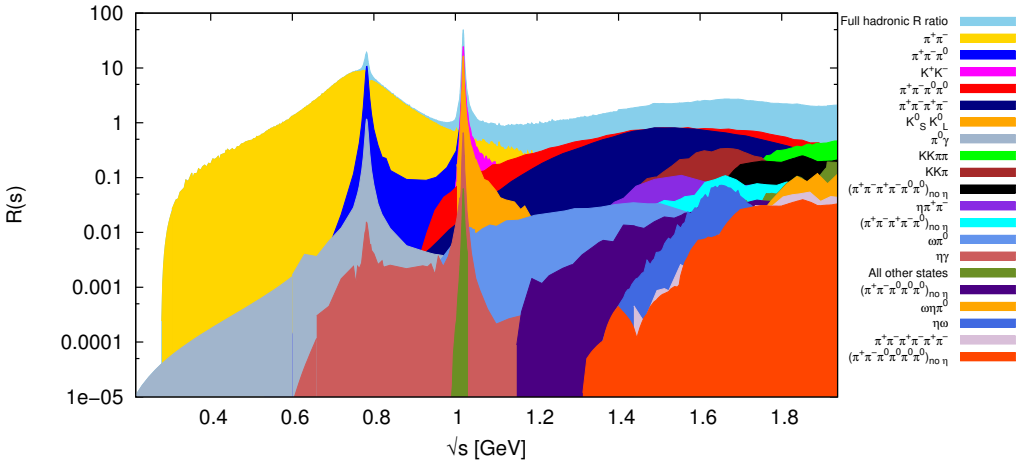


Undressing and FSR introduce uncertainties in $a_\mu^{\rm HVP,LO}$: 0.2 and 0.7 vs. total : 4.0 [10^{-10}]

- lattice $\rightsquigarrow$ $\Delta\alpha^{(5)}_{\rm had}$

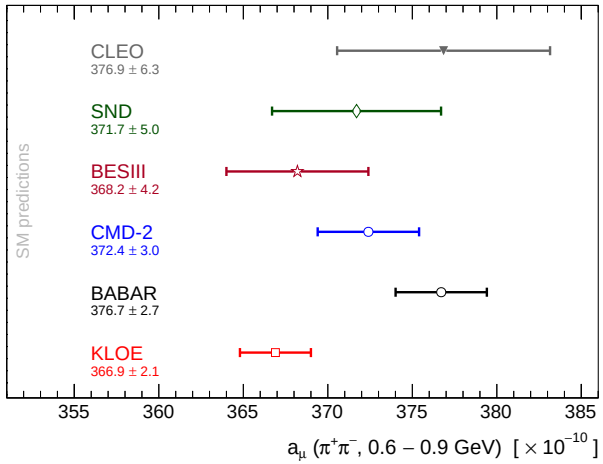
$a_\mu^{\mathrm{HVP,LO}}$: data-driven approach

$$R(s) = \frac{\sigma^{(0)}_{\mathrm{had}}(s)}{4\pi\alpha^2/3s}$$



[KNT19, Phys. Rev. D 101(2020)1, 014029]

$\pi^+\pi^-$ channel



[DHMZ19, Eur. Phys. J. C 80(2020)3, 241]

- pion form factor $F_\pi^V(s)$:

$$\langle \pi^\pm(p')|J^\mu(0)|\pi^\pm(p)\rangle = \pm(p'+p)^\mu\, F_\pi^V((p'-p)^2)$$

$$\sigma_{\mathrm{had}}(e^+e^- \to \pi^+\pi^-) = \frac{\pi\alpha^2}{3s}\left(1-\frac{4M_\pi^2}{s}\right)^{3/2} |F_\pi^V(s)|^2$$

- global fit with contraints from analyticity, unitarity and crossing symmetry

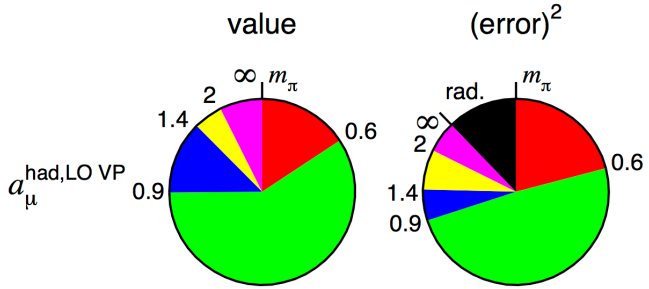
[Colangelo, Hoferichter, Stoffer, 2020]

lattice see e.g. [Mainz, 1910.01083]

$a_\mu^{\mathrm{HVP,LO}}$ & $\Delta\alpha_{\mathrm{had}}^{(5)}$: data-driven approach

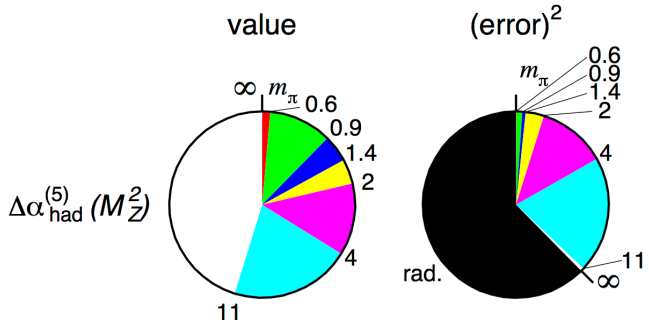
[KNT19, Keshavarzi, Nomura & Teubner, Phys. Rev. D 101(2020)1, 014029]

$$a_\mu^{\mathrm{HVP,LO}} = \frac{m_\mu^2}{12\pi^3} \int_{M_{\pi^0}^2}^{\infty} ds \, \frac{\sigma_{\mathrm{had}}^{(0)}(s)\,\hat{K}(s)}{s}$$



$$\Delta\alpha_{\mathrm{had}}^{(5)}(q^2) = -\frac{q^2}{4\pi^2\alpha}\,\mathrm{P}\int_{M_{\pi^0}^2}^{\infty} ds \, \frac{\sigma_{\mathrm{had}}^{(0)}(s)}{(s-q^2)}$$

- exclusive channels: $\sqrt{s} \lesssim 2\,\mathrm{GeV}$
- inclusive channels: $2 \lesssim \sqrt{s} \lesssim 11\,\mathrm{GeV}$
- use of perturbative QCD: $\sqrt{s} \gtrsim 11\,\mathrm{GeV}$

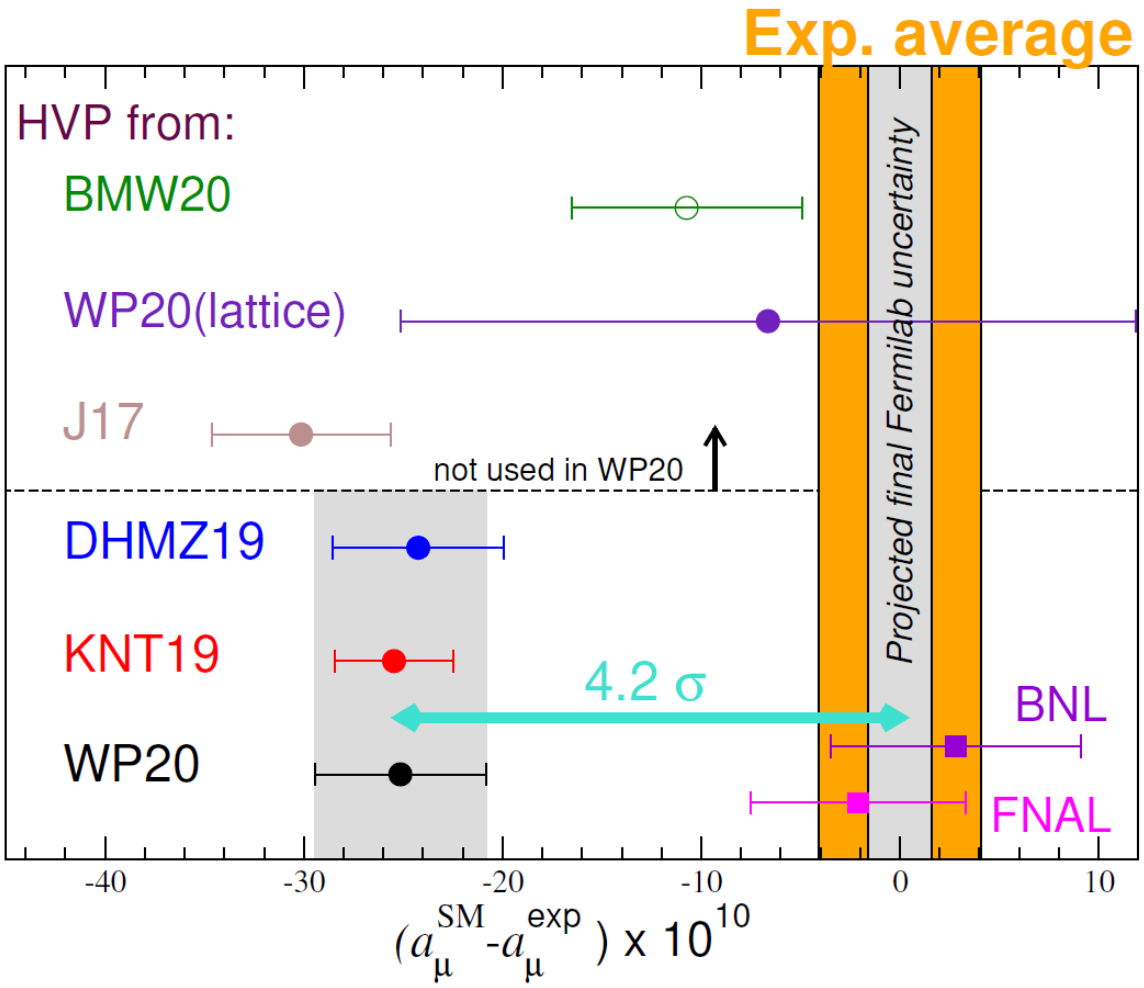


[KNT18, PRD97 114025]

$$a_\mu^{\mathrm{HVP,LO}} = (692.78 \pm 1.21_{\mathrm{stat}} \pm 1.97_{\mathrm{sys}} \pm 0.21_{\mathrm{vp}} \pm 0.70_{\mathrm{fsr}}\,;\ [2.42]_{\mathrm{tot}}) \cdot 10^{-10} \quad [0.35\%]$$

$$\Delta\alpha_{\mathrm{had}}^{(5)}(M_Z^2) = (276.09 \pm 0.26_{\mathrm{stat}} \pm 0.68_{\mathrm{sys}} \pm 0.14_{\mathrm{vp}} \pm 0.83_{\mathrm{fsr}}\,;\ [1.12]_{\mathrm{tot}}) \cdot 10^{-4} \quad [0.41\%]$$

$a_\mu^{\mathrm{HVP,LO}}$: data-driven approach [WP '20]



[M. Hoferichter, $(g-2)_\mu$ workshop, 2021]

$$a_\mu^{\mathrm{HVP,LO}} = 693.1(2.8)_{\mathrm{exp}}(2.8)_{\mathrm{sys}}(0.7)_{\mathrm{DV+QCD}}\,[4.0_{\mathrm{tot}}] \cdot 10^{-10} \quad [0.58\%]$$

lattice QCD+QED

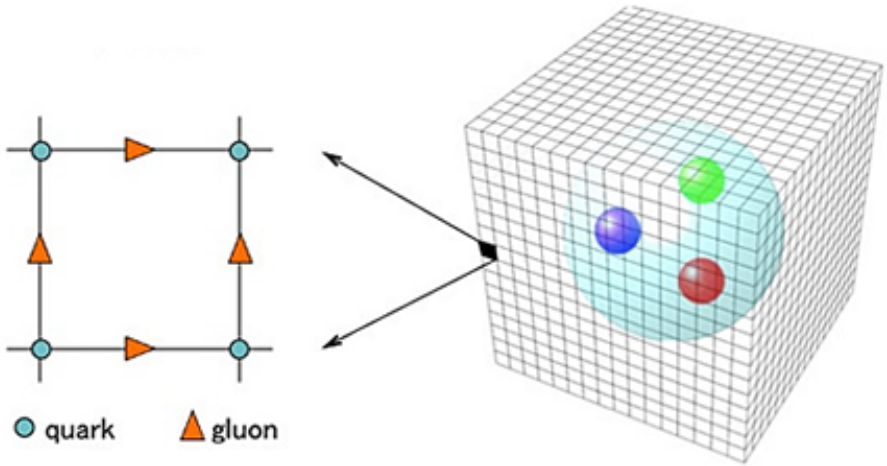
$$\mathcal{L}_{\rm QCD+QED} = \frac{1}{4e^2} F_{\mu\nu}F_{\mu\nu} + \frac{1}{2g^2}\,{\rm Tr}G_{\mu\nu}G_{\mu\nu} + \sum_f \bar\psi_f[\gamma_\mu(\partial_\mu + iq_f A_\mu + iB_\mu) + m_f]\psi_f$$

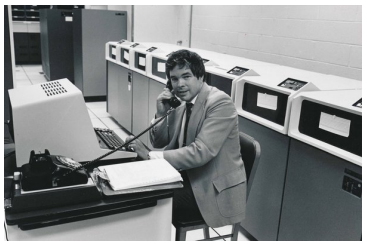
photon field strength tensor : $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

gluon : $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$

regularization of QCD+QED on a Euclidean lattice

lattice spacing a: $\Lambda_{\rm UV} = a^{-1}$





gluonic field: $U_\mu \in {\rm SU}(N_c = 3)$

$$U_\mu(x) = e^{i\,ag\,B_\mu(x)}$$

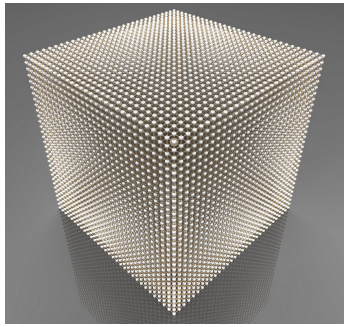
[Kenneth Wilson, 1974]

$$S^{\rm lat}_{\rm gluon} = \frac{6}{g^2}\sum_x \sum_{\mu<\nu} {\rm Re\,Tr}(1 - W^{1\times 1}_{\mu\nu}) \longrightarrow \frac{1}{2g^2}\int d^4x\,{\rm Tr}G_{\mu\nu}G_{\mu\nu}$$

observable $\mathcal{O}$:

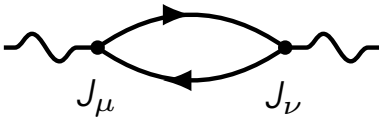
$$\langle \mathcal{O}[U, A, \{\psi_f, \bar\psi_f\}]\rangle = \frac{1}{Z}\int [dU]\,[dA]\,\prod_{\rm f}[d\psi_{\rm f}]\,[d\bar\psi_{\rm f}]\,e^{-S^{\rm lat}_{\rm QCD+QED}}\,\mathcal{O}[U, A, \{\psi_f, \bar\psi_f\}]$$

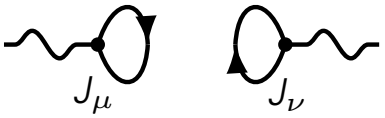
lattice QCD+QED



first principles formulation of QCD+QED : systematically improvable

$$\langle \mathcal{O}[U, A, \{\psi_f, \bar{\psi}_f\}] \rangle \;=\; \frac{1}{Z} \int [dU]\,[dA] \prod_{\mathrm{f}} [d\psi_{\mathrm{f}}]\,[d\bar{\psi}_{\mathrm{f}}]\; e^{-S^{\mathrm{lat}}_{\mathrm{QCD+QED}}}\; \mathcal{O}[U, A, \{\psi_f, \bar{\psi}_f\}]$$





$$\Pi_{\mu\nu}(Q) = \int \mathrm{d}^4x\; e^{iQx}\; \langle J_\mu(x)\, J_\nu(0) \rangle\,, \qquad J_\mu(x) \;=\; \sum_{f=1}^{N_{\mathrm{f}}} Q_f\, \overline{\psi}_f(x) \gamma_\mu \psi_f(x)$$

- statistical uncertainties : Monte Carlo data , signal/noise
- systematic uncertainties :
  - continuum-limit extrapolation ($a \to 0$) : discretization effects
  - finite-volume effects ($L, T \to \infty$) : long-distance $\pi\pi$ effects
  - physical masses for u, d, s, c, b quarks
  - electromagnetic corrections
  - lattice scale setting : a in fm accurately

lattice QCD+QED : $a_\mu^{\mathrm{HVP,\,LO}}$

$$\Pi_{\mu\nu}(Q) = \int \mathrm{d}^4x\, e^{iQx}\, \langle J_\mu(x)\, J_\nu(0)\rangle \;=\; (Q_\mu Q_\nu - Q^2\,\delta_{\mu\nu})\,\Pi(Q^2)$$

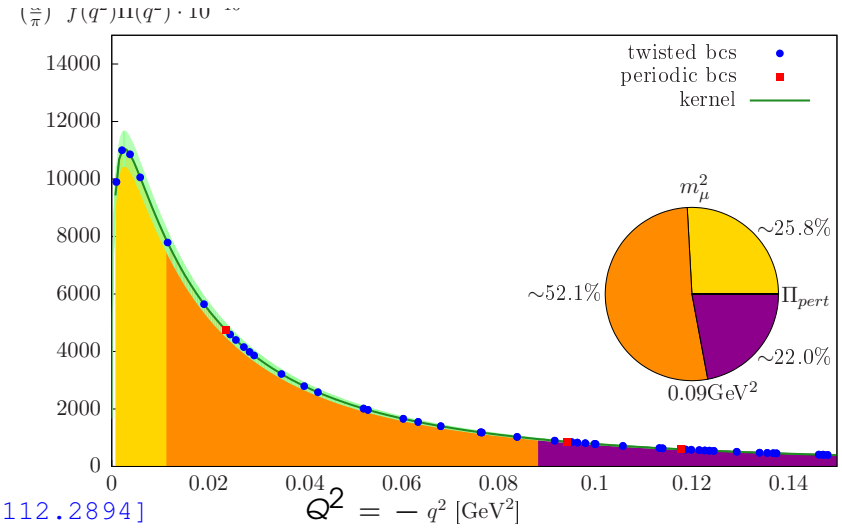
$$J_\mu(x) \;=\; \sum_{f=1}^{N_\mathrm{f}} Q_f\, \overline{\psi}_f(x)\gamma_\mu\psi_f(x)$$

spacelike momenta $Q^2 = -q^2 \geq 0$:

$$a_\mu^{\mathrm{HVP,\,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2\, f(Q^2) \times 4\pi^2 \left[\Pi(Q^2) - \Pi(0)\right]$$

[T. Blum, hep-lat/0212018]

illustration : integrand [light (u, d) quarks, no errors included]



[Mainz, 1112.2894]

$$\Pi(Q^2) = \frac{\sum_\mu \Pi_{\mu\mu}(Q)}{3Q^2}$$

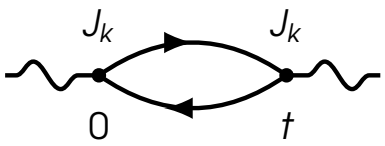
$$m_\mu^2 \approx 0.01\,\mathrm{GeV}^2$$

$$Q_\nu = \frac{2\pi n_\nu}{L}$$

$$\rightsquigarrow L \approx 12\,\mathrm{fm}$$

split integral in windows

$a_\mu^{\mathrm{HVP,\,LO}}$: time-momentum representation



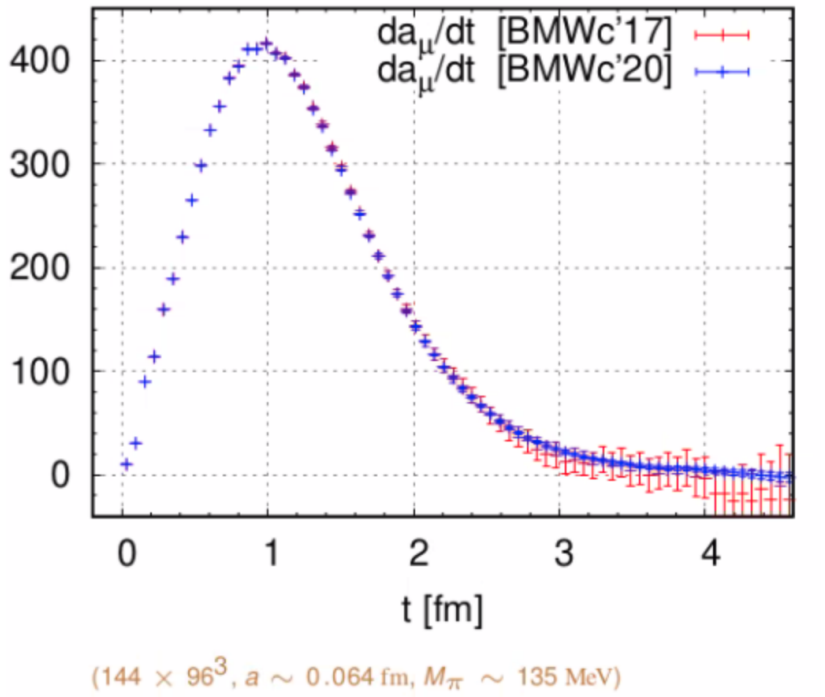
$$C(t) \;=\; -\frac{1}{3}\sum_{k=1}^{3}\int d^3x\,\langle\, J_k(x)\,J_k(0)\,\rangle$$

$$\Pi(Q^2)-\Pi(0) \;=\; \int_0^\infty dt\; C(t)\,\left[t^2-\frac{4}{Q^2}\sin^2\left(\frac{1}{2}Q\,t\right)\right]$$

$\rightsquigarrow$ continuous function of Q^2, including $Q^2=0$

$$a_\mu^{\mathrm{HVP,\,LO}} = \left(\frac{\alpha}{\pi}\right)^2\int_0^\infty dt\; C(t)\,\widetilde{f}(t)$$

integrand light (u,d) quarks $[\times 10^{10}]$



- dominated at long distances by the resonance nature of the ρ meson

$\rightsquigarrow$ split integral in windows $\rightsquigarrow$ bounds

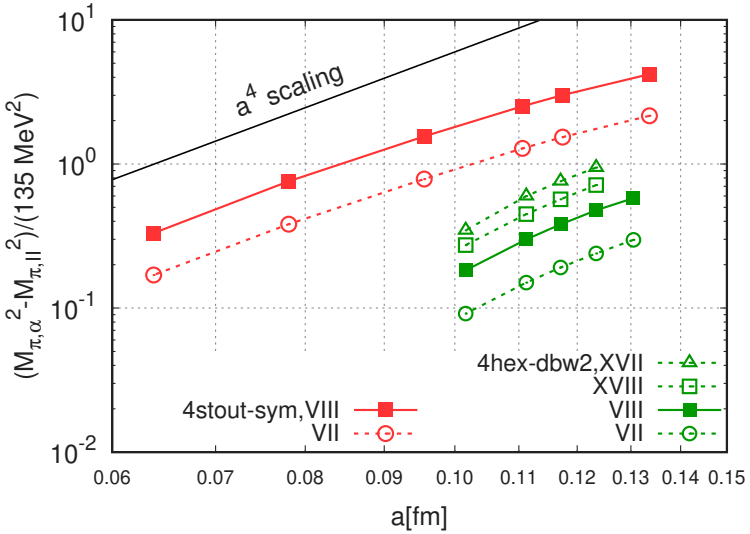
a_μ^{light} : continuum-limit extrapolation [BMW '21]

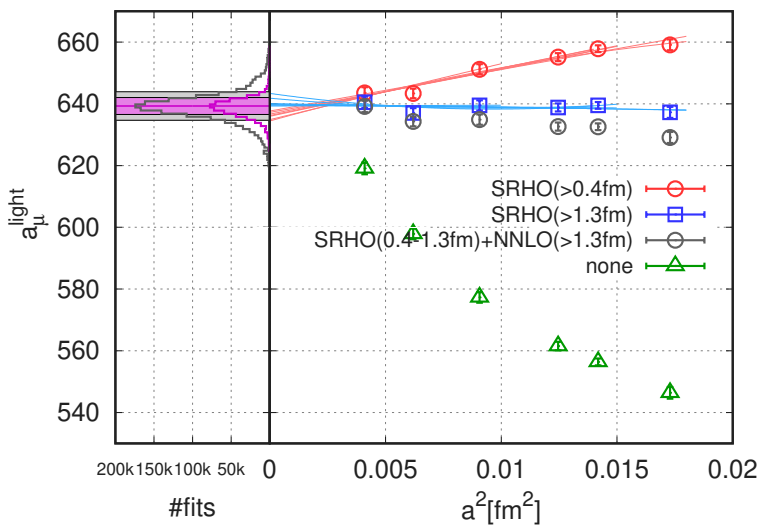
$$\mathcal{L}_{\text{QCD+QED}} = \frac{1}{4e^2} F_{\mu\nu}F_{\mu\nu} + \frac{1}{2g^2}\,\text{Tr}G_{\mu\nu}G_{\mu\nu} + \sum_f \bar\psi_f[\gamma_\mu(\partial_\mu + iq_f A_\mu + iB_\mu) + m_f]\psi_f$$

$$\langle \mathcal{O}[U, A, \{\psi_f, \bar\psi_f\}]\rangle = \frac{1}{Z}\int [dU]\,[dA]\,\prod_{\text{f}}[d\psi_{\text{f}}]\,[d\bar\psi_{\text{f}}]\; e^{-S^{\text{lat}}_{\text{QCD+QED}}}\;\mathcal{O}[U, A, \{\psi_f, \bar\psi_f\}]$$

discretized Dirac operator : (smeared) staggered quarks $\rightsquigarrow$ mix spinor & spacetime indices

$\rightsquigarrow$ additional quark species ("tastes")





$0.064 \le a \le 0.132\,\text{fm}$

model taste-breaking discretization effects : also relevant for finite volume effects

- rho-pion-gamma (RHO) model : 5 parameters $\leftarrow M_\rho^{\text{phys}}, \Gamma_\rho^{\text{phys}}, F_\rho^{\text{phys}}, e^{(\text{phys})}, M_\pi$

  $\rightsquigarrow$ pion-vacuum polarization $\Sigma(q^2, M_\pi^2) \rightsquigarrow \hat\Pi^{\text{RHO}}(q^2) \rightsquigarrow a_\mu^{\text{RHO}}$

- staggered RHO (SRHO) model : $\Sigma(q^2, M_\pi^2) \rightsquigarrow \frac{1}{16}\sum_\alpha \Sigma(q^2, M_{\pi,\alpha}^2, L, T)$ (taste averaged)

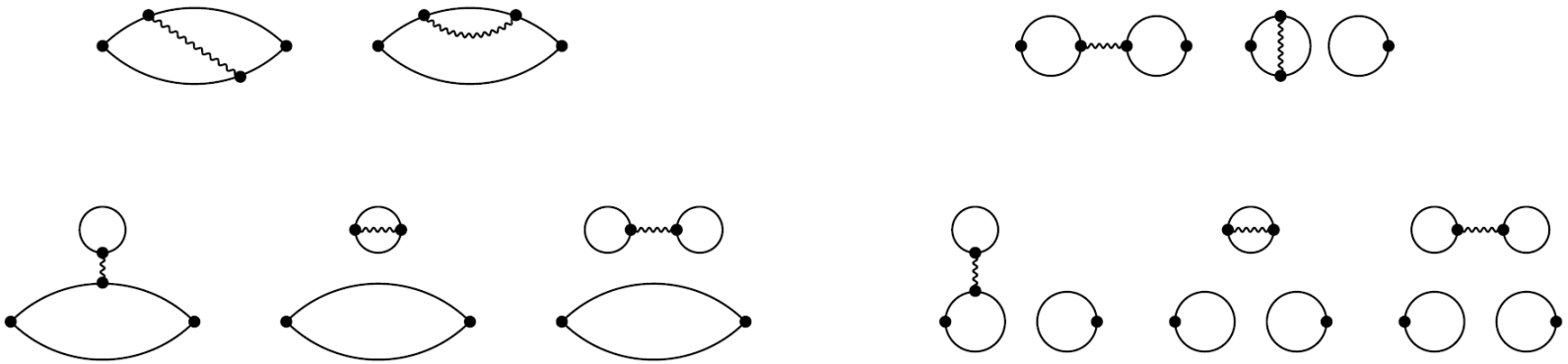
lattice QCD+QED : isospin breaking

$$\mathcal{L}_{\mathrm{QCD+QED}} = \frac{1}{4e^2} F_{\mu\nu}F_{\mu\nu} + \frac{1}{2g^2}\,\mathrm{Tr}G_{\mu\nu}G_{\mu\nu} + \sum_f \bar{\psi}_f[\gamma_\mu(\partial_\mu + iq_f A_\mu + iB_\mu) + m_f]\psi_f$$

$$= \mathcal{L}^{\mathrm{iso}}_{\mathrm{QCD}} + \frac{1}{2}\delta m\,(\bar{d}d - \bar{u}u) + ie\,A_\mu J_\mu\,, \qquad \delta m = m_d - m_u\,, \qquad \alpha = \frac{e^2}{4\pi}$$

$$\langle \mathcal{O}[U, A, \{\psi_f, \bar{\psi}_f\}]\rangle = \frac{1}{Z}\int [dU]\,[dA]\,\prod_{\mathrm{f}}[d\psi_{\mathrm{f}}]\,[d\bar{\psi}_{\mathrm{f}}]\,e^{-S^{\mathrm{lat}}_{\mathrm{QCD+QED}}}\,\mathcal{O}[U, A, \{\psi_f, \bar{\psi}_f\}]$$

electromagnetic corrections at $O(\alpha)$: insertions of the vector current J_μ



strong isospin corrections at $O(\delta m)$: insertions of the scalar operator $\bar{d}d - \bar{u}u$

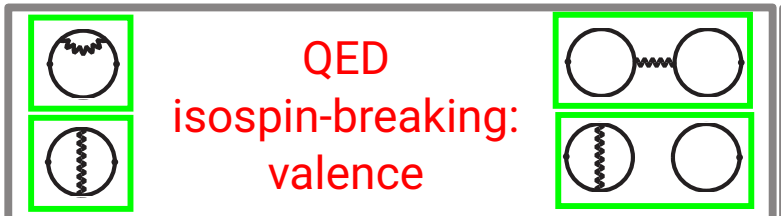
$a_\mu^{\mathrm{HVP,\,LO}}$

[BMW '21]

Isospin symmetric

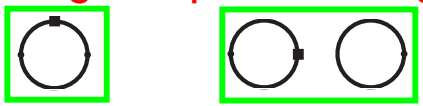
connected light	connected strange	connected charm	disconnected
633.7(2.1)(4.2)	53.393(89)(68)	14.6(0)(1)	-13.36(1.18)(1.36)

QED isospin-breaking: valence



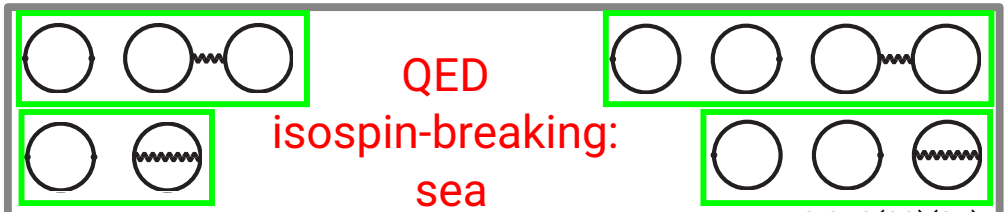
connected -1.23(40)(31) disconnected -0.55(15)(10)

Strong isospin-breaking



connected	disconnected
6.60(63)(53)	-4.67(54)(69)

QED isospin-breaking: sea



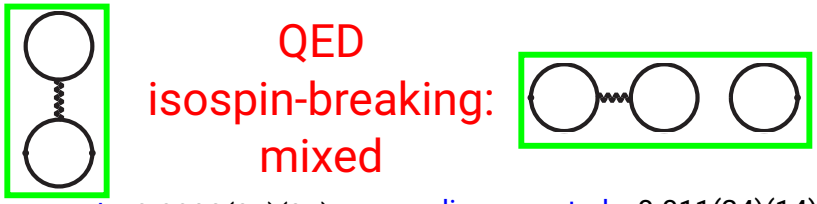
connected 0.37(21)(24) disconnected -0.040(33)(21)

Etc.

bottom; higher order; perturbative

0.11(4)

QED isospin-breaking: mixed



connected -0.0093(86)(95) disconnected 0.011(24)(14)

Finite-size effects

isospin-symmetric

18.7(2.5)

isospin-breaking

0.0(0.1)

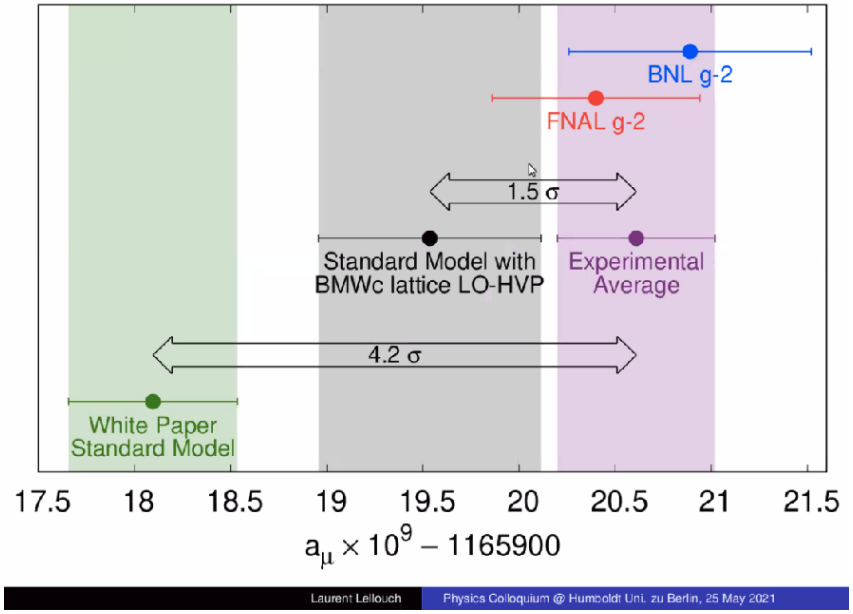
$10^{10}\times a_\mu^{\text{LO-HVP}} = 707.5(2.3)_{\text{stat}}(5.0)_{\text{sys}}[5.5]_{\text{tot}}$

[WP '20]

$10^{10}\times a_\mu^{\mathrm{HVP,LO}} = 693.1\,(2.8)_{\text{exp}}\,(2.8)_{\text{sys}}\,(0.7)_{\text{DV+QCD}}\,[4.0_{\text{tot}}]$ [2.1σ wrt. BMW '21]

$(g-2)_\mu$

$$a_\mu \equiv \frac{g_\mu - 2}{2}$$



$$a_\mu^{\text{exp; FNAL'21}} = (11\,659\,204.0 \pm 5.4) \cdot 10^{-10} \qquad [0.46\,\text{ppm}] \leftarrow \{0.43_{\text{stat.}},\ 0.16_{\text{syst.}},\ 0.03_{\text{fund.prms}}\}\ \text{ppm}$$

$$a_\mu^{\text{exp; BNL'06+FNAL'21}} = (11\,659\,206.1 \pm 4.1) \cdot 10^{-10} \qquad [0.35\,\text{ppm}]$$

$$a_\mu^{\text{SM; WP'20}} = (11\,659\,181.0 \pm 4.3) \cdot 10^{-10} \qquad [0.37\,\text{ppm}] \leftarrow \{0.34_{\text{LO hadronic}},\ \ldots\}\ \text{ppm}$$

$$a_\mu^{\text{SM; BMW'21}} = (11\,659\,195.4 \pm 5.7) \cdot 10^{-10} \qquad [0.49\,\text{ppm}] \leftarrow \{0.47_{\text{LO hadronic}},\ \ldots\}\ \text{ppm}$$

- difference experiment – theory [WP '20]:

$$\Delta a_\mu^{\text{exp}-\text{WP'20}} = (25.1 \pm 5.9) \cdot 10^{-10} \qquad [4.2\sigma]$$

- difference experiment – theory [BMW '21]:

$$\Delta a_\mu^{\text{exp}-\text{BMW'21}} = (10.7 \pm 7.0) \cdot 10^{-10} \qquad [1.5\sigma]$$

[FNAL '21]: Muon *g*-2 collab., 2104.03281 [hep-ex], Phys.Rev.Lett. 126 (2021) 14, 141801, Apr. 7, 2021.

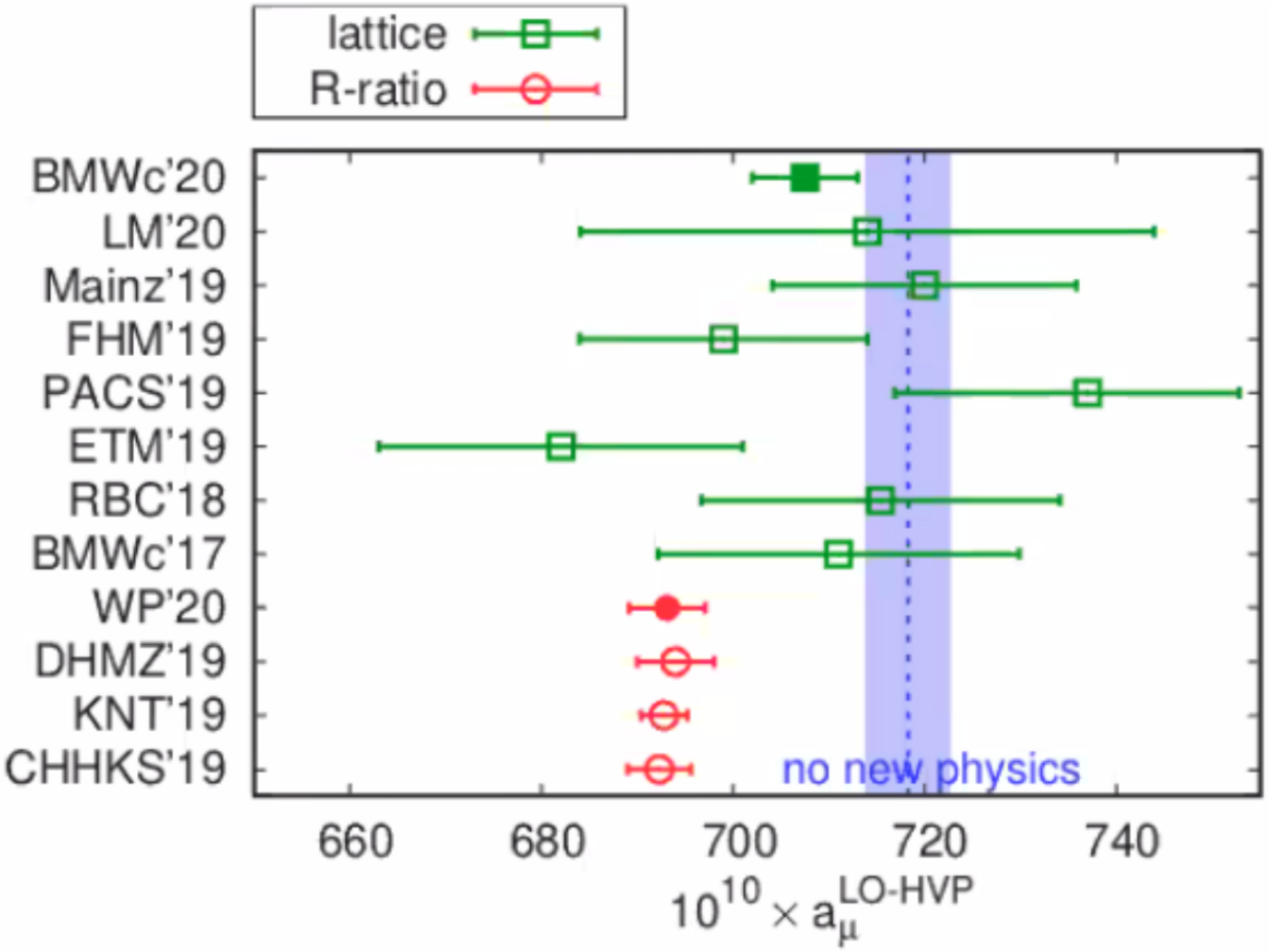
[BNL '06]: Muon *g*-2 E821 collab., hep-ex/0602035, Phys.Rev.D 73 (2006) 072003, Apr. 7, 2006.

[WP '20]: Muon *g*-2 theory initiative White Paper, 2006.04822 [hep-ph], Phys.Rept. 887 (2020) 1-166, Dec. 3, 2020.

[BMW '21]: Budapest Marseille Wuppertal Lattice QCD collab., 2002.12347 (hep-lat), Nature 593 (2021) 7857, 51-55, Apr. 7, 2021.

checks & comparisons of the HVP contribution

$a_\mu^{\mathrm{HVP,\,LO}}$: data-driven & lattice



[L. Lellouch, 2021]

$a_\mu^{\mathrm{HVP,\,LO}}$: spacelike & timelike momenta

$s = q^2 = -Q^2$

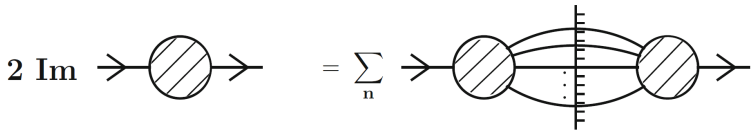
lattice QCD+QED : spacelike $Q^2 > 0$

$$\Pi_{\mu\nu}(Q) = \int \mathrm{d}^4x\, e^{iQx}\, \langle J_\mu(x)\, J_\nu(0)\rangle$$
$$= (Q_\mu Q_\nu - Q^2\, \delta_{\mu\nu})\, \Pi(Q^2)$$

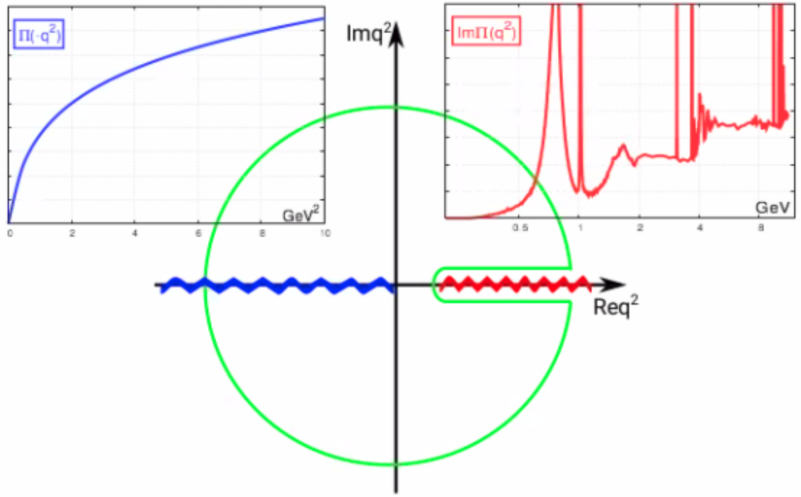
$$C(t) = -\frac{1}{3}\sum_{k=1}^{3}\int d^3x\, \langle\, J_k(x)\, J_k(0)\,\rangle$$

$$a_\mu^{\mathrm{HVP,\,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, C(t)\, \widetilde{f}(t)$$

dispersive data-driven approach : timelike



$$a_\mu^{\mathrm{HVP,LO}} = \frac{m_\mu^2}{12\pi^3}\int_{M_{\pi^0}^2}^{\infty} ds\, \frac{\sigma_{\mathrm{had}}^{(0)}(s)\, \hat{K}(s)}{s}$$



[L. Lellouch, 2021]

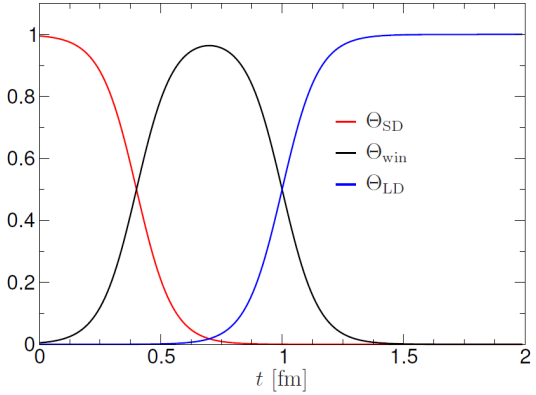
$$C(t) = \frac{1}{16\pi^3\alpha^2}\int_0^\infty d(\sqrt{s})\, s^2\, \sigma_{\mathrm{had}}^{(0)}(s)\, e^{-\sqrt{s}\,t}$$

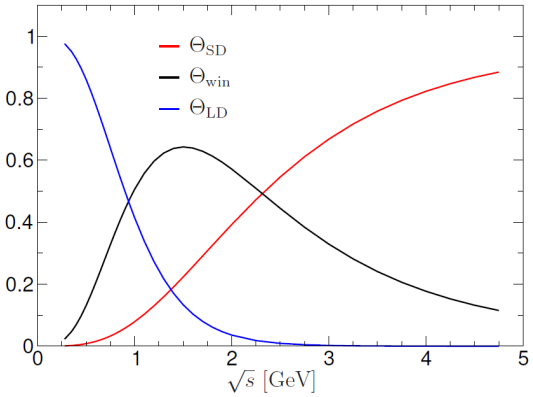
$[a_{\mu,\mathrm{win}}^{\mathrm{light}}]_{\mathrm{iso}}$: window quantities

lattice QCD+QED : spacelike

$$C(t) = -\frac{1}{3}\sum_{k=1}^{3}\int d^3x \,\langle J_k(x)\, J_k(0)\rangle$$

$$a_{\mu,\mathrm{win}}^{\mathrm{HVP,\,LO}}(t_1,t_2) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, C(t)\,\widetilde{f}(t)\, W(t;t_1,t_2)$$



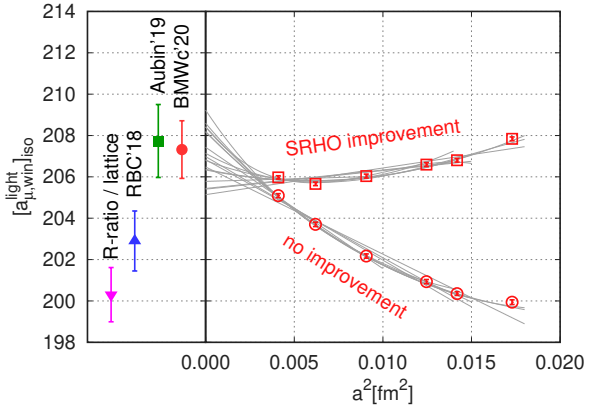


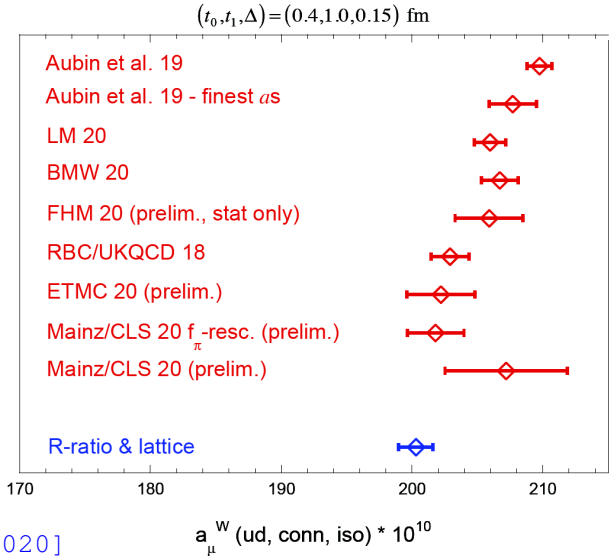
[M. Hoferichter, 2021]

window can be chosen to study KLOE-BaBar tension

dispersive data-driven approach : timelike

$$C(t) = \frac{1}{16\pi^3\alpha^2}\int_0^\infty d(\sqrt{s})\, s^2\, \sigma_{\mathrm{had}}^{(0)}(s)\, e^{-\sqrt{s}\,t}$$





$[a_{\mu,\mathrm{win}}^{\mathrm{light}}]_{\mathrm{iso}}$ BMW *vs.* *R*-ratio : 3.7σ

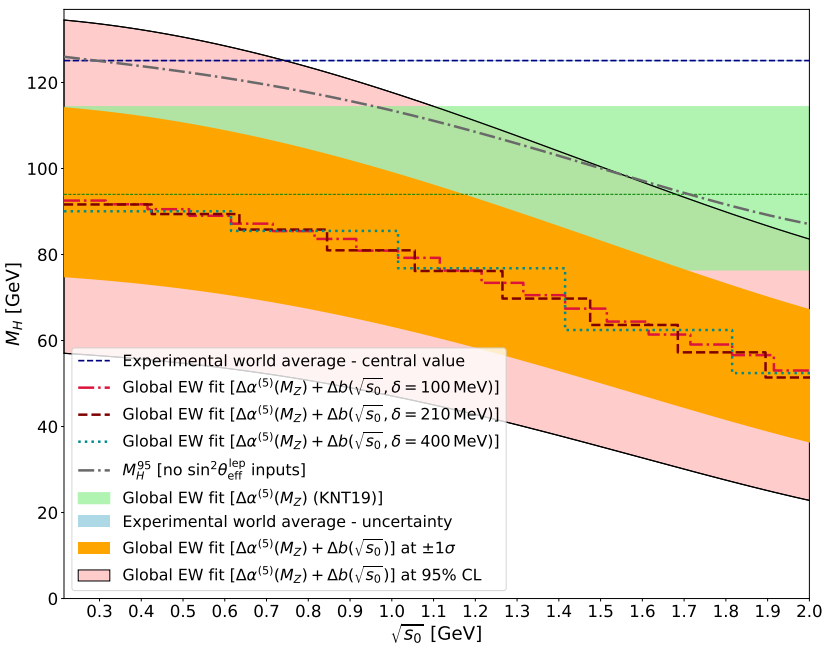
[H. Wittig, 2020]

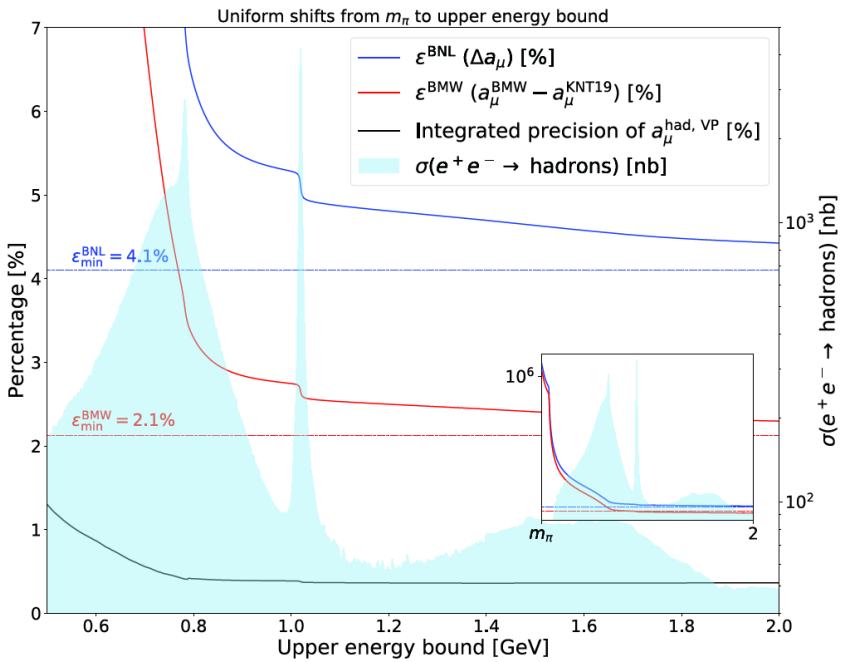
$a_\mu^{\mathrm{HVP,LO}}$ & $\Delta\alpha_{\mathrm{had}}^{(5)}$: electroweak fit

$$a_\mu^{\mathrm{HVP,LO}} = \frac{m_\mu^2}{12\pi^3} \int_{M_{\pi^0}^2}^{\infty} ds \, \frac{\sigma_{\mathrm{had}}^{(0)}(s)\, \hat{K}(s)}{s} \,, \qquad \Delta\alpha_{\mathrm{had}}^{(5)}(q^2) = -\frac{q^2}{4\pi^2\alpha} \, \mathrm{P} \int_{M_{\pi^0}^2}^{\infty} ds \, \frac{\sigma_{\mathrm{had}}^{(0)}(s)}{(s-q^2)}$$

How much should $\sigma_{\mathrm{had}}^{(0)}(s)$ increase to agree with experiment?

$$\Delta\sigma(s) = \epsilon\, \sigma_{\mathrm{had}}^{(0)}(s) \,, \qquad \text{in} \quad (\sqrt{s_0} - \delta/2) \leq \sqrt{s} \leq (\sqrt{s_0} + \delta/2)$$





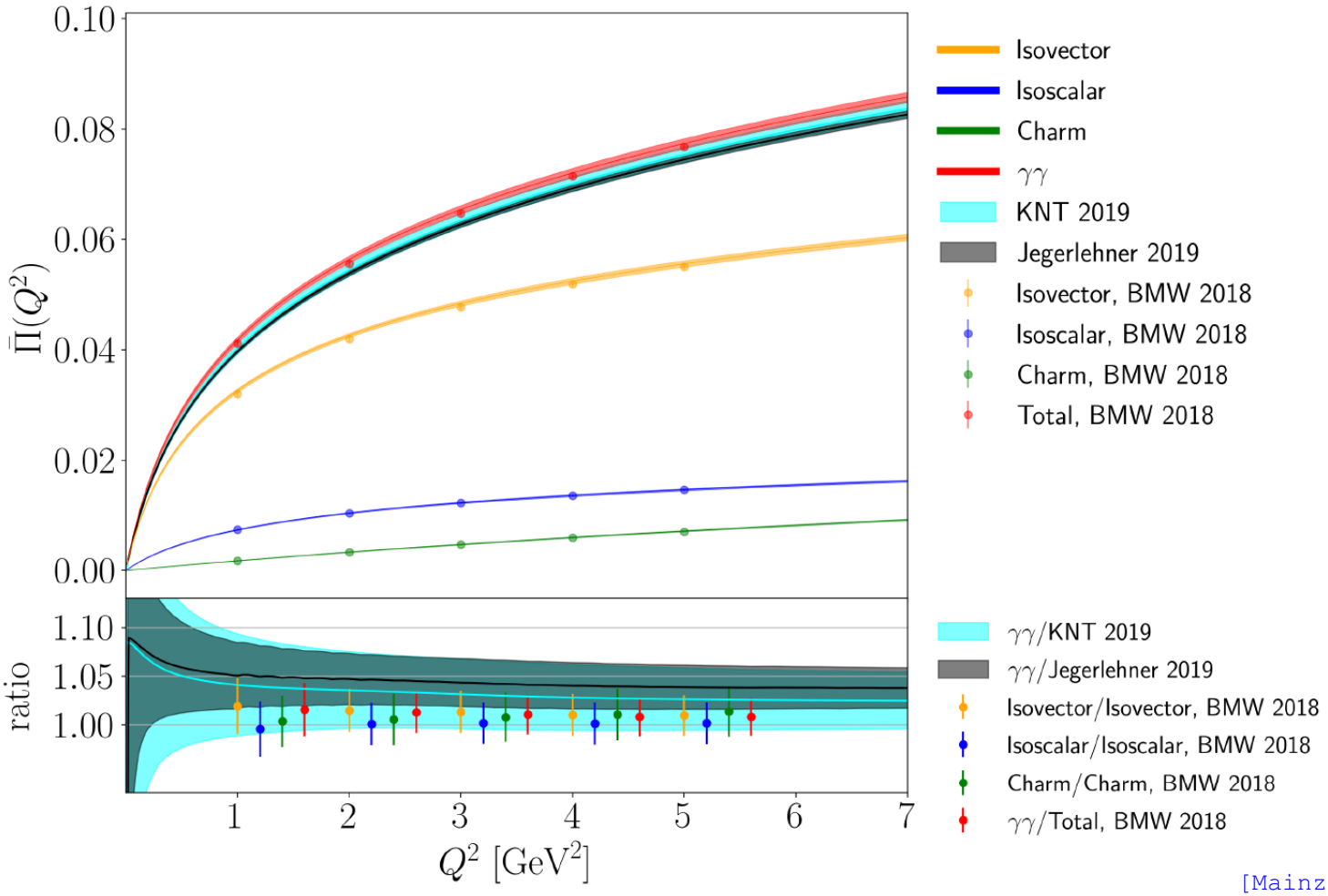
[Keshavarzi, Marciano, Passera, Sirlin, PRD 2020]

see also [Crivellin et al. 2020; De Rafael 2020; Malaescu et al. 2020]

$\Delta\alpha_{\rm had}$

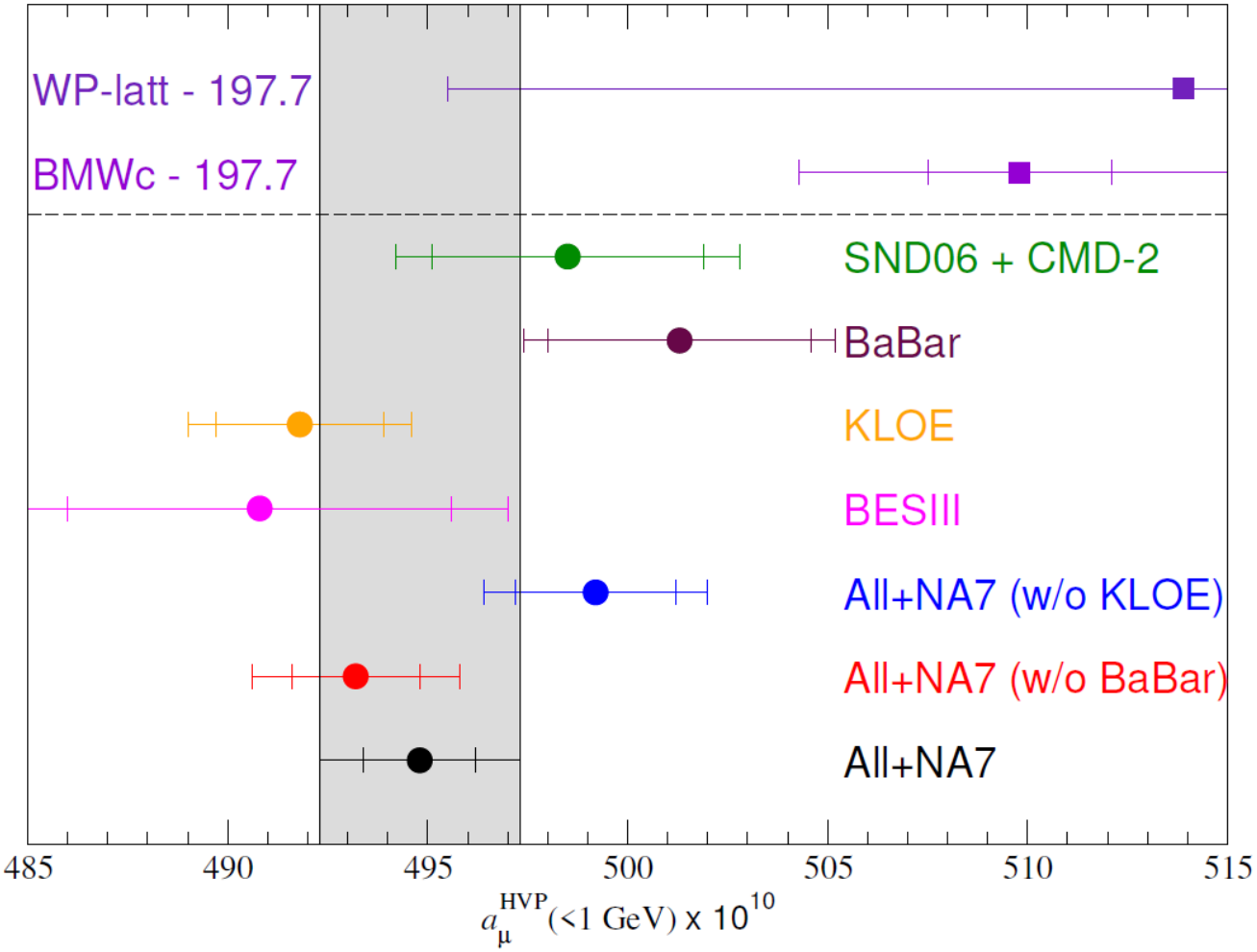
$$\Delta\alpha^{(5)}_{\rm had}(q^2) \;=\; -\frac{q^2}{4\pi^2\alpha}\;\mathrm{P}\int\limits_{M^2_{\pi^0}}^{\infty} ds\,\frac{\sigma^{(0)}_{\rm had}(s)}{(s-q^2)}$$

$$\Delta\alpha_{\rm had}(q^2) \;=\; 4\pi\alpha\,\Big(\Pi(Q^2)-\Pi(0)\Big) \;=\; 4\pi\alpha\,\bar{\Pi}(Q^2)$$



[Mainz, 2020]

$a_\mu^{\mathrm{HVP,LO}}(< 1\,\mathrm{GeV})$: data-driven & lattice



[M. Hoferichter, $(g-2)_\mu$ workshop, 2021]
see also [Colangelo, Hoferichter, Stoffer, 2020]

$$a_\mu^{\mathrm{total}}[\mathrm{WP'20}] - a_\mu^{2\pi,<1\mathrm{GeV}}[\mathrm{WP'20}] = 197.7 \cdot 10^{-10}$$

$a_\mu^{\mathrm{HVP,LO}}$: comparison

status :

- $a_\mu^{\mathrm{HVP,LO}}$ [BMW'21 - WP'20] : $+2.1\,\sigma$, $+2.1\%$
- $a_{\mu,\mathrm{win}}^{\mathrm{HVP,LO}}$ [BMW'21 - R-ratio] : $+3.7\,\sigma$, $+3.0\%$ window mainly in : $0.5 \lesssim \sqrt{s} \lesssim 4\,\mathrm{GeV}$
- $\Delta\alpha_{\mathrm{had}}^{(5)}(q^2 = 1\,\mathrm{GeV}^2)$ [BMW'21 - R-ratio] : $+3.7\,\sigma$, $+2.8\%$
- $\left(\Delta\alpha_{\mathrm{had}}^{(5)}(10\,\mathrm{GeV}^2) - \Delta\alpha_{\mathrm{had}}^{(5)}(1\,\mathrm{GeV}^2)\right)$ [BMW'21 - R-ratio] : $+0.3\,\sigma$, $+0.6\%$

prospects :

- new R-ratio $\pi^+\pi^-$ experimental data + analyses : CMD3, BES-III, BaBar, Belle II
- new lattice results : $a_\mu^{\mathrm{HVP,LO}}$, window quantities, $\Delta\alpha_{\mathrm{had}}^{(5)}$, $F_\pi^V(s)$

  (+ quantities relevant for lattice scale setting)

  various discretizations, isospin breaking corrections , . . .
- interplay : data-driven dispersive approach $\leftrightarrow$ lattice
- MUonE project : extract spacelike HVP from $\mu\, e \to \mu\, e$ elastic scattering experiment

other contributions to $(g-2)_\mu$

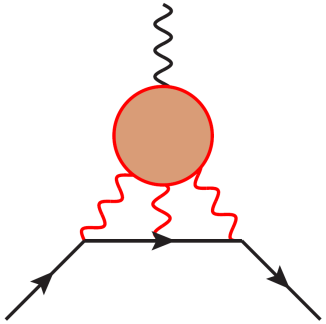
hadronic light-by-light

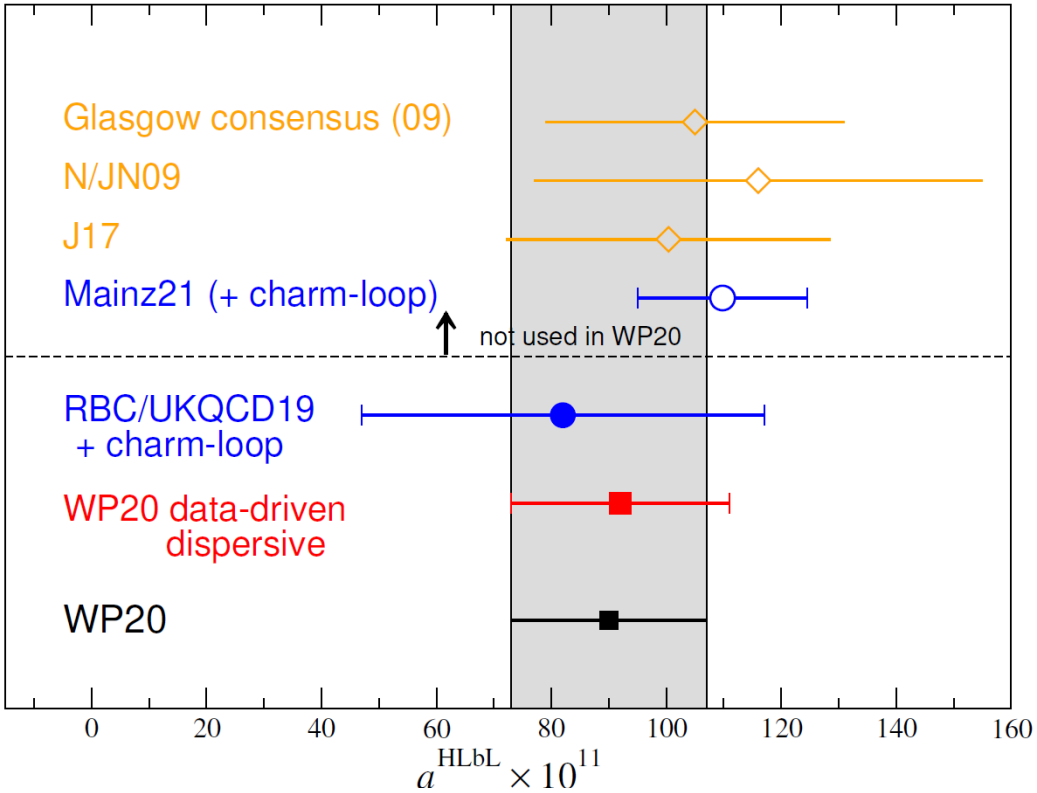
electroweak

QED

a_μ^{HLbL} : hadronic light-by-light

- dispersive approach : data-driven
- lattice QCD+QED





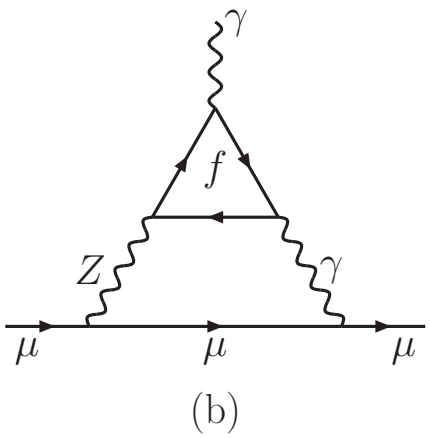
[M. Hoferichter, $(g-2)_\mu$ workshop, 2021]

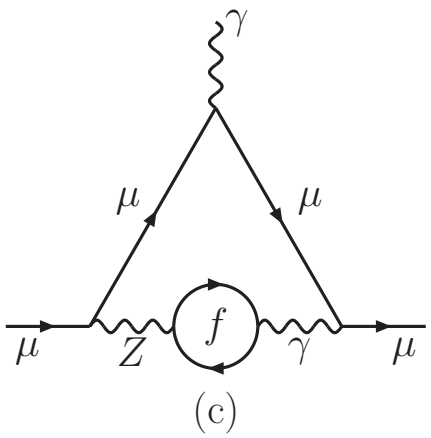
[WP '21]

$$a_\mu^{\mathrm{HLbL}} = (9.2 \pm 1.9) \cdot 10^{-10} \quad [21\%] \quad \rightarrow \quad [\sim 10\%] \quad \text{Fermilab exp. target}$$

$a_\mu^{\rm EW}$ & $a_\mu^{\rm QED}$: electroweak and QED

- electroweak : up to two-loop + leading three-loop logarithms





leading uncertainty from hadronic effects

[WP '21]

$$a_\mu^{\rm EW} = (15.36 \pm 0.1) \cdot 10^{-10}$$

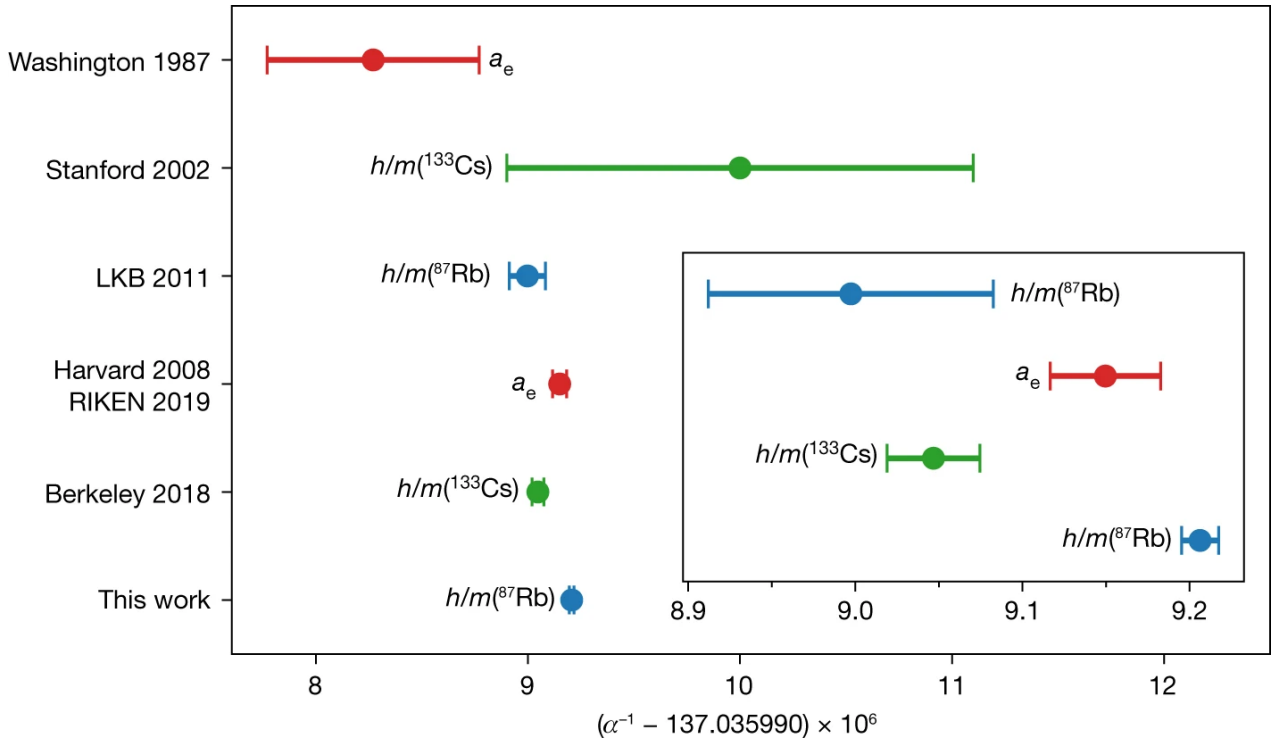
- QED : up to $O(\alpha^5)$

photon and lepton loops

$$a_\mu^{\rm QED}(\alpha(a_e)) = 11\,658\,471.8842\,(7)_{m_\tau}\,(17)_{O(\alpha^4)}\,(6)_{O(\alpha^5)}\,(100)_{O(\alpha^6)}\,(28)_{\alpha(a_e)}\,[106] \cdot 10^{-10}$$

α : fine-structure constant

- caesium (Cs) and rubidium (Rb) atomic recoil experiments
- electron g-2 : $a_e[\mathrm{SM}] \equiv a_e[\mathrm{exp}] \quad \rightsquigarrow \quad \alpha$



[Morel et al., Nature 588, 61-65 (2020)]

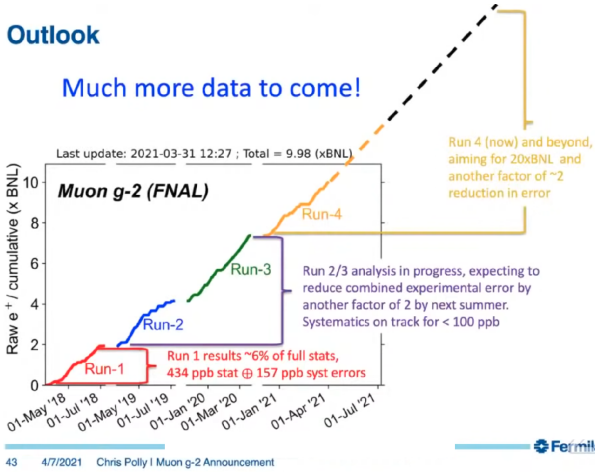
- ratio with reduced hadronic uncertainties :

$$R_{e/\mu} = \frac{a_e^{\mathrm{HVP,LO}}}{a_\mu^{\mathrm{HVP,LO}}} \left(\frac{m_\mu}{m_e}\right)^2$$

conclusion

- Fermilab experiment : will reduce by 4 the uncertainty in the coming years

  $\rightsquigarrow \quad \delta a_\mu \approx 1.5 \cdot 10^{-10}$



- J-PARC experiment

- ongoing work to consolidate the SM prediction of the muon $g-2$

- QED & EW contributions : uncertainties are below Fermilab exp. target

- hadronic light-by-light : desired reduction of the error by ~ 2

- hadronic vacuum polarization :

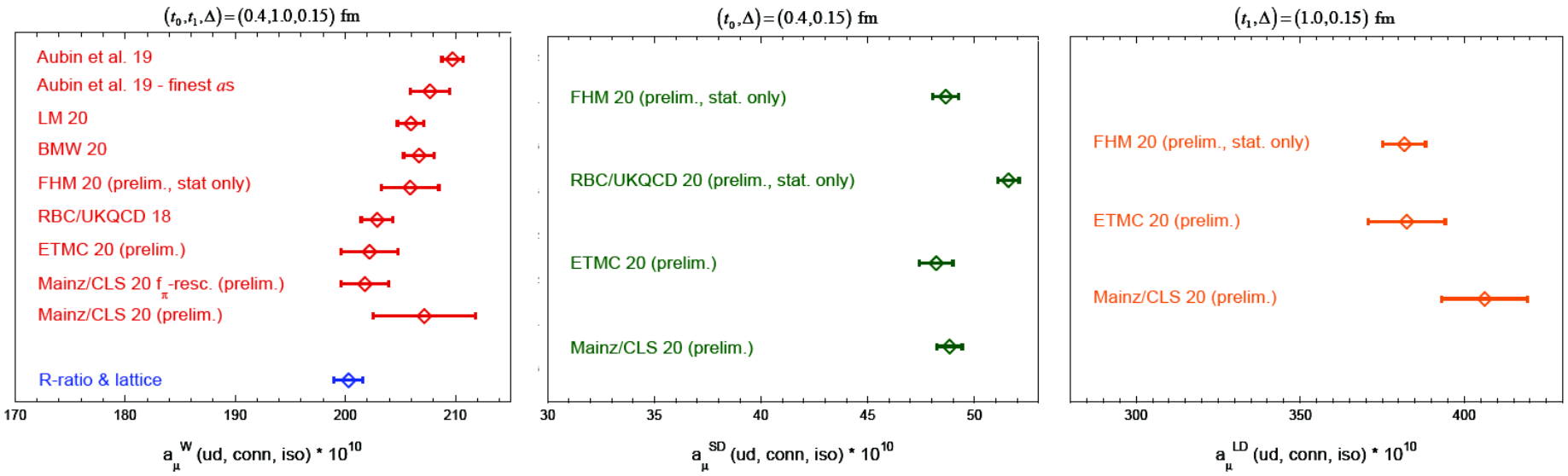
  - interplay : dispersive data driven approach $\leftrightarrow$ lattice MUonE project

  - desired reduction of the error by $3 \div 4$

$a_\mu^{\mathrm{HVP,\,LO}}$: window quantities

"Window" quantities

(Plots from Davide Giusti)



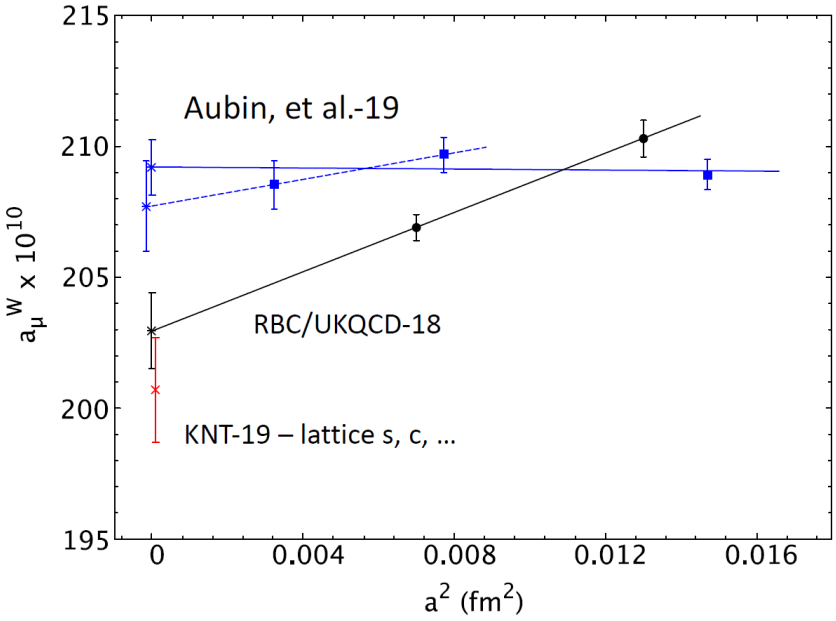
[H. Wittig, 2020]

$[a^{\rm light}_{\mu,\rm win}]_{\rm iso}$: window quantities

lattice QCD+QED : spacelike

$$C(t) = -\frac{1}{3}\sum_{k=1}^{3}\int d^3x\,\langle J_k(x)\,J_k(0)\rangle$$

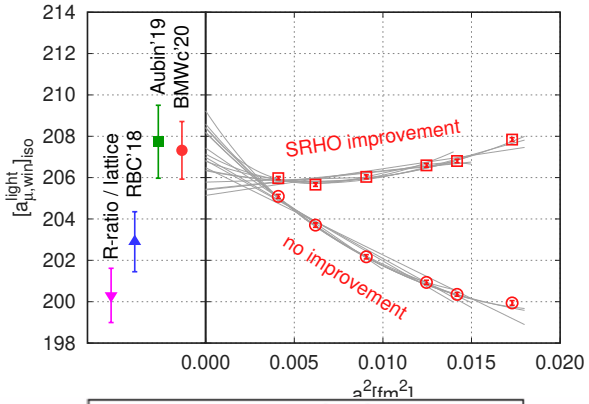
$$a^{\rm HVP,\,LO}_{\mu,\rm win}(t_1,t_2) = \left(\frac{\alpha}{\pi}\right)^2\int_0^\infty dt\,C(t)\,\widetilde{f}(t)\,W(t;t_1,t_2)$$

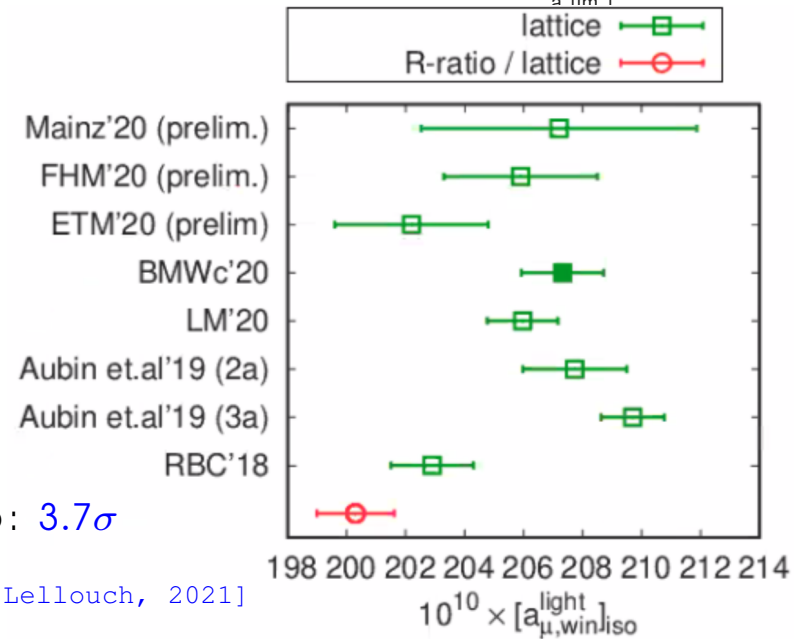


[Aubin et al., 2019]

dispersive data-driven approach : timelike

$$C(t) = \frac{1}{16\pi^3\alpha^2}\int_0^\infty d(\sqrt{s})\,s^2\,\sigma^{(0)}_{\rm had}(s)\,e^{-\sqrt{s}\,t}\,intr$$

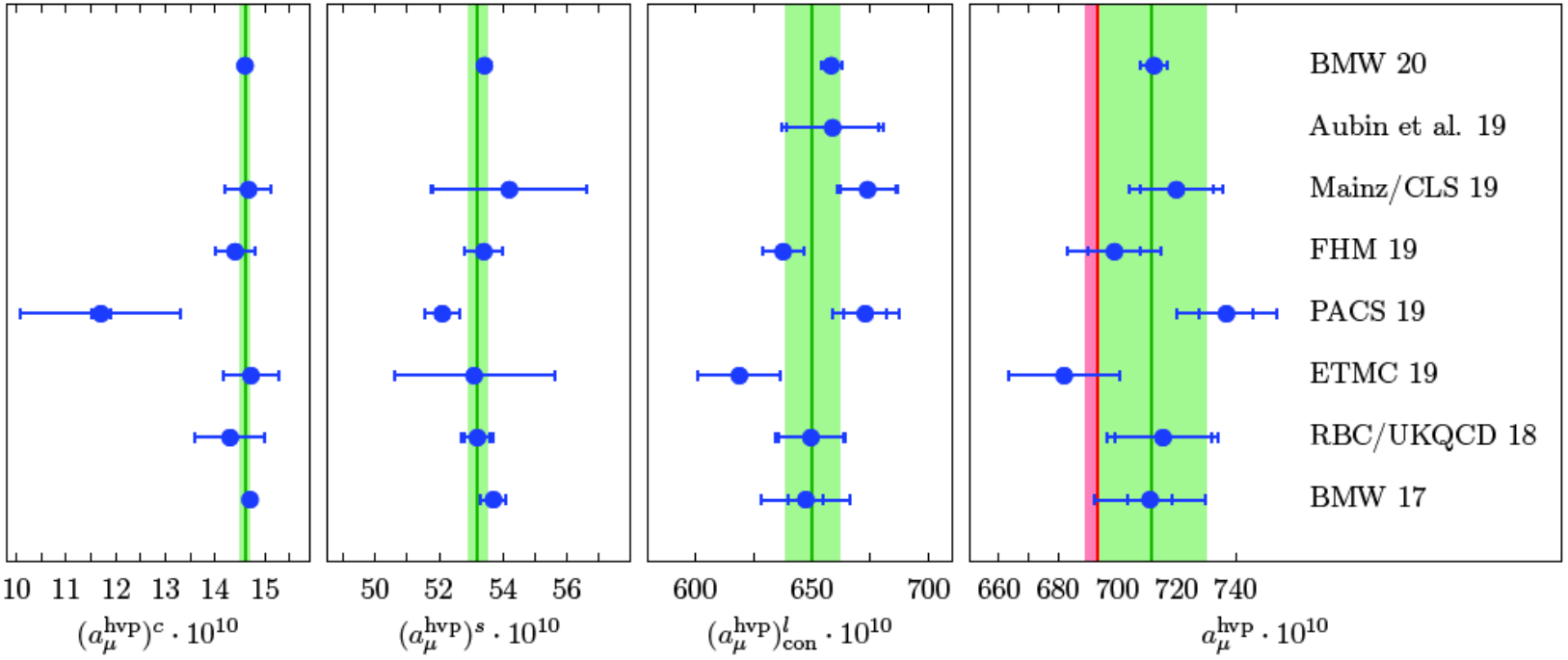


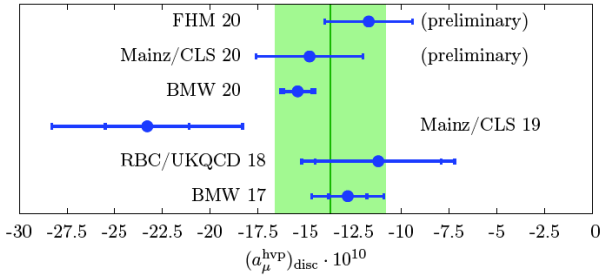


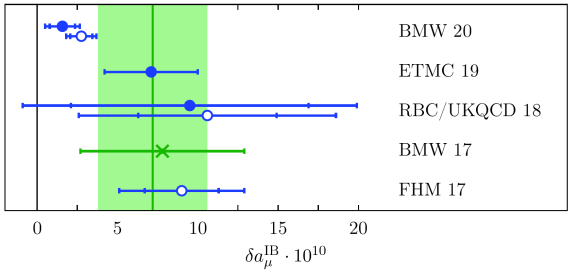
$[a^{\rm light}_{\mu,\rm win}]_{\rm iso}$ BMW *vs.* *R*-ratio : 3.7σ

[L. Lellouch, 2021]

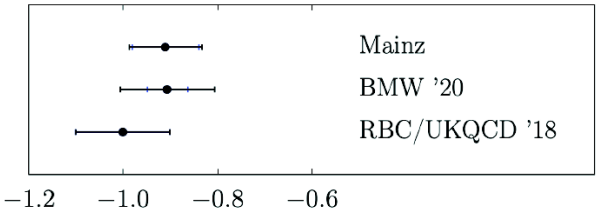
$a_\mu^{\mathrm{HVP,\,LO}}$: comparison lattice







$(a_\mu^{\mathrm{hvp}})^{\mathrm{win,disc}} \times 10^{10}$



[H. Wittig, 2020]

$a_\mu^{\rm light}$: continuum-limit extrapolation [BMW '21]

window quantities : [RBC-UKQCD, 2018]

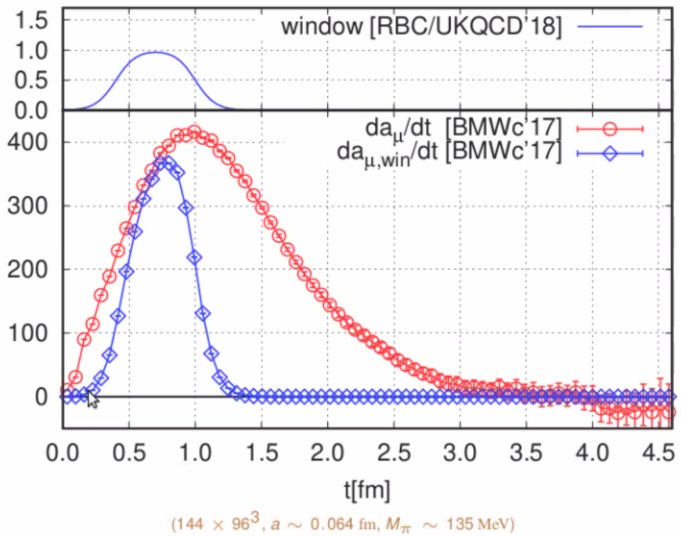
$$a_{\mu,\rm win}^{\rm light}(t_1,t_2) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, C(t)\, \widetilde{f}(t)\, W(t;t_1,t_2)$$

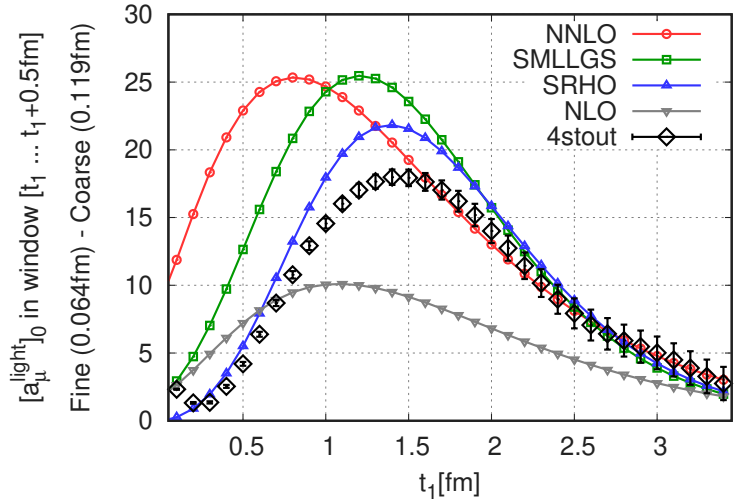
$$W(t;t_1,t_2) \equiv \Theta(t;t_1,\Delta) - \Theta(t;t_2,\Delta) \quad \text{with} \quad \Theta(t;t',\Delta) \equiv \tfrac{1}{2} + \tfrac{1}{2}\tanh[(t-t')/\Delta]$$

$W(t;t_1,t_1+0.5\,\text{fm})$

$\Delta = 0.15\,\text{fm}$

$$\Delta a_{\mu,\rm win}(t_1) \equiv a_{\mu,\rm win}^{\rm light}(t_1, a[\text{fine}]) - a_{\mu,\rm win}^{\rm light}(t_1, a[\text{coarse}])$$





model taste-breaking discretization effects :

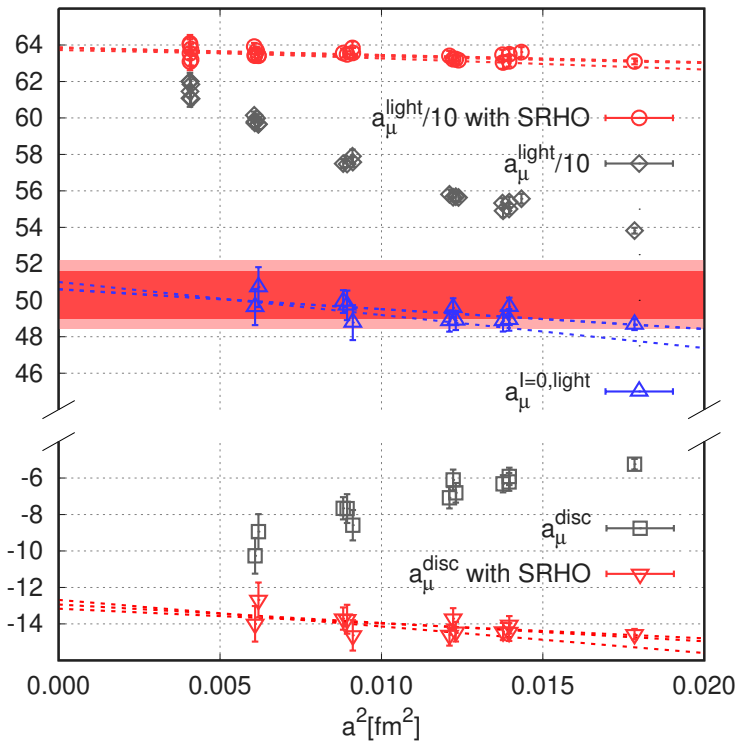
$$[a_\mu^{\rm light}]_0(L,T,a) \;\to\; [a_\mu^{\rm light}]_0(L,T,a) + \tfrac{10}{9}\left[a_{\mu,t\geq t_{\rm sep}}^{\rm RHO}(L_{\rm ref},T_{\rm ref}) - a_{\mu,t\geq t_{\rm sep}}^{\rm SRHO}(L,T,a)\right]$$

$a_\mu^{I=0}$: continuum-limit extrapolation [BMW '21]

isoscalar ($I = 0$) contribution to $a_\mu^{\mathrm{HVP,LO}}$: three pions contribute

(*vs.* two pions, in $I = 1$ isovector)

$$a_\mu^{I=0} \equiv \frac{1}{10} a_\mu^{\mathrm{light}} + a_\mu^{\mathrm{disc}} + \cdots$$



model taste-breaking discretization effects :

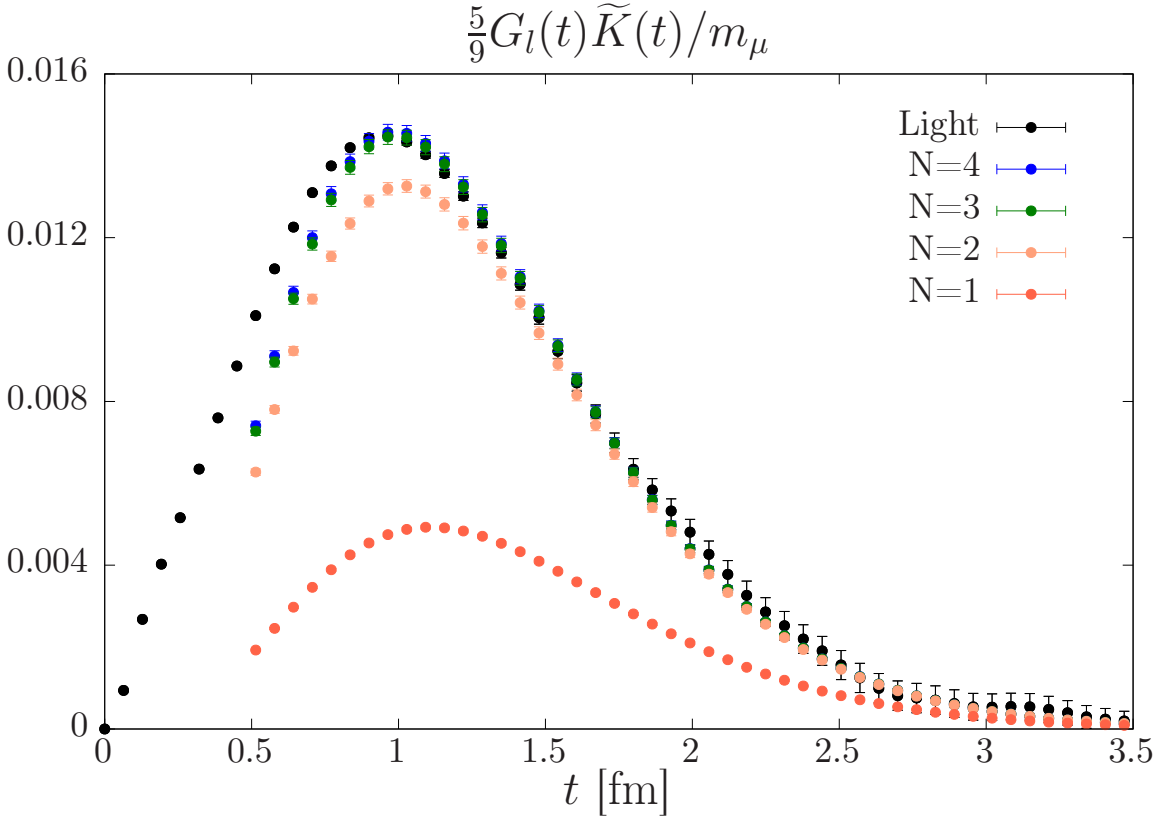
$$[a_\mu^{\mathrm{light}}]_0(L,T,a) \;\rightarrow\; [a_\mu^{\mathrm{light}}]_0(L,T,a) + \tfrac{10}{9}\left[a^{\mathrm{RHO}}_{\mu,t\geq t_{\mathrm{sep}}}(L_{\mathrm{ref}},T_{\mathrm{ref}}) - a^{\mathrm{SRHO}}_{\mu,t\geq t_{\mathrm{sep}}}(L,T,a)\right]$$

a_μ^{light} : reconstruction of $\langle J_k J_k \rangle$ lattice correlator

spectroscopy $\rightsquigarrow$ $\{E_n\}$, $n = 1, 2, \ldots, N$

$$G_\ell(t) = -\frac{1}{3}\sum_{k=1}^{3}\int d^3x \, \langle J_k(x)\, J_k(0) \rangle = \sum_{n=0}^{\infty} \frac{Z_n^2}{2E_n}\, e^{-E_n t}$$

integrand of a_μ^{light}



[Mainz, 2019]

$\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$: spacelike calculation

$$\alpha(Q^2) = \frac{\alpha}{1 - \Delta\alpha_{\rm QED}(Q^2)}$$

- Combine experimental data and perturbation theory (PT)

$$\begin{aligned}\Delta\alpha^{(5)}_{\rm had}(M_Z^2) = {} & \Delta\alpha^{(5)}_{\rm had}(-M_0^2)^{\rm exp} \\ & + \left[\Delta\alpha^{(5)}_{\rm had}(-M_Z^2) - \Delta\alpha^{(5)}_{\rm had}(-M_0^2)\right]^{\rm pQCD} \\ & + \left[\Delta\alpha^{(5)}_{\rm had}(M_Z^2) - \Delta\alpha^{(5)}_{\rm had}(-M_Z^2)\right]^{\rm pQCD}\end{aligned}$$

with M_0^2 sufficiently large

$a_\mu^{\mathrm{HVP,\,LO}}$: data-driven DHMZ19 & KNT19

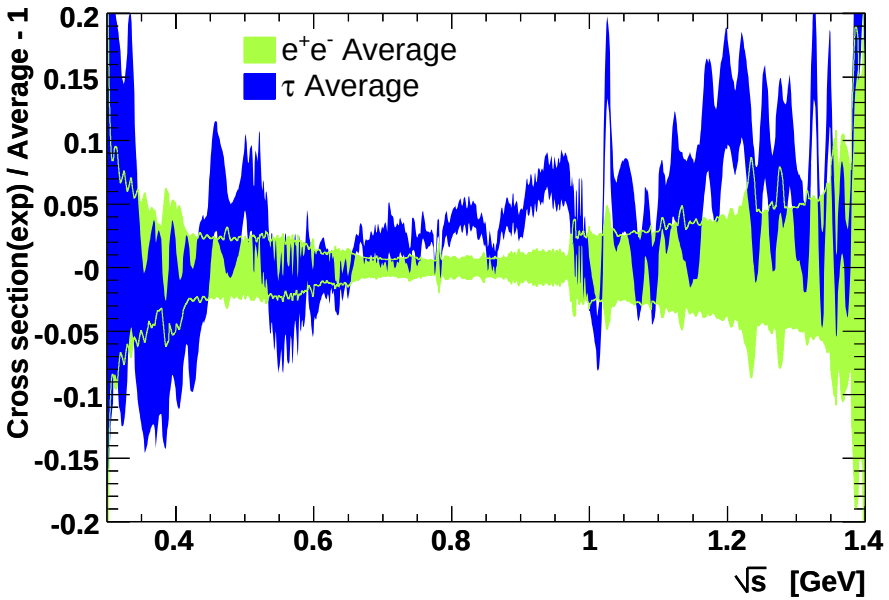
		DHMZ19	KNT19	Difference
$\pi^+\pi^-$		507.85(0.83)(3.23)(0.55)	504.23(1.90)	3.62
$\pi^+\pi^-\pi^0$		46.21(0.40)(1.10)(0.86)	46.63(94)	−0.42
$\pi^+\pi^-\pi^+\pi^-$		13.68(0.03)(0.27)(0.14)	13.99(19)	−0.31
$\pi^+\pi^-\pi^0\pi^0$		18.03(0.06)(0.48)(0.26)	18.15(74)	−0.12
K^+K^-		23.08(0.20)(0.33)(0.21)	23.00(22)	0.08
K_SK_L		12.82(0.06)(0.18)(0.15)	13.04(19)	−0.22
$\pi^0\gamma$		4.41(0.06)(0.04)(0.07)	4.58(10)	−0.17
Sum of the above		626.08(0.95)(3.48)(1.47)	623.62(2.27)	2.46
[1.8, 3.7] GeV (without $c\bar{c}$)		33.45(71)	34.45(56)	−1.00
$J/\psi,\,\psi(2S)$		7.76(12)	7.84(19)	−0.08
[3.7, ∞) GeV		17.15(31)	16.95(19)	0.20
Total $a_\mu^{\mathrm{HVP,LO}}$	$694.0(1.0)_{\mathrm{stat.}}(3.5)_{\mathrm{syst.channel}}(1.6)_{\mathrm{syst.common}}(0.1)_{\psi}(0.7)_{\mathrm{DV}}[4.0]$	692.8(2.4)	1.2

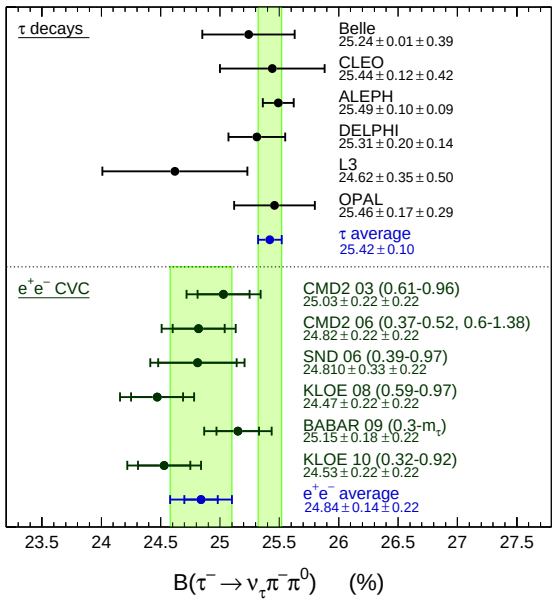
$$a_\mu^{\mathrm{HVP,LO}} = 693.1(2.8)_{\mathrm{exp}}(2.8)_{\mathrm{sys}}(0.7)_{\mathrm{DV+QCD}}\,[4.0_{\mathrm{tot}}]\cdot 10^{-10} \quad [0.58\%]$$

[WP '21]

$a_\mu^{\mathrm{HVP,\,LO}}$: hadronic τ decays

- semileptonic τ decays : $\tau^- \to \bar{\nu}\, W^- \to \bar{\nu} + (\bar{u}d) \quad \rightsquigarrow \quad \tau^- \to \bar{\nu} + \text{hadrons}$





[WP'21]

- differences wrt. e^+e^- data after isospin breaking corrections
- no $\omega - \rho$ mixing in $I = 1$ τ-data

$$a_\mu^{\mathrm{HVP,LO}}[e^+e^-] = 691.1(4.0) \cdot 10^{-10}$$

$$a_\mu^{\mathrm{HVP,LO}}[\tau] = 703.0(4.4) \cdot 10^{-10}$$

$$\Delta a_\mu^{\mathrm{HVP,LO}}[\tau - (e^+e^-)] = \quad 11.9 \cdot 10^{-10} \quad [\sim 2\sigma]$$

- model-dependence on $\gamma - \rho$ mixing $\rightsquigarrow$ lattice