Braneworld gravity = infinite derivative gravity

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based on arXiv: 2310.09333 w/ Aguilar, Bueno, Hennigar, Llorens
+ upcoming work w/ Bueno and Hennigar

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Higher-derivative gravity with covariant derivatives

$$I = \frac{1}{16\pi G} \int \mathrm{d}^D x \sqrt{|g|}\, \mathcal{L}(g^{ab}, R_{abcd}, \nabla_a)$$

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General form of linearized EOMs

$$\mathcal{E}_{ab} \equiv \sum_{l=0} \ell^{2l} \left[\alpha_l \bar{\Box}^l G^{(1)}_{ab} + \beta_l \bar{\Box}^l R^{(1)} \bar{g}_{ab} + \gamma_{l+1} \ell^2 \Box^l [\bar{g}_{ab} \bar{\Box} - \bar{\nabla}_a \bar{\nabla}_b] R^{(1)} \right] = 0$$

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$$\mathcal{E}_{ab} \equiv \left[f_1(\ell^2\bar{\Box})G^{(1)}_{ab} + f_2(\ell^2\bar{\Box})R^{(1)}\bar{g}_{ab} + f_3(\ell^2\bar{\Box})[\bar{g}_{ab}\bar{\Box} - \bar{\nabla}_a\bar{\nabla}_b]R^{(1)} \right] = 0$$

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Bianchi identity $\nabla^a\mathcal{E}_{ab} = 0$ leads to

$$f_2\left(\ell^2\bar{\Box}\right) = \frac{D-2}{2D}\left[f_1\left(\ell^2\bar{\Box}\right) - f_1\left(\ell^2\bar{\Box} + 2D\chi\right)\right] + \frac{D-1}{D\ell^2}\left[f_3\left(\ell^2\bar{\Box} + 2D\chi\right)\left(\ell^2\bar{\Box} - \chi D\right) - f_3\left(\ell^2\bar{\Box}\right)\ell^2\bar{\Box}\right]$$

Connection with the Lagrangian?

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These equations can be derived from a general quadratic Lagrangian

$$\mathcal{L}_{\text{eff}} = \lambda \left[\frac{(D-1)(D-2)}{\ell^2} + R + \ell^2 R F_1(\ell^2 \bar{\Box}) R + \ell^2 R_{ab} F_2(\ell^2 \bar{\Box}) R^{ab} + \ell^2 R_{abcd} F_3(\ell^2 \bar{\Box}) R^{abcd} \right]$$

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For a flat background we get

$$\begin{aligned} f_1 &= \frac{\lambda}{2} \left[1 + \left[4F_3(\ell^2 \bar{\Box}) + F_2(\ell^2 \bar{\Box}) \right] \ell^2 \bar{\Box} \right] \\ f_2 &= 0 \\ f_3 &= -\frac{\lambda}{2} \ell^2 \left[2F_1(\ell^2 \bar{\Box}) + F_2(\ell^2 \bar{\Box}) + 2F_3(\ell^2 \bar{\Box}) \right] \end{aligned}$$

Degrees of freedom

Trace of the equations:

$$-\frac{\lambda}{4}\left[(D-2)-\ell^2\bar{\Box}\left[4F_3(\ell^2\bar{\Box})+DF_2(\ell^2\bar{\Box})+4(D-1)F_1(\ell^2\bar{\Box})\right]\right]R^{(1)}=0$$

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Traceless part is given by

$$\frac{\lambda}{2}\left\{\left[1+\left[4F_3(\ell^2\bar{\Box})+F_2(\ell^2\bar{\Box})\right]\ell^2\bar{\Box}\right]R^{(1)}_{\langle ab\rangle}\right.$$
$$\left.-\ell^2\left[2F_1(\ell^2\bar{\Box})+F_2(\ell^2\bar{\Box})+2F_3(\ell^2\bar{\Box})\right]\bar{\nabla}_{\langle a}\bar{\nabla}_{b\rangle}R^{(1)}\right\}=0$$

Massive gravitons: roots of $1+\left[4F_3(\ell^2\bar{\Box})+F_2(\ell^2\bar{\Box})\right]\ell^2\bar{\Box}\equiv\mathcal{F}(\bar{\Box})$

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Possible resolutions:

- Nonlocal gravity: $\mathcal{F}(\Box)$ is an entire functions with no zeroes [Mazumdar+...]
- Einstein-like gravities: $4F_3(\ell^2\bar{\Box})+F_2(\ell^2\bar{\Box})=0$ [Bueno, PAC]

Consider Einstein gravity in AdS_{D+1}

$$I_{\text{Einstein}} = \frac{1}{16\pi G_{D+1}}\left[\int_{\mathcal{M}} \mathrm{d}^{D+1}X\sqrt{-\hat{g}}\left(\hat{R} + \frac{D(D-1)}{\ell^2}\right) + 2\int_{\partial\mathcal{M}} \mathrm{d}^{D}x\sqrt{-g}\,K\right]$$

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We add a brane at a cutoff $\rho=\ell/\epsilon$ before the AdS boundary. We have

$$\boxed{I_{\rm Einstein}+I_{\rm brane}=I_{\rm bgrav}+I_{\rm CFT}}$$

$$I_{\rm brane}=-T\int_{\rho=\ell/\varepsilon}\mathrm{d}^{D}x\sqrt{-g}\,,\quad I_{\rm bgrav}=\frac{1}{16\pi G_{D+1}}\int\mathrm{d}^{D}x\sqrt{-g}\mathcal{L}$$

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$$I_{\text{brane}} = -T\int_{\rho=\ell/\varepsilon} \mathrm{d}^{D}x\sqrt{-g}\,, \quad I_{\text{bgrav}} = \frac{1}{16\pi G_{D+1}}\int \mathrm{d}^{D}x\sqrt{-g}\mathcal{L}$$

Variation with respect to the boundary metric yields

$$\Pi_{ab} = K_{ab} - g_{ab}(K - T)$$

where Π_{ab} = boundary stress-energy tensor, equivalent to the equations of motion of I_{bgrav}

$$\Pi^{ab} \equiv \frac{2}{\sqrt{-g}}\frac{\delta I_{\text{bgrav}}}{\delta g_{ab}}$$

Induced Gravity on the brane

Induced gravity [Kraus, Larsen, Siebelink]

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$$\frac{1}{D-1}(\Pi + DT)^2 - (\Pi_{ab} + Tg_{ab})(\Pi^{ab} + Tg^{ab}) - R = \frac{D(D-1)}{\ell^2}$$

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Choosing $T = (D-1)/\ell$ we get $\Lambda = 0$

$$\boxed{\Pi = \frac{\ell}{2}\left[R + \Pi_{ab}\Pi^{ab} - \frac{1}{D-1}\Pi^2\right]}$$

This identity **fixes the induced gravity theory**

In order to derive the form of the equations, we expand them in derivatives

$$\mathcal{L}=\sum_{n=1}^{\infty}\ell^{2n-1}\mathcal{L}_{(n)}\,,\quad \Pi^{ab}=\sum_{n=1}^{\infty}\ell^{2n-1}\Pi^{ab}_{(n)}$$

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Then, we get

$$\Pi_{(1)} = \frac{R}{2}$$

$$\Pi_{(n)} = \frac{1}{2}\sum_{i=1}^{n-1}\left[\Pi_{(i)\,ab}\Pi^{ab}_{(n-i)} - \frac{1}{D-1}\Pi_{(i)}\Pi_{(n-i)}\right]\,, \quad n \geq 2$$

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The other ingredient we need to solve this recursive relation is

$$\Pi_{(n)} = (D-2n)\mathcal{L}_{(n)} + \text{total derivative}$$

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$$\mathcal{L}_{(1)}=\frac{R}{2(D-2)}\,,\quad \Pi_{(1)\,ab}=-\frac{1}{D-2}G_{ab}$$

Induced Gravity on the brane

Focus on the quadratic part of the Lagrangian

$$\mathcal{L}_{(n)} = \alpha_n R \Box^{n-2} R + \beta_n R^{ab} \Box^{n-2} R_{ab} + \mathcal{O}(R^3)$$

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Recurrence relation becomes

$$\alpha_n = -\frac{D}{4(D-1)}\beta_n \,, \quad \beta_n = \frac{2}{(D-2n)}\left[\frac{\beta_{n-1}}{(D-2)} + \sum_{i=2}^{n-2} \beta_i \beta_{n-i}\right]$$

with $\beta_2 = \frac{1}{2(D-2)^2(D-4)}$.

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The recurrence relation becomes a differential equation

$$f'(x) = (4-D)\beta_2 x^{3-D} - \frac{2}{D-2}xf(x) - 2x^{D-1}f(x)^2$$

Induced Gravity on the brane

$$I^{(2)}_{\rm bgrav} = \frac{1}{16\pi G_{D+1}} \int \mathrm{d}^D x \sqrt{-g} \left[\frac{\ell R}{2(D-2)} + \sum_{n=2}^{\infty} \beta_n \ell^{2n-1} \left(R^{ab} \Box^{n-2} R_{ab} - \frac{D}{4(D-1)} R \Box^{n-2} R \right) \right]$$

Induced Gravity on the brane

$$I_{\rm bgrav}^{(2)} = \frac{1}{16\pi G_D}\int \mathrm{d}^D x\sqrt{-g}\left[R + \ell^2 R^{ab} F\left(\ell^2\Box\right)\left(R_{ab} - \frac{D}{4(D-1)} g_{ab} R\right)\right]$$

where

$$F(\ell^2\Box) = 2(D-2)\sum_{n=2}^{\infty}\beta_n\left(\ell^2\Box\right)^{n-2}$$

and $G_D = 2(D-2)G_{D+1}/\ell$.

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and $G_D = 2(D-2)G_{D+1}/\ell$.

$f(x) = \frac{1}{2(D-2)}x^{4-D}F(x^2) \Rightarrow F(x)$ satisfies the equation

$$\boxed{F'(x) = (D-4)\frac{F(x)-F(0)}{2x} - \frac{1}{2(D-2)}\left(2F(x) + xF(x)^2\right)}$$

where $F(0) = 2(D-2)\beta_2 = \frac{1}{(D-2)(D-4)}$

Induced Gravity on the brane

$$F_D(x) = \frac{D(D-2)}{x^2} - \frac{1}{x} - \frac{(D-2)Y_{\frac{D+2}{2}}\left(\sqrt{x}\right)}{x^{3/2}Y_{\frac{D}{2}}\left(\sqrt{x}\right)}$$

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The theory is well-defined in odd dimensions

$$F_3(x) = -\frac{\sin\left(\sqrt{x}\right)}{x\sin\left(\sqrt{x}\right) + \sqrt{x}\cos\left(\sqrt{x}\right)} \approx -1 + \frac{2x}{3} - \frac{7x^2}{15} + \frac{34x^3}{105} + \ldots$$

$$F_5(x) = \frac{\cos\left(\sqrt{x}\right)}{(3-x)\cos\left(\sqrt{x}\right) + 3\sqrt{x}\sin\left(\sqrt{x}\right)} \approx \frac{1}{3} - \frac{2x}{9} + \frac{x^2}{27} + \frac{2x^3}{405} + \ldots$$

$$F_7(x) = -\frac{\sqrt{x}\sin\left(\sqrt{x}\right) + \cos\left(\sqrt{x}\right)}{(x-15)\sqrt{x}\sin\left(\sqrt{x}\right) + 3(2x-5)\cos\left(\sqrt{x}\right)} \approx \frac{1}{15} + \frac{2x}{75} - \frac{13x^2}{1125} + \frac{22x^3}{16875} + .$$

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Singularities in even dimensions

$$F_4(x) = \frac{8}{x^2} - \frac{1}{x} - \frac{2Y_3(\sqrt{x})}{x^{3/2}Y_2(\sqrt{x})} \approx \frac{1}{4}\left[-2\gamma_{\mathrm{E}} - \log(x/4)\right] + \frac{1}{8}\left[-1 + \gamma_{\mathrm{E}} + \log(x/4)\right]x + \ldots$$

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$F_1 = F$, $F_2 = -D/(4(D-1))F$, $F_3 = 0$
Trace of linearized equations

$$-\frac{(D-2)}{64\pi G}R^{(1)} = 0$$

No scalar modes!

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$$\frac{1}{32\pi G}\left[1 + F(\ell^2\bar{\Box})\ell^2\bar{\Box}\right] G^{(1)}_{ab} = 0$$

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Propagator

$$P_D(k) = \frac{64\pi G_D}{(D-2)}\left[\frac{i\ell k Y_{\frac{D+2}{2}}(i\ell k)}{Y_{\frac{D}{2}}(i\ell k)} - D\right]^{-1}$$

Poles of the propagator represent propagating modes with mass $m^2 = -k^2$

Five dimensions

$$\frac{P_5(k)}{64\pi G_5} = \frac{1}{\ell^2 k^2} + \frac{1}{3 - 3\ell k \tanh(\ell k)}$$

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Infinitely massive modes

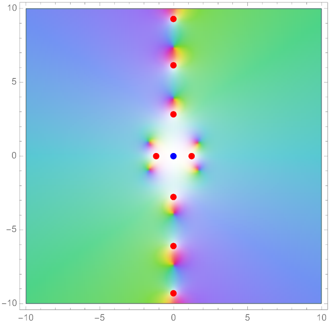
$$\Delta m_n \sim \frac{\pi}{\ell}$$

And one tachyonic mode

$$m_{\rm t}^2 \approx -\frac{1.43923}{\ell^2}$$

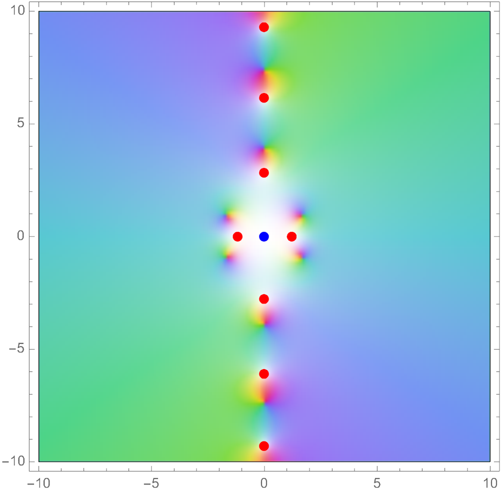
All modes have negative kinetic energy!

$$\frac{P_5(k^2 \to -m_j^2)}{64\pi G_5} = -\frac{2}{3\ell^2[k^2 + m_j^2]}$$



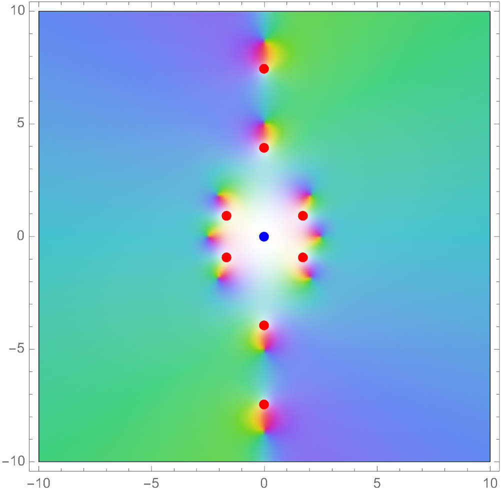
Can the coupling to the CFT alleviate this "pathological" behavior?

Higher dimensions: $D = 5$

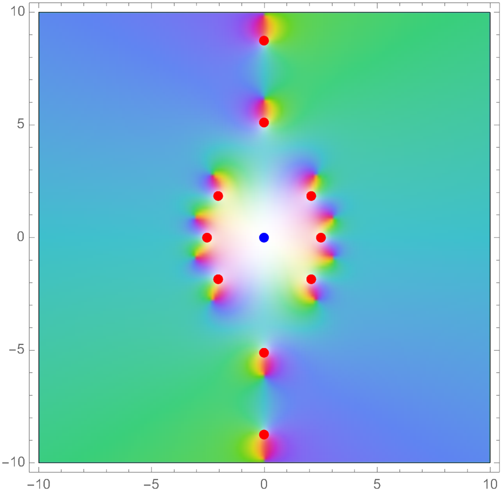


Induced Gravity on the brane

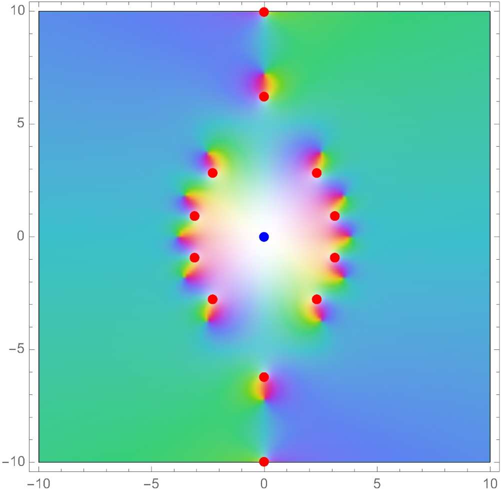
Higher dimensions: $D = 7$



Higher dimensions: $D = 9$



Higher dimensions: $D = 11$



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Induced Gravity on the brane

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The theory has negative-definite energy!

$$\mathcal{L} \to -\mathcal{L}$$

Nonlocal Massive Gravity in $D = 3$

We define NLMG as the 3D braneworld gravity with a global sign change

$$I_{\rm NLMG} = \frac{1}{16\pi G}\int \mathrm{d}^3x\sqrt{-g}\left[-R+\ell^2 R^{ab}F\left(\ell^2\Box\right)\left(R_{ab}-\frac{3}{8}g_{ab}R\right)\right]$$

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- It generalizes New Massive Gravity [Bergshoeff, Hohm, Townsend]

$$\mathcal{L} = -R + \ell^2\left(R^{ab}R_{ab} - \frac{3}{8}R^2\right) + \mathcal{O}(\ell^4)$$

- It propagates an infinite tower of positive-energy massive gravitons
- Dynamical gravity in $D = 3$

Newtonian potential

$$ds^2 = -(1 + 2\Phi)\,\mathrm{d}t^2 + (1 - 2\Psi)\,\mathrm{d}\vec{x}^2$$

Assuming a pressureless stress energy tensor with energy density $\rho = T_{tt}$, these potentials satisfy the equations

$$\begin{aligned} \Box\Psi + \frac{1}{2}F\left(\ell^2\Box\right)\ell^2\Box^2(\Phi + \Psi) &= -8\pi G\rho \\ \Box(\Psi - \Phi) &= -8\pi G\rho \end{aligned}$$

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$$\Psi - \Phi \propto \log(r/M) \quad \Rightarrow \text{large gauge transformation}$$

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For the massive gravities:

$$\Phi_{\mathrm{NMG}} = -4\pi G M K_0(mr)$$

$$\Phi_{\mathrm{NLMG}} = -4\pi G\ell M \int_0^\infty \mathrm{d}k\, J_0(kr)\tanh(\ell k)$$

NONLOCAL MASSIVE GRAVITY IN $D = 3$

Asymptotic behavior

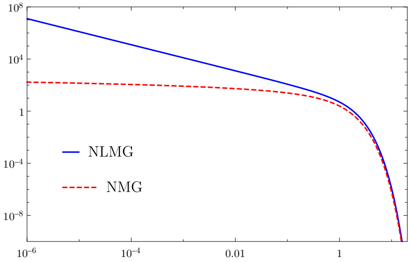
$$\Phi_{\rm NMG} \sim r^{-1/2} e^{-mr}$$

$$\Phi_{\rm NLMG} \sim r^{-1/2} e^{-m_1 r}$$

Small r behavior

$$\Phi_{\rm NMG} \sim \log(r)$$

$$\Phi_{\rm NLMG} \sim -\frac{4\pi G \ell M}{r}$$



- **We recover the $1/r$ behavior of 4 dimensions!**
- **Genuine nonlocal effect**: cannot be obtained with a finite number of derivatives
- Could there be black hole solutions?

Conclusions

- Induced gravity on the brane is an infinite-derivative (nonlocal) gravity
- These theories are not “healthy”: negative energy modes. What is the meaning of this?
- Do the massive modes agree with the Kaluza-Klein modes (in the Randall-Sundrum sense)?
- In 3D, the braneworld gravity with reversed sign is a well-behaved theory which generalizes NMG. What other aspects of the theory could we study?
- Extensions: spectrum of braneworld theories in AdS/dS, Lovelock branworlds.

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Thank you for your attention!