Conformal Bounds from Entanglement

Pablo Bueno

GRASS-SYMBHOL Meeting 2023
November 16$^{\text{th}}$ 2023

Universitat de Barcelona

Ministerio de Ciencia e Innovación
Agencia Estatal de Investigación

Based on:

- [PB, Horacio Casini, Oscar Lasso Andino, Javier Moreno]
  Phys.Rev.Lett. 131 (2023) 17, 171601

1. Introduction

Entanglement = non-separability (of quantum states)

Entanglement = non-separability (of quantum states)

Consider a finite quantum system made of two subsystems A, B in some state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

Entanglement $=$ non-separability (of quantum states)

Consider a finite quantum system made of two subsystems A, B in some state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

- If $|\psi\rangle$ can be written as $|\psi\rangle = |\phi\rangle_A \otimes |\tilde{\phi}\rangle_B \implies |\psi\rangle$ is called separable.

Entanglement = non-separability (of quantum states)

Consider a finite quantum system made of two subsystems A, B in some state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

- If $|\psi\rangle$ can be written as $|\psi\rangle = |\phi\rangle_A \otimes |\tilde{\phi}\rangle_B \implies |\psi\rangle$ is called separable.
- If $|\psi\rangle$ cannot be written as $|\psi\rangle = |\phi\rangle_A \otimes |\tilde{\phi}\rangle_B \implies |\psi\rangle$ is called entangled.

Entanglement $=$ non-separability (of quantum states)

Consider a finite quantum system made of two subsystems A, B in some state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

- If $|\psi\rangle$ can be written as $|\psi\rangle = |\phi\rangle_A \otimes |\tilde{\phi}\rangle_B \implies |\psi\rangle$ is called separable.
- If $|\psi\rangle$ cannot be written as $|\psi\rangle = |\phi\rangle_A \otimes |\tilde{\phi}\rangle_B \implies |\psi\rangle$ is called entangled.

In the latter case, the state of each subsystem cannot be fully described without the other.

Entanglement = non-separability (of quantum states)

Consider a finite quantum system made of two subsystems *A*, *B* in some state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

- If $|\psi\rangle$ can be written as $|\psi\rangle = |\phi\rangle_A \otimes |\tilde{\phi}\rangle_B \implies |\psi\rangle$ is called separable.
- If $|\psi\rangle$ cannot be written as $|\psi\rangle = |\phi\rangle_A \otimes |\tilde{\phi}\rangle_B \implies |\psi\rangle$ is called entangled.

In the latter case, the state of each subsystem cannot be fully described without the other. The two form a single inseparable entity

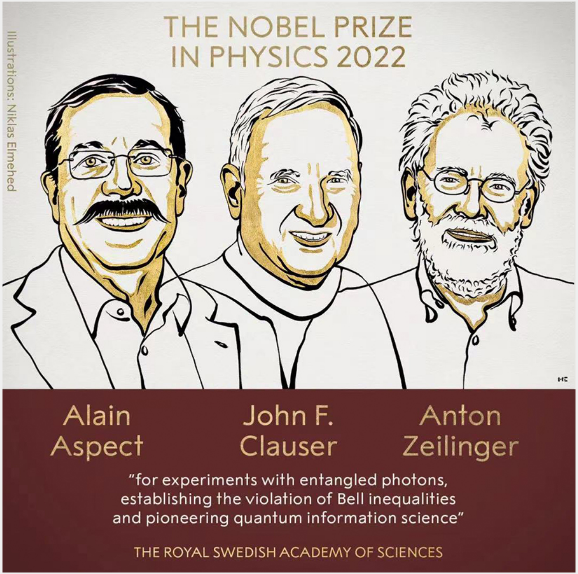
Entanglement = non-separability (of quantum states)

Consider a finite quantum system made of two subsystems A, B in some state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

- If $|\psi\rangle$ can be written as $|\psi\rangle = |\phi\rangle_A \otimes |\tilde{\phi}\rangle_B \quad \Longrightarrow \quad |\psi\rangle$ is called separable.
- If $|\psi\rangle$ cannot be written as $|\psi\rangle = |\phi\rangle_A \otimes |\tilde{\phi}\rangle_B \quad \Longrightarrow \quad |\psi\rangle$ is called entangled.

In the latter case, the state of each subsystem cannot be fully described without the other. The two form a single inseparable entity $\Leftrightarrow$ taking partial traces we loose information.

Entanglement is real!



Von Neumann entropy $\Leftrightarrow$ standard notion of entropy associated to any quantum state ρ:

$$S_{\mathrm{vN}}(\rho) \equiv -\operatorname{Tr} \rho \log \rho$$

Von Neumann entropy $\Leftrightarrow$ standard notion of entropy associated to any quantum state ρ:

$$S_{\mathrm{vN}}(\rho) \equiv -\operatorname{Tr} \rho \log \rho$$

- $S_{\mathrm{vN}}(\rho) \geq 0$ for any state.

Von Neumann entropy $\Leftrightarrow$ standard notion of entropy associated to any quantum state ρ:

$$S_{\mathrm{vN}}(\rho) \equiv -\operatorname{Tr} \rho \log \rho$$

- $S_{\mathrm{vN}}(\rho) \geq 0$ for any state.
- $S_{\mathrm{vN}}(\rho) = 0$ if ρ is pure.

Given a system composed of two subsystems A and B in some pure state ρ_{AB},

Given a system composed of two subsystems A and B in some pure state ρ_{AB}, the Entanglement Entropy of A with respect to B is defined as the von Neumann entropy of ρ_A:

Quantifying entanglement

Given a system composed of two subsystems A and B in some pure state ρ_{AB}, the Entanglement Entropy of A with respect to B is defined as the von Neumann entropy of ρ_A:

$$S(A) \equiv S_{\mathrm{vN}}(\rho_A) = -\operatorname{Tr}_A \rho_A \log \rho_A$$

where $\rho_A \equiv \operatorname{Tr}_B \rho_{AB}$ is the reduced density matrix.

Quantifying entanglement

Given a system composed of two subsystems A and B in some pure state ρ_{AB}, the Entanglement Entropy of A with respect to B is defined as the von Neumann entropy of ρ_A:

$$S(A) \equiv S_{\mathrm{vN}}(\rho_A) = -\operatorname{Tr}_A \rho_A \log \rho_A$$

where $\rho_A \equiv \operatorname{Tr}_B \rho_{AB}$ is the reduced density matrix.

- $S(A)$ quantifies "how entangled" is A with B.

Given a system composed of two subsystems A and B in some pure state ρ_{AB}, the Entanglement Entropy of A with respect to B is defined as the von Neumann entropy of ρ_A:

$$S(A) \equiv S_{\mathrm{vN}}(\rho_A) = -\operatorname{Tr}_A \rho_A \log \rho_A$$

where $\rho_A \equiv \operatorname{Tr}_B \rho_{AB}$ is the reduced density matrix.

- $S(A)$ quantifies "how entangled" is A with B.
- If ρ_{AB} is separable, ρ_A will be pure and then $S(A) = 0$.

Given a system composed of two subsystems A and B in some pure state ρ_{AB}, the Entanglement Entropy of A with respect to B is defined as the von Neumann entropy of ρ_A:

$$S(A) \equiv S_{\mathrm{vN}}(\rho_A) = -\operatorname{Tr}_A \rho_A \log \rho_A$$

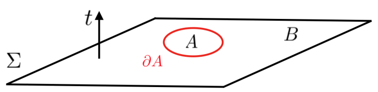
where $\rho_A \equiv \operatorname{Tr}_B \rho_{AB}$ is the reduced density matrix.

- $S(A)$ quantifies “how entangled” is A with B.
- If ρ_{AB} is separable, ρ_A will be pure and then $S(A) = 0$.
- By definition it satisfies $S(A) = S(B)$.

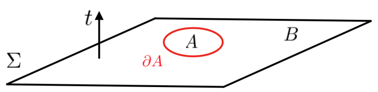
⋆ Entanglement in QFT

- The natural subsystems in QFT are spacetime regions,

- The natural subsystems in QFT are spacetime regions, *e.g.*, fix some time-slice Σ, divide it in two regions A and B:

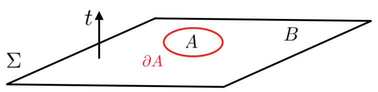


- The natural subsystems in QFT are spacetime regions, *e.g.*, fix some time-slice Σ, divide it in two regions A and B:



Associated to A there is an algebra of operators $\mathcal{A}(A)$ (associated to B there is another).

- The natural subsystems in QFT are spacetime regions, *e.g.*, fix some time-slice Σ, divide it in two regions A and B:



  Associated to A there is an algebra of operators $\mathcal{A}(A)$ (associated to B there is another).
- Given a global state and some region A, one would like to associate a density matrix to $\mathcal{A}(A)$ and compute functionals such as the EE...

Entanglement in QFT

- In QFT the entanglement entropy of subregions is divergent in any state:

$$S(A) = +\infty\,.$$

Entanglement in QFT

- In QFT the entanglement entropy of subregions is divergent in any state:

$$S(A) = +\infty\,.$$

- Any state is intrinsically and infinitely entangled across $\mathcal{A}(A)$ and $\mathcal{A}(B)$.

Entanglement in QFT

- In QFT the entanglement entropy of subregions is divergent in any state:

$$S(A) = +\infty\,.$$

- Any state is intrinsically and infinitely entangled across $\mathcal{A}(A)$ and $\mathcal{A}(B)$.
- We can either regulate the theory (e.g., in the lattice) or consider alternative well-defined measures.

- In QFT the entanglement entropy of subregions is divergent in any state:

$$S(A) = +\infty\,.$$

- Any state is intrinsically and infinitely entangled across $\mathcal{A}(A)$ and $\mathcal{A}(B)$.
- We can either regulate the theory (e.g., in the lattice) or consider alternative well-defined measures.
- In a general QFT in d dimensions, in any state, the EE of any spacetime region A has the structure:

$$S^{(d)}(A) = b_{d-2}\frac{L^{d-2}}{\delta^{d-2}} + b_{d-4}\frac{L^{d-4}}{\delta^{d-4}} + \cdots + \begin{cases} b_1 \frac{L}{\delta} + (-1)^{\frac{d-1}{2}} s^{\rm univ}\,, & (\text{odd } d)\,, \\ b_2 \frac{L^2}{\delta^2} + (-1)^{\frac{d-2}{2}} s^{\rm univ} \log\left(\frac{L}{\delta}\right) + b_0\,, & (\text{even } d)\,. \end{cases}$$

where L is some characteristic length of A and δ is a UV regulator.

Let us focus on $d = 3$ conformal field theories (CFTs).

Let us focus on $d=3$ conformal field theories (CFTs).

- The vacuum EE of some region A is given by:

$$S^{(d=3)}(A) = b_1 \cdot \frac{\mathrm{perimeter}(\partial A)}{\delta} - F(A) + \mathcal{O}(\delta)$$

Let us focus on $d = 3$ conformal field theories (CFTs).

- The vacuum EE of some region A is given by:

$$S^{(d=3)}(A) = b_1 \cdot \frac{\text{perimeter}(\partial A)}{\delta} - F(A) + \mathcal{O}(\delta)$$

- We will be interested in the universal term, $F(A)$.

Let us focus on $d = 3$ conformal field theories (CFTs).

- The vacuum EE of some region A is given by:

$$S^{(d=3)}(A) = b_1 \cdot \frac{\text{perimeter}(\partial A)}{\delta} - F(A) + \mathcal{O}(\delta)$$

- We will be interested in the universal term, $F(A)$.
- It is not possible to resolve A with more precision than the one determined by δ:

Let us focus on $d = 3$ conformal field theories (CFTs).

- The vacuum EE of some region A is given by:

$$S^{(d=3)}(A) = b_1 \cdot \frac{\text{perimeter}(\partial A)}{\delta} - F(A) + \mathcal{O}(\delta)$$

- We will be interested in the universal term, $F(A)$.
- It is not possible to resolve A with more precision than the one determined by δ: perimeter and perimeter$\cdot(1 + a\delta)$ with $a \sim \mathcal{O}(1)$ cannot be distinguished.

Let us focus on $d = 3$ conformal field theories (CFTs).

- The vacuum EE of some region A is given by:

$$S^{(d=3)}(A) = b_1 \cdot \frac{\text{perimeter}(\partial A)}{\delta} - F(A) + \mathcal{O}(\delta)$$

- We will be interested in the universal term, $F(A)$.
- It is not possible to resolve A with more precision than the one determined by δ: perimeter and perimeter$\cdot(1 + a\delta)$ with $a \sim \mathcal{O}(1)$ cannot be distinguished. This uncertainty pollutes $F(A)$ via the area-law term:

$$F(A) \to F(A) - a \cdot b_1 \cdot \text{perimeter}(\partial A)$$

In order to define $F(A)$ rigorously, we can use mutual information,

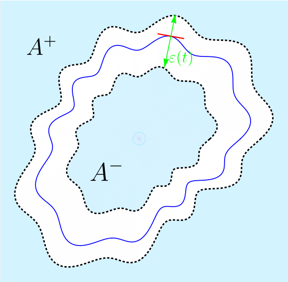
$$I(A, B) \equiv S(A) + S(B) - S(A \cup B) \, ,$$

which is well-defined in the continuum.

In order to define $F(A)$ rigorously, we can use mutual information,

$$I(A, B) \equiv S(A) + S(B) - S(A \cup B) \,,$$

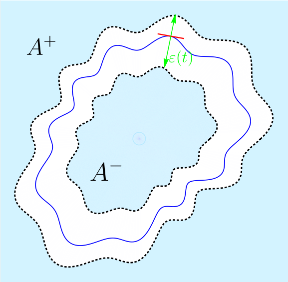
which is well-defined in the continuum.



In order to define $F(A)$ rigorously, we can use mutual information,

$$I(A,B) \equiv S(A) + S(B) - S(A \cup B)\,,$$

which is well-defined in the continuum.



Robust definition of $F(A)$:

$$I(A^+,A^-) = \kappa \int_{\partial A} \frac{\mathrm{d}s}{\varepsilon(s)} - 2F(A) + \mathcal{O}(\varepsilon)\,.$$

[Casini, Huerta, Myers, Yale]

2. Entanglement entropy shape dependence

Which shape minimizes $F(A)$?

Which shape minimizes $F(A)$?

Obvious candidate $\Leftrightarrow A =$ round disk

Which shape minimizes $F(A)$?

Obvious candidate $\Leftrightarrow A = \text{round disk}$

- Let $F_o \equiv F(\text{round disk})$.

Obvious candidate $\Leftrightarrow A =$ round disk

- Let $F_0 \equiv F(\text{round disk})$.
- This coincides with the Euclidean free energy on the round sphere for general theories: [Casini, Huerta, Myers]

$$F_0 = -\log Z_{\mathbb{S}^3} \quad \forall\ \mathrm{CFT}_3$$

Which shape minimizes $F(A)$?

Obvious candidate $\Leftrightarrow A =$ round disk

- Let $F_\circ \equiv F(\mathrm{round\ disk})$.
- This coincides with the Euclidean free energy on the round sphere for general theories: [Casini, Huerta, Myers]

$$F_\circ = -\log Z_{\mathbb{S}^3} \quad \forall\ \mathrm{CFT}_3$$

- "Natural" to expect that $F(A) \geq F_\circ\ \forall$ region A.

$$\text{Obvious candidate} \Leftrightarrow A = \text{round disk}$$

- Let $F_0 \equiv F(\text{round disk})$.
- This coincides with the Euclidean free energy on the round sphere for general theories: [Casini, Huerta, Myers]

$$F_0 = -\log Z_{\mathbb{S}^3} \quad \forall \ \mathrm{CFT}_3$$

- "Natural" to expect that $F(A) \geq F_0 \ \forall$ region A. Previous evidence from holographic theories [Alexakis, Mazzeo; Astaneh, Gibbons, Solodukhin], Obvious candidate $\Leftrightarrow$ A = round disk

- Let $F_0 \equiv F(\text{round disk})$.
- This coincides with the Euclidean free energy on the round sphere for general theories: [Casini, Huerta, Myers]

$$F_0 = -\log Z_{\mathbb{S}^3} \quad \forall\ \mathrm{CFT}_3$$

- "Natural" to expect that $F(A) \geq F_0$ $\forall$ region A. Previous evidence from holographic theories [Alexakis, Mazzeo; Astaneh, Gibbons, Solodukhin], small deformations of disk regions [Mezei], Obvious candidate $\Leftrightarrow A = \text{round disk}$

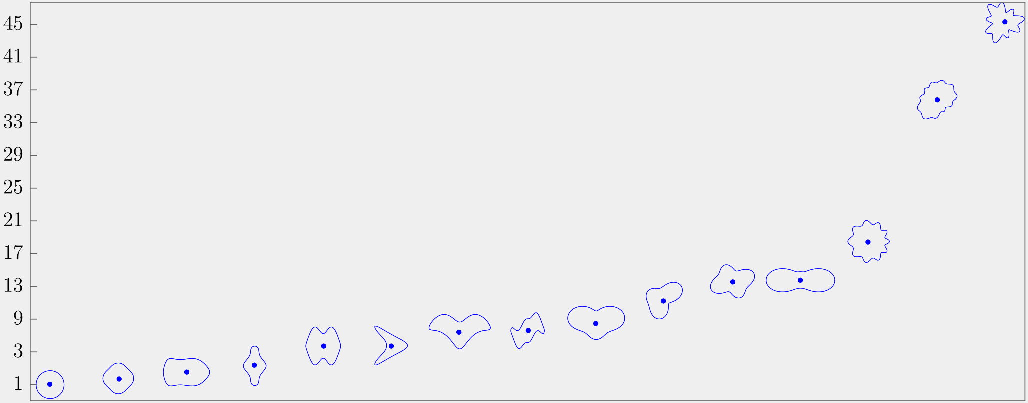
- Let $F_0 \equiv F(\text{round disk})$.
- This coincides with the Euclidean free energy on the round sphere for general theories: [Casini, Huerta, Myers]

$$F_0 = -\log Z_{\mathbb{S}^3} \quad \forall\ \mathrm{CFT}_3$$

- "Natural" to expect that $F(A) \geq F_0\ \forall$ region A. Previous evidence from holographic theories [Alexakis, Mazzeo; Astaneh, Gibbons, Solodukhin], small deformations of disk regions [Mezei], regions with sharp features,

the EMI model...

[PB, Casini, Moreno, Lasso Andino]



Disks globally maximize the EE in three dimensions

General proof from strong subaddtivitiy of EE.

[PB, Casini, Moreno, Lasso Andino]

General proof from strong subaddtivitiy of EE.

[PB, Casini, Moreno, Lasso Andino]

$$\forall\ \text{CFT}_3, \quad \forall \text{ region } A: \quad \frac{F(A)}{F_0} \geq 1,$$

General proof from strong subaddtivitiy of EE.

[PB, Casini, Moreno, Lasso Andino]

$$\forall\ \mathrm{CFT}_3, \quad \forall\ \text{region}\ A: \quad \frac{F(A)}{F_0} \geq 1\,, \quad \text{with} \quad \frac{F(A)}{F_0} = 1 \Leftrightarrow A = \text{round disk}$$

General proof from strong subaddtivitiy of EE.

[PB, Casini, Moreno, Lasso Andino]

$$\forall\ \mathrm{CFT}_3, \quad \forall\ \text{region } A: \quad \frac{F(A)}{F_0} \geq 1\,, \quad \text{with} \quad \frac{F(A)}{F_0} = 1 \Leftrightarrow A = \text{round disk}$$

For regions with $n_{\partial A}$ connected boundaries, the bound can be improved:

General proof from strong subaddtivitiy of EE.

[PB, Casini, Moreno, Lasso Andino]

$$\forall\ \mathrm{CFT}_3, \quad \forall \text{ region } A: \quad \frac{F(A)}{F_0} \geq 1\,, \quad \text{with} \quad \frac{F(A)}{F_0} = 1 \Leftrightarrow A = \text{round disk}$$

For regions with $n_{\partial A}$ connected boundaries, the bound can be improved:

$$\forall\ \mathrm{CFT}_3, \quad \forall \text{ region } A: \quad F(A) \geq n_{\partial A} F_0$$

3. Conformal bounds from entanglement

⋆ Conformal bounds

- Originally obtained from positivity of energy flux escaping at infinity for states resulting from local insertions of the stress tensor on the vacuum

- Originally obtained from positivity of energy flux escaping at infinity for states resulting from local insertions of the stress tensor on the vacuum $\Rightarrow$ constraints on **local** correlators' $\langle TT \rangle$ and $\langle TTT \rangle$ coefficients.

[Hofman, Maldacena]

Conformal bounds

- Originally obtained from positivity of energy flux escaping at infinity for states resulting from local insertions of the stress tensor on the vacuum $\Rightarrow$ constraints on **local** correlators' $\langle TT \rangle$ and $\langle TTT \rangle$ coefficients.

  [Hofman, Maldacena]

- Rigorous proof using bootstrap methods [Hofman, Li, Meltzer, Poland, Rejon-Barrera]

- Originally obtained from positivity of energy flux escaping at infinity for states resulting from local insertions of the stress tensor on the vacuum $\Rightarrow$ constraints on **local** correlators' $\langle TT \rangle$ and $\langle TTT \rangle$ coefficients.

  [Hofman, Maldacena]

- Rigorous proof using bootstrap methods [Hofman, Li, Meltzer, Poland, Rejon-Barrera]

Prototypical example in $d = 4$ for trace-anomaly coefficients

$$\langle T^{\mu}_{\mu} \rangle = -\frac{a}{16\pi^2}\mathcal{X}_4 + \frac{c}{16\pi^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$$

- Originally obtained from positivity of energy flux escaping at infinity for states resulting from local insertions of the stress tensor on the vacuum $\Rightarrow$ constraints on **local** correlators' $\langle TT\rangle$ and $\langle TTT\rangle$ coefficients.

  [Hofman, Maldacena]

- Rigorous proof using bootstrap methods [Hofman, Li, Meltzer, Poland, Rejon-Barrera]

Prototypical example in $d = 4$ for trace-anomaly coefficients

$$\langle T^{\mu}_{\mu}\rangle = -\frac{a}{16\pi^2}\mathcal{X}_4 + \frac{c}{16\pi^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$$

Universal bound:

$$\boxed{\left.\frac{c}{a}\right|_{\textbf{Maxwell}} \leq \frac{c}{a} \leq \left.\frac{c}{a}\right|_{\textbf{free scalar}} \qquad \forall\ \mathrm{CFT}_4}$$

3.1 A new conjecture

- **Conjecture:**

[PB, Casini, Moreno, Lasso Andino]

$$\left.\frac{F(A)}{F_0}\right|_{\text{Maxwell}} \leq \frac{F(A)}{F_0} \leq \left.\frac{F(A)}{F_0}\right|_{\text{free scalar}} \quad \forall\ \text{CFT}_3 \quad \forall\ \text{region } A$$

3.2 Hints from four dimensions

In $d = 4$, the universal term is local in nature and appears as the coefficient of a logarithmic divergence. [Solodukhin; Perlmutter, Rangamani, Rota]

$$\frac{S^{4d}_{\text{univ}}(A)}{a} = \frac{1}{\pi}\left[\mathcal{W}_{\partial A} + \left(\frac{c}{a} - 1\right)\frac{\mathcal{K}_{\partial A}}{2}\right],$$

In $d = 4$, the universal term is local in nature and appears as the coefficient of a logarithmic divergence. [Solodukhin; Perlmutter, Rangamani, Rota]

$$\frac{S^{4d}_{\text{univ}}(A)}{a} = \frac{1}{\pi}\left[\mathcal{W}_{\partial A} + \left(\frac{c}{a} - 1\right)\frac{\mathcal{K}_{\partial A}}{2}\right],$$

$\mathcal{W}_{\partial A}$ and $\mathcal{K}_{\partial A}$ are fixed positive definite and positive semidefinite respectively,

In $d = 4$, the universal term is local in nature and appears as the coefficient of a logarithmic divergence. [Solodukhin; Perlmutter, Rangamani, Rota]

$$\frac{S^{4d}_{\text{univ}}(A)}{a} = \frac{1}{\pi}\left[\mathcal{W}_{\partial A} + \left(\frac{c}{a} - 1\right)\frac{\mathcal{K}_{\partial A}}{2}\right],$$

$\mathcal{W}_{\partial A}$ and $\mathcal{K}_{\partial A}$ are fixed positive definite and positive semidefinite respectively, so an analogous conjecture to ours:

$$\left.\frac{S^{4d}_{\text{univ}}(A)}{a}\right|_{\textbf{Maxwell}} \leq \frac{S^{4d}_{\text{univ}}(A)}{a} \leq \left.\frac{S^{4d}_{\text{univ}}(A)}{a}\right|_{\textbf{free scalar}} \qquad \forall\ \text{CFT}_4 \quad \forall\ \text{region } A$$

In $d = 4$, the universal term is local in nature and appears as the coefficient of a logarithmic divergence. [Solodukhin; Perlmutter, Rangamani, Rota]

$$\frac{S^{4d}_{\rm univ}(A)}{a} = \frac{1}{\pi}\left[\mathcal{W}_{\partial A} + \left(\frac{c}{a} - 1\right)\frac{\mathcal{K}_{\partial A}}{2}\right],$$

$\mathcal{W}_{\partial A}$ and $\mathcal{K}_{\partial A}$ are fixed positive definite and positive semidefinite respectively, so an analogous conjecture to ours:

$$\left.\frac{S^{4d}_{\rm univ}(A)}{a}\right|_{\textbf{Maxwell}} \leq \frac{S^{4d}_{\rm univ}(A)}{a} \leq \left.\frac{S^{4d}_{\rm univ}(A)}{a}\right|_{\textbf{free scalar}} \quad \forall\ \mathrm{CFT}_4 \quad \forall\ \text{region } A$$

is trivially equivalent to the HM bounds!

3.3 Orbifold theories and multicomponent regions

Consider a theory $\mathbb{O}$ obtained from quotienting some complete theory $\mathbb{C}$ by some finite symmetry group G.

Consider a theory O obtained from quotienting some complete theory C by some finite symmetry group G.

- The mutual information is given by [Casini, Huerta, Magan, Pontello]

$$I^{\mathrm{O}}(A^+, A^-) = I^{\mathrm{C}}(A^+, A^-) - n_{\partial A} \log |G| + \Delta\,, \qquad \Delta \geq 0$$

Consider a theory O obtained from quotienting some complete theory C by some finite symmetry group G.

- The mutual information is given by [Casini, Huerta, Magan, Pontello]

$$I^{\mathrm{O}}(A^+, A^-) = I^{\mathrm{C}}(A^+, A^-) - n_{\partial A} \log |G| + \Delta\,, \qquad \Delta \geq 0$$

- Using the definition of $F(A)$ from the MI, one finds

$$n_{\partial A} \leq \left.\frac{F(A)}{F_0}\right|_{\mathrm{O}} \leq \frac{F(A)|_{\mathrm{C}} + \frac{n_{\partial A}}{2} \log |G|}{F_0|_{\mathrm{C}} + \frac{1}{2} \log |G|} \leq \left.\frac{F(A)}{F_0}\right|_{\mathrm{C}}$$

Consider a theory $\mathbb{O}$ obtained from quotienting some complete theory $\mathbb{C}$ by some finite symmetry group G.

- The mutual information is given by [Casini, Huerta, Magan, Pontello]

$$I^{\mathbb{O}}(A^+, A^-) = I^{\mathbb{C}}(A^+, A^-) - n_{\partial A} \log|G| + \Delta\,, \qquad \Delta \geq 0$$

- Using the definition of $F(A)$ from the MI, one finds

$$n_{\partial A} \leq \left.\frac{F(A)}{F_0}\right|_{\mathbb{O}} \leq \frac{F(A)|_{\mathbb{C}} + \frac{n_{\partial A}}{2} \log|G|}{F_0|_{\mathbb{C}} + \frac{1}{2} \log|G|} \leq \left.\frac{F(A)}{F_0}\right|_{\mathbb{C}}$$

- Hence, the ratio $F(A)/F_0$ for the parent theory is always greater than the one for the orbifold theory.

- The same happens for infinite symmetry groups.

- The same happens for infinite symmetry groups.
- The Maxwell theory is an orbifold of the free scalar under $\mathbb{R}$ implementing $\phi \to \phi + \delta$.

- The same happens for infinite symmetry groups.
- The Maxwell theory is an orbifold of the free scalar under $\mathbb{R}$ implementing $\phi \to \phi + \delta$.
- One has: $F(A)|_{\textbf{Maxwell}} = F(A)|_{\textbf{free scalar}} + n_{\partial A}/4 \log(-\log(\delta))$,

- The same happens for infinite symmetry groups.
- The Maxwell theory is an orbifold of the free scalar under $\mathbb{R}$ implementing $\phi \to \phi + \delta$.
- One has: $F(A)|_{\textbf{Maxwell}} = F(A)|_{\textbf{free scalar}} + n_{\partial A}/4 \log(-\log(\delta))$, and from this:

$$\left.\frac{F(A)}{F_0}\right|_{\textbf{Maxwell}} = n_{\partial A}$$

- The same happens for infinite symmetry groups.
- The Maxwell theory is an orbifold of the free scalar under $\mathbb{R}$ implementing $\phi \to \phi + \delta$.
- One has: $F(A)|_{\textbf{Maxwell}} = F(A)|_{\textbf{free scalar}} + n_{\partial A}/4 \log(-\log(\delta))$, and from this:

$$\left.\frac{F(A)}{F_0}\right|_{\textbf{Maxwell}} = n_{\partial A}$$

- Hence, the lower bound is equivalent to the improved general bound for topologically non-trivial regions.

Conjecture

$$\left.\frac{F(A)}{F_0}\right|_{\textbf{Maxwell}} \leq \frac{F(A)}{F_0} \leq \left.\frac{F(A)}{F_0}\right|_{\textbf{free scalar}} \qquad \forall \text{ CFT}_3 \quad \forall \text{ region } A$$

Conjecture

$$n_{\partial A} \leq \frac{F(A)}{F_0} \leq \frac{F(A)}{F_0}\bigg|_{\textbf{free scalar}} \qquad \forall\ \mathrm{CFT}_3 \quad \forall\ \text{region } A$$

Conjecture

$$\boxed{n_{\partial A} \leq \frac{F(A)}{F_0} \leq \left.\frac{F(A)}{F_0}\right|_{\textbf{free scalar}} \qquad \forall\ \mathrm{CFT}_3 \quad \forall\ \text{region } A}$$

Lower bound not conjectural

$$\boxed{n_{\partial A} \leq \frac{F(A)}{F_0} \leq \left.\frac{F(A)}{F_0}\right|_{\textbf{free scalar}} \qquad \forall\ \mathrm{CFT}_3 \quad \forall\ \text{region } A}$$

Lower bound not conjectural (follows from the general shape-dependence results).

3.4 Disconnected regions

Disconnected components and large separations

- Consider region with two disconnected components: $A = A_1 \cup A_2$.

Disconnected components and large separations

- Consider region with two disconnected components: $A = A_1 \cup A_2$.
- Then, $S(A_1 \cup A_2) = S(A_1) + S(A_2) - I(A_1, A_2)$.

- Consider region with two disconnected components: $A = A_1 \cup A_2$.
- Then, $S(A_1 \cup A_2) = S(A_1) + S(A_2) - I(A_1, A_2)$. If A_1 and A_2 are disks:

$$\frac{F(A_1 \cup A_2)}{F_0} = 2 + \frac{I(A_1, A_2)}{F_0}$$

- Consider region with two disconnected components: $A = A_1 \cup A_2$.
- Then, $S(A_1 \cup A_2) = S(A_1) + S(A_2) - I(A_1, A_2)$. If A_1 and A_2 are disks:

$$\frac{F(A_1 \cup A_2)}{F_0} = 2 + \frac{I(A_1, A_2)}{F_0}$$

- For long separations, free scalar provides greatest value of $I(A_1, A_2)$.

- Consider region with two disconnected components: $A = A_1 \cup A_2$.
- Then, $S(A_1 \cup A_2) = S(A_1) + S(A_2) - I(A_1, A_2)$. If A_1 and A_2 are disks:

$$\frac{F(A_1 \cup A_2)}{F_0} = 2 + \frac{I(A_1, A_2)}{F_0}$$

- For long separations, free scalar provides greatest value of $I(A_1, A_2)$. In general: [Cardy; Agon, Faulkner]

$$I(A_1, A_2) \sim |r_{A_1} - r_{A_2}|^{-4\Delta_{\mathrm{CFT}_3}} \quad \text{where} \quad \Delta_{\mathrm{CFT}_3} \equiv \text{smallest scaling dimension}$$

- Consider region with two disconnected components: $A = A_1 \cup A_2$.
- Then, $S(A_1 \cup A_2) = S(A_1) + S(A_2) - I(A_1, A_2)$. If A_1 and A_2 are disks:

$$\frac{F(A_1 \cup A_2)}{F_0} = 2 + \frac{I(A_1, A_2)}{F_0}$$

- For long separations, free scalar provides greatest value of $I(A_1, A_2)$. In general: [Cardy; Agon, Faulkner]

  $I(A_1, A_2) \sim |r_{A_1} - r_{A_2}|^{-4\Delta_{\mathrm{CFT}_3}}$ where $\Delta_{\mathrm{CFT}_3} \equiv$ smallest scaling dimension

- Now, $\forall$ CFT_3 one has

- Consider region with two disconnected components: $A = A_1 \cup A_2$.
- Then, $S(A_1 \cup A_2) = S(A_1) + S(A_2) - I(A_1, A_2)$. If A_1 and A_2 are disks:

$$\frac{F(A_1 \cup A_2)}{F_0} = 2 + \frac{I(A_1, A_2)}{F_0}$$

- For long separations, free scalar provides greatest value of $I(A_1, A_2)$. In general: [Cardy; Agon, Faulkner]

$$I(A_1, A_2) \sim |r_{A_1} - r_{A_2}|^{-4\Delta_{\mathrm{CFT}_3}} \quad \text{where} \quad \Delta_{\mathrm{CFT}_3} \equiv \text{smallest scaling dimension}$$

- Now, $\forall$ CFT_3 one has

$$\Delta_{\mathrm{CFT}_3} \geq \Delta_{\text{free scalar}} = \frac{(d-2)}{2} \qquad \text{(unitarity bound)}$$

Disconnected components and large separations

- Consider region with two disconnected components: $A = A_1 \cup A_2$.
- Then, $S(A_1 \cup A_2) = S(A_1) + S(A_2) - I(A_1, A_2)$. If A_1 and A_2 are disks:

$$\frac{F(A_1 \cup A_2)}{F_0} = 2 + \frac{I(A_1, A_2)}{F_0}$$

- For long separations, free scalar provides greatest value of $I(A_1, A_2)$. In general: [Cardy; Agon, Faulkner]

$$I(A_1, A_2) \sim |r_{A_1} - r_{A_2}|^{-4\Delta_{\mathrm{CFT}_3}} \quad \text{where} \quad \Delta_{\mathrm{CFT}_3} \equiv \text{smallest scaling dimension}$$

- Now, $\forall$ CFT_3 one has

$$\Delta_{\mathrm{CFT}_3} \geq \Delta_{\text{free scalar}} = \frac{(d-2)}{2} \qquad \text{(unitarity bound)}$$

- Then, $F(A_1 \cup A_2)/F_0$ is absolutely maximized by the free scalar.

- Consider region with two disconnected components: $A = A_1 \cup A_2$.
- Then, $S(A_1 \cup A_2) = S(A_1) + S(A_2) - I(A_1, A_2)$. If A_1 and A_2 are disks:

$$\frac{F(A_1 \cup A_2)}{F_0} = 2 + \frac{I(A_1, A_2)}{F_0}$$

- For long separations, free scalar provides greatest value of $I(A_1, A_2)$. In general: [Cardy; Agon, Faulkner]

$$I(A_1, A_2) \sim |r_{A_1} - r_{A_2}|^{-4\Delta_{\mathrm{CFT}_3}} \quad \text{where} \quad \Delta_{\mathrm{CFT}_3} \equiv \text{smallest scaling dimension}$$

- Now, $\forall$ CFT_3 one has

$$\Delta_{\mathrm{CFT}_3} \geq \Delta_{\text{free scalar}} = \frac{(d-2)}{2} \qquad \text{(unitarity bound)}$$

- Then, $F(A_1 \cup A_2)/F_0$ is absolutely maximized by the free scalar.
- Also holds for general shapes if it holds for A_1 and A_2 individually.

- Let $A_1(\delta, \epsilon)$ be the causal cone of some disk region A to which one has removed a conical frustrum of angle ϵ and radial height δ.

- Let $A_1(\delta, \epsilon)$ be the causal cone of some disk region A to which one has removed a conical frustrum of angle ϵ and radial height δ. Let A_2 be some other disk region and $A = A_1(\delta, \epsilon) \cup A_2$.

- Let $A_1(\delta,\epsilon)$ be the causal cone of some disk region A to which one has removed a conical frustrum of angle ϵ and radial height δ. Let A_2 be some other disk region and $A = A_1(\delta,\epsilon) \cup A_2$. Then, the so-called “pinching property” implies that: [Casini, Teste, Torroba]

$$\frac{F(\lim_{\epsilon\to 0}\lim_{\delta\to 0} A)}{F_0} = 2\,, \qquad \text{(interacting CFTs)}$$

$$\frac{F(\lim_{\epsilon\to 0}\lim_{\delta\to 0} A)}{F_0} = 2 + \frac{I(A_1,A_2)}{F_0}\,, \qquad \text{(free CFTs)}$$

- Let $A_1(\delta, \epsilon)$ be the causal cone of some disk region A to which one has removed a conical frustrum of angle ϵ and radial height δ. Let A_2 be some other disk region and $A = A_1(\delta, \epsilon) \cup A_2$. Then, the so-called "pinching property" implies that: [Casini, Teste, Torroba]

$$\frac{F(\lim_{\epsilon \to 0} \lim_{\delta \to 0} A)}{F_0} = 2\,, \qquad \text{(interacting CFTs)}$$

$$\frac{F(\lim_{\epsilon \to 0} \lim_{\delta \to 0} A)}{F_0} = 2 + \frac{I(A_1, A_2)}{F_0}\,, \qquad \text{(free CFTs)}$$

regardless of the separation between A_1 and A_2.

- Let $A_1(\delta,\epsilon)$ be the causal cone of some disk region A to which one has removed a conical frustrum of angle ϵ and radial height δ. Let A_2 be some other disk region and $A = A_1(\delta,\epsilon) \cup A_2$. Then, the so-called "pinching property" implies that: [Casini, Teste, Torroba]

$$\frac{F(\lim_{\epsilon\to 0}\lim_{\delta\to 0} A)}{F_0} = 2\,, \qquad \text{(interacting CFTs)}$$

$$\frac{F(\lim_{\epsilon\to 0}\lim_{\delta\to 0} A)}{F_0} = 2 + \frac{I(A_1,A_2)}{F_0}\,, \qquad \text{(free CFTs)}$$

regardless of the separation between A_1 and A_2.

- In this case, $F(A)/F_0$ is smaller for any interacting CFT than for any free one.

- Now, strong numerical evidence suggests that:

[Agon, PB, Lasso Andino, Vilar Lopez]

$$\left.\frac{I(A_1, A_2)}{F_0}\right|_{\text{free fermion}} < \left.\frac{I(A_1, A_2)}{F_0}\right|_{\text{free scalar}}$$

for arbitrary spatial regions A_1, A_2.

- Now, strong numerical evidence suggests that:

  [Agon, PB, Lasso Andino, Vilar Lopez]

$$\left.\frac{I(A_1,A_2)}{F_0}\right|_{\text{free fermion}} < \left.\frac{I(A_1,A_2)}{F_0}\right|_{\text{free scalar}}$$

  for arbitrary spatial regions A_1, A_2.
- Once again the free scalar provides an absolute maximum for $F(A)/F_0$.

3.5 Connected regions

⋆ Small deformations of a disk region

- Consider general slightly deformed disks

$$\frac{r(\theta)}{R} = 1 + \frac{\epsilon}{\sqrt{\pi}} \sum_{\ell} [a_{\ell,(c)} \cos(\ell\theta) + a_{\ell,(s)} \sin(\ell\theta)] \,, \quad (\epsilon \ll 1)$$

- Consider general slightly deformed disks

$$\frac{r(\theta)}{R} = 1 + \frac{\epsilon}{\sqrt{\pi}} \sum_{\ell} [a_{\ell,(c)} \cos(\ell\theta) + a_{\ell,(s)} \sin(\ell\theta)] \,, \quad (\epsilon \ll 1)$$

- Then, at leading order in ϵ, we have [Mezei; Faulkner, Leigh, Parrikar]

$$\frac{F(A)}{F_0} = 1 + \frac{\pi^3}{24} \frac{C_T}{F_0} \sum_{\ell} \ell(\ell^2 - 1) \left[a^2_{\ell,(c)} + a^2_{\ell,(s)}\right] \epsilon^2 \,,$$

- Consider general slightly deformed disks

$$\frac{r(\theta)}{R} = 1 + \frac{\epsilon}{\sqrt{\pi}} \sum_{\ell} [a_{\ell,(c)} \cos(\ell\theta) + a_{\ell,(s)} \sin(\ell\theta)] \,, \quad (\epsilon \ll 1)$$

- Then, at leading order in ϵ, we have [Mezei; Faulkner, Leigh, Parrikar]

$$\frac{F(A)}{F_0} = 1 + \frac{\pi^3}{24} \frac{C_T}{F_0} \sum_{\ell} \ell(\ell^2 - 1) \left[a^2_{\ell,(c)} + a^2_{\ell,(s)}\right] \epsilon^2 \,,$$

where C_T controls, for a general CFT, the stress-tensor two-point function,

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle_{\mathbb{R}^3} = \frac{C_T}{x^6} \left[I_{\mu(\rho} I_{\sigma)\nu} - \frac{\delta_{\mu\nu}\delta_{\rho\sigma}}{3} \right] \,,$$

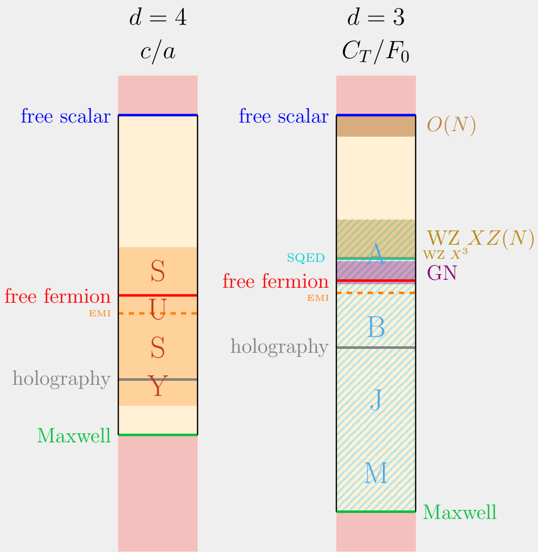
From our general conjecture it follows that:

$$0 \leq \frac{C_T}{F_0} \leq \left.\frac{C_T}{F_0}\right|_{\text{free scalar}} = \frac{3}{4\pi^2 \log 2 - 6\zeta[3]} \simeq 0.14887\ldots$$

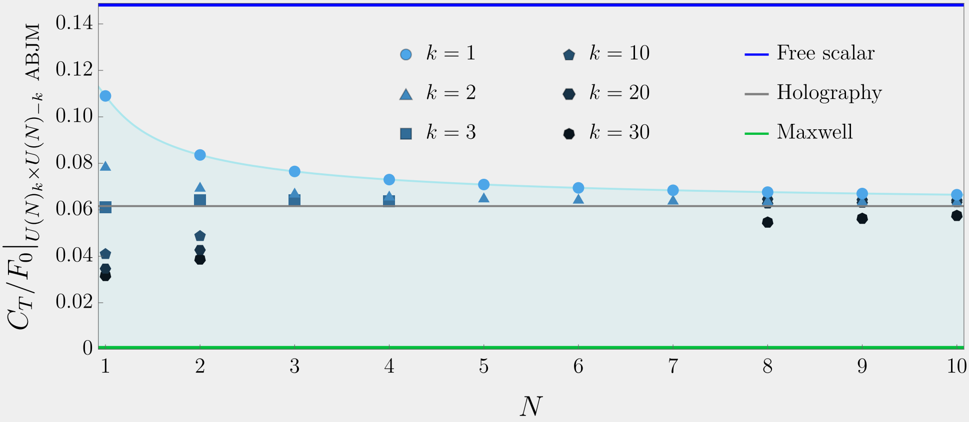
From our general conjecture it follows that:

$$0 \leq \frac{C_T}{F_0} \leq \frac{C_T}{F_0}\bigg|_{\text{free scalar}} = \frac{3}{4\pi^2 \log 2 - 6\zeta[3]} \simeq 0.14887\ldots$$

- New three-dimensional version of HM bounds!


$d = 4$
c/a
free scalar
S
free fermion
EMI
U
S
holography
Y
Maxwell
$d = 3$
C_T/F_0
free scalar
$O(N)$
WZ $XZ(N)$
SQED
A
WZ X^3
GN
free fermion
EMI
B
holography
J
M
Maxwell

ABJM model

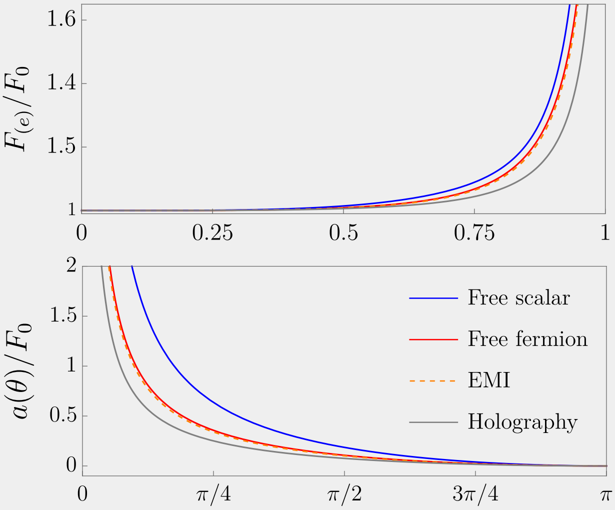

0.14
0.12
0.10
0.08
0.06
0.04
0.02
0
$C_T/F_0|_{U(N)_k \times U(N)_{-k}}$ ABJM
1
2
3
4
5
6
7
8
9
10
N
$k = 1$
$k = 2$
$k = 3$
$k = 10$
$k = 20$
$k = 30$
Free scalar
Holography
Maxwell

⋆ Ellipses and corners

- Moving from the perturbed-disks regime, values of $F(A)/F_0$ for more complicated regions exist in some cases, at least for a few theories.

- Moving from the perturbed-disks regime, values of $F(A)/F_0$ for more complicated regions exist in some cases, at least for a few theories.
- The conjectural bounds are always satisfied.

- Moving from the perturbed-disks regime, values of $F(A)/F_0$ for more complicated regions exist in some cases, at least for a few theories.
- The conjectural bounds are always satisfied.



4. Future

Future

- Find additional evidence/general proof/counterexample

Future

- Find additional evidence/general proof/counterexample
- More restrictive upper bound for SUSY theories?

Future

- Find additional evidence/general proof/counterexample
- More restrictive upper bound for SUSY theories?
- Analogous conjecture in $d = 5$ CFTs?

Future

- Find additional evidence/general proof/counterexample
- More restrictive upper bound for SUSY theories?
- Analogous conjecture in $d = 5$ CFTs?
- Bounds on other ratios of seemingly unrelated universal quantities?

The End