Carroll Fermions

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based upon work done with

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Motivation

Carroll symmetries refer to a particular Inönü-Wigner contraction of the Poincaré group in which the speed of light c is taken to zero

Levy-Leblond (1965); Gupta (1966)

- conformal Carroll = BMS

  Duval, Gibbons, Horvathy (2014)
- null hypersurfaces
- flat space and celestial holography

  lectures by Pasterski (2021), Raclariu (2021) and review by Donnay (2023)
- black hole horizons

  Donnay, Marteau (2019)
- dark matter, cosmology

  de Boer, Hartong, Obers, Sybesma, Vandoren (2021) and (2023)
- hydrodynamics

  Ciambelli, Marteau, Petkou, Petropoulos, Siampos (2018)
- tensionless limit of strings

  Bagchi, Chakrabortty and Parekh (2016)

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Electric Carroll Scalars

Our starting point is the following Lagrangian in Hamiltonian form for a 4D relativistic free real scalar Φ with mass M:

$$\mathcal{L} = \frac{1}{c}\Pi_\Phi \partial_t \Phi - \frac{1}{2}\Pi_\Phi^2 - \frac{1}{2}\partial_a \Phi \partial^a \Phi \mp \frac{M^2 c^2}{2}\Phi^2 \,, \qquad a = 1, 2, 3$$

After taking the upper sign, making the following redefinitions

$$\Pi_\Phi = \pi \,, \qquad \Phi = c\,\phi \,, \qquad M = \frac{m}{c^2}$$

and taking the limit that $c \to 0$, we obtain the following Lagrangian describing an electric Carroll scalar :

$$\mathcal{L}_{\text{el. scalar}} = \pi \partial_t \phi - \frac{1}{2}\pi^2 - \frac{m^2}{2}\phi^2 \quad \text{or} \quad \mathcal{L}_{\text{el. scalar}} = \frac{1}{2}\left(\partial_t \phi\right)^2 - \frac{m^2}{2}\phi^2$$

The electric Carroll particle has non-zero energy but cannot move

Magnetic Carroll Scalars

de Boer, Hartong, Obers, Sybesma, Vandoren (2021)

We start from the same Lagrangian

$$\mathcal{L} = \frac{1}{c}\Pi_\Phi \partial_t \Phi - \frac{1}{2}\Pi_\Phi^2 - \frac{1}{2}\partial_a \Phi \partial^a \Phi \mp \frac{M^2 c^2}{2}\Phi^2 \,, \qquad a = 1, 2, 3$$

but now take the lower sign and redefine

$$\Pi_\Phi = c\,\pi \,, \qquad \Phi = \phi \,, \qquad M = m$$

Taking the limit that $c \to 0$ we thus obtain the following Lagrangian for a magnetic Carroll scalar :

$$\mathcal{L}_{\text{magn. scalar}} = \pi \partial_t \phi - \frac{1}{2}\partial_a \phi \partial^a \phi + \frac{m^2}{2}\phi^2$$

under Carroll boosts: $\pi \quad \to \quad \phi \quad \to \quad 0$: reducible but undecomposable representation

The magnetic Carroll particle can move but has zero energy

Outline

Earlier work

- various limit procedures: on equations of motion, in Lagrangian or Hamiltonian

Bagchi, Mehra, Nandi (2019); Bagchi, Grümiller, Nandi (2022)
Banerjee, Dutta, Mondal (2023); Koutrolikos, Najfizadeh (2023)
L. Mele (Master thesis 2023)

- using degenerate Clifford algebras

Bagchi, A. Banerjee, R. Basu, M. Islam and S. Mondal (2023)
Stakenborg (Master thesis 2023)

our contribution

- obstacle: for relativistic fermion we have $\Pi_\Psi \sim \Psi$

  resolution: start from two Dirac spinors and decompose each of them into two independent projections $\rightarrow$ 4 independent spinors

- we show that in even dimensions truncations to a minimal formulation are possible

- generalization to curved background using results on Carroll geometry

Un-projected Carroll Fermions

Our starting point is a relativistic complex Dirac spinor Ψ in D-dimensional Minkowski spacetime with Lorentz transformation rule

$$\delta\Psi(x) = \Xi^A \partial_A \Psi(x) - \frac{1}{4}\Lambda_{AB}\Gamma^{AB}\Psi(x)$$

with

$$\delta X^A \equiv X'^A - X^A = -\Xi^A\,, \qquad \Xi^A = \Lambda^A{}_B X^B$$

To obtain a Carroll fermion we decompose $A = (0, a)$, redefine the parameters and coordinates as follows:

$$\Lambda^{ab} = \lambda^{ab}\,, \qquad \Lambda^{0a} = \frac{1}{\tilde{c}}\lambda^{0a}\,, \qquad X^0 = \frac{t}{\tilde{c}}\,, \qquad X^a = x^a$$

and take the limit that $\tilde{c} \equiv 1/c \to \infty$. In this way we obtain a Carroll fermion $\Psi = \psi$ with transformation rule

$$\delta\psi = \xi^0 \frac{\partial\psi}{\partial t} + \xi^a \frac{\partial\psi}{\partial x^a} - \frac{1}{4}\lambda^{ab}\Gamma_{ab}\psi\,,$$

where the parameters ξ^0, ξ^a parametrize spatial rotations and Carroll boosts

Projected Carroll Fermions

To obtain non-trivial internal Carroll boosts we decompose the Dirac spinor Ψ, covariant w.r.t. spatial rotations, into two independent components $\Psi_\pm$ and redefine these two components differently as follows:

$$\Psi_\pm = \tilde{c}^{\pm 1/2+\epsilon}\,\frac{1}{2}\left(1 \pm i\Gamma^0\right)\psi_\pm \qquad \leftrightarrow \qquad \psi_\pm = \tilde{c}^{\mp 1/2-\epsilon}\,\frac{1}{2}\left(1 \pm i\Gamma^0\right)\Psi_\pm$$

$$\Downarrow$$

$$\begin{aligned}
\delta\psi_+ &= \xi^0\frac{\partial\psi_+}{\partial t} + \xi^a\frac{\partial\psi_+}{\partial x^a} - \frac{1}{4}\lambda^{ab}\Gamma_{ab}\psi_+\,, \\
\delta\psi_- &= \xi^0\frac{\partial\psi_-}{\partial t} + \xi^a\frac{\partial\psi_-}{\partial x^a} - \frac{1}{4}\lambda^{ab}\Gamma_{ab}\psi_- - \frac{1}{2}\lambda^{0a}\Gamma_{0a}\psi_+
\end{aligned}$$

The projected spinors $\psi_\pm$ form a reducible but indecomposable representation of the homogeneous Carroll group

A Parent Lagrangian

Our starting point is the following off-diagonal Lagrangian for two Dirac spinors Ψ and $\mathbf{X}$:

$$\mathcal{L}_{\text{off-diag}} = \bar{\mathbf{X}}\Gamma^A\partial_A\Psi - \frac{M}{\tilde{c}}\bar{\mathbf{X}}\Psi + \text{h.c.}\,,$$

where M is a complex parameter with the dimension of mass

To create sufficient freedom for defining two different limits we introduce the following projected spinors

$$\Psi_\pm = P_\pm\Psi\,, \qquad \mathbf{X}_\pm = P_\pm\mathbf{X}\,,$$

so that we have four projected spinors that we can scale independently

alternative method : start from a single Dirac Lagrangian and rewrite it in Hamiltonian form using Lagrange multipliers imposing the second-class constraints

The Electric Carroll Limit

The electric Carroll limit is defined by the rescaling $M = \tilde{c}^2 m$ together with:

$$\Psi_+ = \sqrt{\tilde{c}}\,\tilde{c}^{-1}\psi_+\,, \qquad \Psi_- = \frac{1}{\sqrt{\tilde{c}}}\,\tilde{c}^{-1}\psi_-\,,$$

$$\mathrm{X}_+ = \sqrt{\tilde{c}}\,\tilde{c}^{-1}\chi_+\,, \qquad \mathrm{X}_- = \frac{1}{\sqrt{\tilde{c}}}\,\tilde{c}^{-1}\chi_-\,.$$

After taking the limit that $\tilde{c} \to \infty$ the spinors χ_- and ψ_- drop out and we find

$$\mathcal{L}_{\text{off-diag}} = \bar{\chi}_+\Gamma^0\dot{\psi}_+ - m\bar{\chi}_+\psi_+ + \text{h.c.}$$

with

$$\delta\psi_+ = \xi^0\dot{\psi}_+ + \xi^a\partial_a\psi_+ - \frac{1}{4}\lambda_{ab}\Gamma^{ab}\psi_+\,,$$

$$\delta\chi_+ = \xi^0\dot{\chi}_+ + \xi^a\partial_a\chi_+ - \frac{1}{4}\lambda_{ab}\Gamma^{ab}\chi_+\,.$$

This suggests the truncation $\chi_+ = \psi_+$ after which we obtain the following electric Carroll Lagrangian in diagonal form:

$$\boxed{\mathcal{L}_{\text{electric Carroll}} = 2\bar{\psi}_+\Gamma^0\dot{\psi}_+ - \mathfrak{Re}(m)\bar{\psi}_+\psi_+}$$

Bagchi, Grumiller, Nandi (2022)

where $\mathfrak{Re}(m) = m + m^*$ is a real mass parameter

The Magnetic Carroll Limit I

The magnetic Carroll limit is defined by the rescaling $M = \tilde{c}^2 m$ together with the following twisted rescalings:

$$\Psi_+ = \sqrt{\tilde{c}}\,\tilde{c}^{-1/2}\psi_+\,, \qquad \Psi_- = \frac{1}{\sqrt{\tilde{c}}}\,\tilde{c}^{-1/2}\psi_-\,,$$

$$\mathrm{X}_+ = \frac{1}{\sqrt{\tilde{c}}}\,\tilde{c}^{-1/2}\chi_+\,, \qquad \mathrm{X}_- = \sqrt{\tilde{c}}\,\tilde{c}^{-1/2}\chi_-\,.$$

After taking the limit $\tilde{c} \to \infty$ all four projected spinors survive and we obtain:

$$\mathcal{L}_{\text{off-diag}} = \bar{\chi}_+\Gamma^0\dot{\psi}_+ + \bar{\chi}_-\Gamma^0\dot{\psi}_- + \bar{\chi}_-\Gamma^a\partial_a\psi_+ - m(\bar{\chi}_+\psi_+ + \bar{\chi}_-\psi_-) + \text{h.c.}\ .$$

with

$$\delta\psi_+ = \xi^0\dot{\psi}_+ + \xi^a\partial_a\psi_+ - \frac{1}{4}\lambda_{ab}\Gamma^{ab}\psi_+ \quad \text{and similar for } \chi_-$$

$$\delta\psi_- = \xi^0\dot{\psi}_- + \xi^a\partial_a\psi_- - \frac{1}{4}\lambda_{ab}\Gamma^{ab}\psi_- - \frac{1}{2}\lambda_{0a}\Gamma^{0a}\psi_+ \quad \text{and similar for } \chi_+$$

The Magnetic Carroll Limit II

This suggests the following truncations in even dimensions:

$$\chi_\pm = \Gamma_\star \psi_\mp$$

with

$$\Gamma_\star = (-\mathrm{i})^{\frac{D}{2}+1}\Gamma^0\Gamma^1\cdots\Gamma^{D-1}$$

This leads to the following minimal Lagrangian :

$$\boxed{\mathcal{L}_{\text{magn. Carroll, 1}} = 2\bar\psi_-\Gamma^0\Gamma_\star\dot\psi_+ + 2\bar\psi_+\Gamma^0\Gamma_\star\dot\psi_- + 2\bar\psi_+\Gamma^a\Gamma_\star\partial_a\psi_+ + i\mathfrak{Im}(m)(\bar\psi_+\Gamma_\star\psi_- + \bar\psi_-\Gamma_\star\psi_+)}$$

where $\mathfrak{Im}(m) = -i(m - m^*)$ is a real mass parameter

One may obtain a second magnetic Carroll Lagrangian with a different mass term by inserting a Γ_* in the mass term of the parent Lagrangian:

$$\boxed{\mathcal{L}_{\text{ magnetic Carroll, 2}} = 2\bar\psi_-\Gamma^0\Gamma_\star\dot\psi_+ + 2\bar\psi_+\Gamma^0\Gamma_\star\dot\psi_- + 2\bar\psi_+\Gamma^a\Gamma_\star\partial_a\psi_+ + i\mathfrak{Im}(m)\bar\psi_+\psi_+}$$

Outline

Electric Carroll Supersymmetry

Our starting point is a relativistic Wess-Zumino (WZ) multiplet in flat $4D$ spacetime with field content

$$\{Z\,,\Psi_L\,,F\}$$

By making the redefinitions

$$X^0 = \frac{t}{\tilde{c}}\,, \qquad X^a = x^a\,, \qquad Z = \frac{z}{\tilde{c}}\,, \qquad \Psi = \frac{\psi}{\sqrt{\tilde{c}}}\,, \qquad F = f\,, \qquad \mathcal{E} = \frac{\varepsilon}{\sqrt{\tilde{c}}}$$

and rescaling $M = m\tilde{c}^2$, we obtain in the limit that $\tilde{c} \to \infty$, the following $\mathcal{N} = 1$ electric Carroll WZ Lagrangian

$$\mathcal{L}_{\text{electric Carroll WZ}} = \dot{z}\dot{z}^* - \bar{\psi}\Gamma^0\dot{\psi}_L + ff^* + \left(mfz - \frac{m}{2}\bar{\psi}\psi_L + \text{h.c.}\right)$$

Bagchi, Grumiller, Nandi (2022)

which realizes the following supersymmetry commutators:

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)] = \frac{1}{2}\left(\bar{\varepsilon}_2\Gamma^0\varepsilon_1\right)\frac{\partial}{\partial t}$$

Magnetic Carroll Supersymmetry

work in progress by A. Fontanella, J. Rosseel + E.B. (2023)

There are several issues to consider:

1. projected spinors do not satisfy Majorana condition $\rightarrow$ we start from on-shell $\mathcal{N}=2$ hypermultiplet with field content $\{Z^i, \Psi\}$ $(i=1,2)$
2. tachyonic hypermultiplet requires insertions of Γ_5
3. we need first-order formulation introducing new auxiliary fields G^i_μ
4. supersymmetry parameter $\mathcal{E}_i$ of hypermultiplet is a symplectic Majorana spinor $\rightarrow$ we need modified projector:

$$\mathcal{E}_{i\pm} = \mathcal{E}_i \pm \Gamma^0 \epsilon_{ij} \mathcal{E}_j$$

5. supersymmetry rule needs to be modified with a field-dependent on-shell trivial symmetry

All these manipulations lead to the following Lagrangian:

$$\mathcal{L}_{\rm hyper} = G_{\mu i}\partial^\mu Z^i + G_\mu{}^i \partial^\mu Z_i + G^i_\mu G^\mu_i + \left(\bar\Psi \Gamma^\mu \Gamma_5 \partial_\mu \Psi + \text{h.c.}\right) + \left(\frac{M}{\tilde c}\right)^2 Z_i Z^i + \frac{M}{\tilde c}\bar\Psi\Psi$$

Taking the Limit

Redefining

$$x^0 = \frac{t}{\tilde{c}}\,, \qquad Z^i = \sqrt{\tilde{c}}\, z^i\,, \qquad G_0^i = \frac{1}{\sqrt{\tilde{c}}}\, g_0^i\,, \qquad G_a^i = \sqrt{\tilde{c}}\, g_a^i\,,$$

$$\Psi_\pm = \tilde{c}^{\pm 1/2}\psi_\pm\,, \qquad \mathcal{E}_{i+} = \epsilon_{i+}\,, \qquad \mathcal{E}_{i-} = \tilde{c}^{-1}\epsilon_{i-}$$

and taking the limit that $\tilde{c} \to \infty$ we obtain the following magnetic Carroll hyper Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{magn. Carroll hyper}} &= g_{0i}\dot{z}^i + g_0^i z_i + g_{ai}\partial^a Z^i + g_a^i\partial^a Z_i + g_{ai}g^{ai} + m^2 z_i z^i \\ &\quad + \bar{\psi}_+\Gamma^0\Gamma_5\dot{\psi}_- + \bar{\psi}_-\Gamma^0\Gamma_5\dot{\psi}_+ + \bar{\psi}_+\Gamma^k\Gamma_5\partial_k\psi_+ - m\bar{\psi}_+\psi_+\end{aligned}$$

Note that the auxiliary field G_0 has become a Lagrange multiplier g_0

This Lagrangian is invariant under a well-defined $\mathcal{N} = 2$ Carroll supersymmetry

Outline

Carroll Geometry I

A metric-compatible spin-connection $\Omega_\mu{}^{AB} = -\Omega_\mu{}^{BA}$ with torsion $T_{\mu\nu}{}^A$ satisfies the following first Cartan structure equations:

$$T_{\mu\nu}{}^A = 2\partial_{[\mu} E_{\nu]}{}^A - 2\Omega_\mu{}^{AB} E_{\nu]b}$$

In general relativity we have

1. All spin-connection components can be solved for in terms of the Vierbeine $E_\mu{}^A$ and the torsion tensors $T_{\mu\nu}{}^A$
2. Each torsion tensor component contains a spin-connection field

This is no longer the case in Carroll gravity!

The Carroll Vierbeine $(\tau_\mu\,, e_\mu{}^a)$ and Carroll spin-connections $(\omega_\mu{}^{ab}\,, \omega_\mu{}^{0a})$ together with their transformation rules can be obtained by making the following redefinitions:

$$E_\mu{}^0 = \frac{1}{\tilde{c}}\tau_\mu\,, \qquad E_\mu{}^a = e_\mu{}^a\,, \qquad E_0{}^\mu = \tilde{c}\tau^\mu\,, \qquad E_a{}^\mu = e_a{}^\mu\,, \qquad T_{\mu\nu}{}^0 = t_{\mu\nu}{}^0\,,$$

$$\Omega_\mu{}^{ab} = \omega_\mu{}^{ab}\,, \qquad \Omega_\mu{}^{a0} = \frac{1}{\tilde{c}}\omega_\mu{}^{a0}\,, \qquad \Lambda^{ab} = \lambda^{ab}\,, \qquad \Lambda^{0a} = \frac{1}{\tilde{c}}\lambda^{0a}\,, \qquad T_{\mu\nu}{}^a = t_{\mu\nu}{}^a$$

and taking the limit that $\tilde{c} \to \infty$.

Carroll Geometry II

We define

$$X_0 \equiv \tau^\mu X_\mu\,, \quad X_a \equiv e_a{}^\mu X_\mu\,, \qquad X_{0a} \equiv \tau^\mu e_a{}^\nu X_{\mu\nu}\,, \quad X_{ab} \equiv e_a{}^\mu e_b{}^\nu X_{\mu\nu}\,.$$

After taking the limit we find that in Carroll geometry:

1. Not all spin-connection components can be solved for in terms of the Vierbeine $(\tau_\mu\,, e_\mu{}^a)$ and the torsion tensors $t_{\mu\nu}{}^0\,, t_{\mu\nu}{}^a$. In particular, the spin-connection components $\omega^{(a,0b)}$ remain independent.

2. Not all torsion tensor components contain a spin-connection field. In particular, the components $t_{0(a,b)}$ do not contain any spin-connection component. Such tensor components are called intrinsic torsion tensors

Setting intrinsic torsion tensors equal to zero leads to geometric constraints. This leads to the following four distinct Carrol geometries with each have a specific geometric interpretation:

Four Carroll Geometries

Figueroa-O'Farrill (2020)

Carroll 1: all intrinsic torsion tensors are non-zero.
Carroll 2: $t_{0a}{}^a = 0$.
Carroll 3: $t_{0\{a,b\}} = 0$.
Carroll 4: $t_{0a}{}^a = t_{0\{a,b\}} = 0$.

Magnetic Carroll Gravity

Our starting point is the Einstein-Hilbert action in a first-order formulation with zero torsion ($T_{\mu\nu}{}^{A} = 0$):

$$S_{\text{EH}} = \frac{1}{16\pi G_N}\int \mathrm{d}^4x\, E E_A{}^{\mu} E_B{}^{\nu} R_{\mu\nu}{}^{AB}(\Omega)$$

Taking the Carroll limit of this action, along with $G_N = G_C/\tilde{c}$, leads to the following first-order magnetic Carroll gravity action:

$$S_{\text{magn. Carroll grav.}} = \frac{1}{16\pi G_C}\int \mathrm{d}^4x\, e\Big(e_a{}^{\mu} e_b{}^{\nu} R_{\mu\nu}(J)^{ab} + 2\tau^{\mu} e_a{}^{\nu} R_{\mu\nu}(G)^{0a}\Big)$$

In this action the spin-connection components $\omega^{(a,0b)}$ only occur linearly. They are therefore independent and, furthermore, occur as Lagrange multipliers leading to the geometric constraints :

$$t_{0a}{}^{a} = t_0{}^{\{a,b\}} = 0 : \quad \text{Carroll 4 geometry}$$

Gomis, Rollier, Rosseel, ter Veldhuis + E.B. (2017), Hansen, Obers, Oling, Søgaard (2021)
Henneaux, Salgado-Rebolledo (2021); Campoleoni, Henneaux, Pekar, Pérez, Salgado-Rebolledo (2022)

Outline

Coupling Fermions to Gravity

One can couple fermions to general relativity in two different ways:

1. Using a first-order Palatini formulation or
2. Coupling directly in a second-order formulation

The difference is that the Palatini formulation leads to fermion bilinear torsion contributions

Passing to a second-order formulation this leads to quartic fermion terms

What is the Carrollian analogue of these two inequivalent ways of coupling fermions to gravity?

Coupling Electric Carroll Fermions in First-Order Formulation

The coupling of fermions to first-order gravity with zero torsion is described by

$$S = S_{\text{EH}} + \int \mathrm{d}^4x\, E \left[\bar{\Psi} E_A{}^{\mu} \Gamma^A (\partial_\mu \Psi + \frac{1}{4} \Omega_\mu{}^{BC} \Gamma_{BC} \Psi) - \frac{M}{\tilde{c}} \bar{\Psi}\Psi + \text{h.c.} \right]$$

Taking the electric Carroll limit of the above action, we obtain

$$S = S_{\text{magn. Carroll grav.}} + \int \mathrm{d}^4x\, e \left[\bar{\psi} \Gamma^0 \left(\partial_0 + \frac{1}{4} \omega_0{}^{ab} \Gamma_{ab} \right) \psi - m\bar{\psi}\psi + \text{h.c.} \right]$$

The equations of motion of the spin-connections lead to a bilinear fermion contribution to the $\omega^{[a,0b]}$ spin-connection components instead of $\omega_0{}^{ab}$. Therefore, passing to a second-order formulation we do not have quartic fermions

The independent spin-connections $\omega^{(a,0b)}$ do not occur in the coupling to the fermions $\rightarrow$ there is no bilinear fermion contribution to the intrinsic torsion tensor components $t_{0(a,b)}$ which therefore remain zero $\rightarrow$ the action describes a Carroll 4 geometry

Coupling Magnetic Carroll Fermions in First-Order Formulation

Coupling a tachyonic Dirac spinor to general relativity in the first-order formulation and taking the magnetic Carroll limit, we obtain:

$$S = S_{\text{magn. Carroll grav.}} + \int \mathrm{d}^4x\, e \Big[\bar{\psi}_+\Gamma^0\Gamma_5\tau^\mu D_\mu\psi_- + \bar{\psi}_-\Gamma^0\Gamma_5\tau^\mu D_\mu\psi_+ + \bar{\psi}_+\Gamma^a\Gamma_5 e_a{}^\mu D_\mu\psi_+ - m\bar{\psi}_+\psi_+ + \text{h.c.}\Big]\,,$$

with

$$D_\mu\psi_+ \equiv \partial_\mu\psi_+ + \frac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}\psi_+\,, \qquad D_\mu\psi_- \equiv \partial_\mu\psi_- + \frac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}\psi_- + \frac{1}{2}\omega_\mu{}^{0a}\Gamma_{0a}\psi_+\,.$$

The equations of motion of the spin-connections lead to a bilinear fermion contributions to $\omega^{[a,0b]}$ and $\omega_a{}^{bc}$ and these do lead, after passing to the second-order formulation, to quartic fermion terms

Like in the electric case, the independent spin-connections $\omega^{(a,0b)}$ do not occur in the coupling to the fermions $\rightarrow$ the action describes a Carroll 4 geometry

Coupling Carroll Fermions in Second-order Formulation

Taking the limit of $\Omega_\mu{}^{AB}$ in a first-order formulation and then pass to a second-order formulation is not the same as taking the limit directly in a second-order formulation :

$$\Omega_\mu{}^{AB}(E) \quad \to \quad \tilde{c}^2 t_0{}^{(a,b)} + \omega_\mu{}^{ab}(e)\,, \omega_\mu{}^{a0}(e,\tau)$$

When taking the Carroll limit this leads to a new divergent term:

$$S \propto \tilde{c}^2 \int \mathrm{d}^4x\, e \,\left(t_0{}^{(a,b)} t_{0(a,b)} - t_{0a}{}^a t_{0b}{}^b\right) + \mathcal{O}(\tilde{c}^0)$$

and therefore no fermions survive! There are no spin-connections !

One can now Accept, Eliminate or Cancel.

Accept : the new leading term leads to three versions of Electric Carroll gravity with Carroll 1, 2 or 3 Geometry :

$$S_{\text{electric Carroll Grav.}} = \frac{1}{16\pi G_C} \int \mathrm{d}^4x\, e \,\left(t_0{}^{(a,b)} t_{0(a,b)} - t_{0a}{}^a t_{0b}{}^b\right)$$

Henneaux (1979); Hartong (2015); Hansen, Obers, Oling, Søgaard (2021)

Eliminate or Cancel

Eliminate : Apply a Hubbard-Stratonovich transformation to rewrite the leading order term introducing auxiliary fields $\chi^{ab} = \chi^{ba}$ and χ as follows:

$$\int \mathrm{d}^4x\, e \left(\chi^{ab} t_{0(a,b)} - \chi t_{0a}{}^{a} - \frac{1}{4\tilde{c}^2} \chi^{ab} \chi_{(ab)} + \frac{1}{4\tilde{c}^2} \chi^2 \right) .$$

Taking the (electric of magnetic) Carroll limit, the sub-leading terms now lead to actions with fermions that closely resemble the ones obtained in the first-order formulation of magnetic Carroll gravity with a Carroll 4 geometry.

The role of the independent spin-connection $\omega^{(a,0b)}$ is now played by the auxiliary field in the Hubbard-Stratonovich transformation which survives as a Lagrange multiplier $\chi^{\{ab\}}$ and χ

Cancel : Add a term to the action to cancel the leading divergence.

Outline

Future Directions

- there is a parallel story for Galilei symmetries. For instance,

$$S_{\text{el. Galilei. grav.}} = \frac{1}{16\pi G_G}\int \mathrm{d}^4x\, e\, t^{ab}t_{ab}\,,$$

where $t_{ab} = e_a{}^\mu e_b{}^\nu(\partial_\mu\tau_\nu - \partial_\nu\tau_\mu)$ and τ_μ is the clock function

- from particles to extended objects:

$$P_\pm = \frac{1}{2}(1\pm\Gamma_{01})\ :\text{strings} \qquad \text{and} \qquad P_\pm = \frac{1}{2}(1\pm\Gamma_{012})\ :\text{membranes}$$

- the fact that our results follow from taking a limit guarantees the emergence of a conformal Carroll symmetry $\rightarrow$ relations with

BMS symmetry, flat space/celestial holography, infinite-dim. symmetries!