Higher-form symmetries and generalized Komar charge in KK theory of gravity

Gabriele Barbagallo

Based on

[arXiv:2506.15615 [hep-th]]

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Patrick Meessen and Tomás Ortín

GRASS-SYMBHOL Meeting 2025
Toledo, 11th to 13th November

Instituto de Física Teórica UAM-CSIC

CSIC Consejo Superior de Investigaciones Científicas

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The Noether current is

$$\mathbf{J}[\xi] = \mathrm{d}\mathbf{Q}[\xi],$$

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If ξ is a Killing vector k:

$$\mathrm{d}\mathbf{Q}[k] = \mathbf{J}[k] \doteq \iota_k \mathbf{L} \doteq \mathrm{d}\omega_k.$$

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The Komar charge only exists in stationary or axisymmetric spacetimes.

Application of the Komar charge: the Smarr formula

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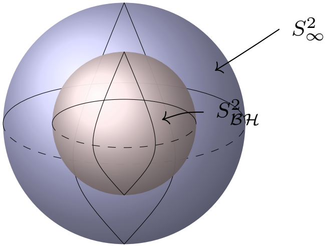

Assuming that the horizon is bifurcate ($k\big|_{\mathcal{BH}} = 0$), choose a spacelike hypersurface Σ^3 with boundary $\partial\Sigma^3 = S^2_{\mathcal{BH}} \cup S^2_\infty$. Integrating $\mathrm{d}\mathbf{K}(k) \doteq 0$ over Σ^3 and applying Stokes' theorem,

$$0 \doteq \int_{\Sigma^3} \mathrm{d}\mathbf{K}(k) = \int_{S^2_\infty} \mathbf{K}(k) - \int_{S^2_{\mathcal{BH}}} \mathbf{K}(k).$$

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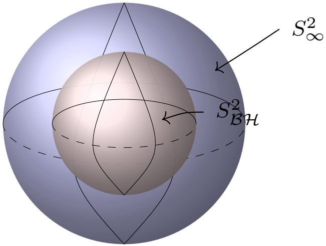

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Smarr formula

$$\int_{S^2_{\mathcal{BH}}} \mathbf{K}(k) \doteq \int_{S^2_\infty} \mathbf{K}(k)$$

Application of the Komar charge: the Smarr formula

$$\mathrm{d}s^2 = \lambda(r)\mathrm{d}t^2 - \frac{1}{\lambda(r)}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)$$

The timelike Killing vector is

$$k = \partial_t = k^\mu\partial_\mu, \qquad k^\mu = \delta^\mu_t$$

$$\hat{k} \equiv k_\mu\mathrm{d}x^\mu = g_{\mu t}\mathrm{d}x^\mu = g_{tt}\mathrm{d}t = \lambda(r)\mathrm{d}t.$$

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The Komar of pure gravity in $d = 4$ is

$$\mathbf{K}[k] = -\frac{1}{16\pi G_N^{(4)}} \star (e^a \wedge e^b) P_{k\,ab}$$

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$$\boxed{M=2TS}$$

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Setup: 5-D GR + KK Bounday conditions

KK paradigm

Physics in the uncompactified (lower) dimensions is just a manifestation of Physics in the total spacetime manifold (higher dimensions)

Setup: 5-D GR + KK Bounday conditions

The 5-D geometry of 4-D stationary black holes [C. Gómez-Fayrén, P. Meessen, T. Ortín, M. Zatti] [G.W. Gibbons, D.L. Wiltshire]

4-D stationary black hole

For the rigidity theorem, it admits a timelike Killing vector m at infinity and a rotational one n. The Killing

$$l \equiv m - \Omega_{\mathcal{H}}\, n, \qquad l^2 \stackrel{\mathcal{H}}{=} 0$$

identifies $\mathcal{H}$ as a Killing horizon.

5-D KK uplift

The 5-D solution has the extra isometry $\hat{k} = \partial_{\underline{z}}$ and remains stationary ($m = \partial_t$). The horizon is the local product of the 4-D horizon with S^1. The uplift of l satisfying

$$\hat{l}^2 \stackrel{\mathcal{H}}{=} 0 \qquad \mathcal{L}_{\hat{l}}\hat{g}_{\hat{\mu}\hat{\nu}} = 0$$

is

$$\hat{l} \equiv l - \hat{k}$$

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$$\hat{l} \qquad \equiv \underset{\boxed{\neq M}}{\underset{\downarrow}{l}} \; - \; \underset{\boxed{\hat{P}_z \sim q}}{\underset{\downarrow}{\hat{k}}}$$

Motivation

Problem

The 5-D Komar charge does *not* measure the 4-D mass

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$$\mathrm{d}s^2_{(5)} = \frac{k_\infty}{k}\,\mathrm{d}s^2_{E(4)} - k^2\big(\mathrm{d}z + \sqrt{k_\infty}A_E\big)^2$$

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$$\hat{\mathbf{K}} \overset{r\to\infty}{\sim} r^2 \partial_r \hat{g}_{tt}$$

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Conclusion: the naive 5-D Komar integral returns a combination of the mass M and the scalar charge Σ.

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Goal: find a consistent modification of the Komar charge to isolate the 4-D mass M.

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- Now the Komar gives the mass

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5D GR with Kaluza–Klein boundary conditions

5D Einstein–Hilbert action

$$\hat{S}[\hat{e}] = \frac{1}{16\pi G_N^{(5)}} \int \hat{\star}(\hat{e}^{\hat{a}} \wedge \hat{e}^{\hat{b}}) \wedge \hat{R}_{\hat{a}\hat{b}}$$

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Kaluza–Klein boundary conditions

We assume a *spatial* Killing vector with closed orbits, $\hat{k} = \partial_{\underline{z}}$. Work in coordinates adapted to $\hat{k}$, so the metric and fields are z-independent:

$$\partial_{\underline{z}}\, \hat{g}_{\hat{\mu}\hat{\nu}} = 0, \qquad z \sim z + 2\pi\ell,$$

where ℓ is some length scale

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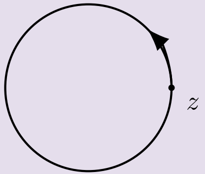
Compact direction

$$\hat{k} = \partial_{\underline{z}}$$



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(KK metric)

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$$k = \sqrt{\left|\hat{g}_{\underline{z}\underline{z}}\right|}$$

(KK scalar)

$$S[e, A, k] = \frac{2\pi\ell}{16\pi G_N^{(5)}} \int \Big\{ k \big[-\star(e^a \wedge e^b) \wedge R_{ab} + \tfrac{1}{2} k^2 F \wedge \star F \big] + \mathrm{d}\big[2 \star dk\big] \Big\}$$

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Field rescalings

$$g_{\mu\nu}\equiv\frac{k_\infty}{k}\,g_{E\,\mu\nu}$$

$$e^a{}_\mu\equiv\sqrt{\frac{k_\infty}{k}}\,e_E{}^a{}_\mu$$

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Einstein-frame

$$S[g_E, A_E, k] \sim$$
$$\int d^dx\,\sqrt{|g_E|}\,\Big\{ R(g_E) + \frac{d-1}{d-2}\, k^{-2}(\partial k)^2 - \tfrac{1}{4}\, k^{2\frac{d-1}{d-2}} F_E^2 \Big\}$$

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$$S[g_E, A_E, k] \sim \int d^d x \sqrt{|g_E|} \left\{ R(g_E) + \frac{d-1}{d-2} k^{-2} (\partial k)^2 - \tfrac{1}{4} k^{2\frac{d-1}{d-2}} F_E^2 \right\}$$

Global symmetries: global rescalings of k and A

$$\delta_\gamma g_{E\,\mu\nu} = 0, \quad \delta_\gamma A_E = \gamma A_E, \quad \delta_\gamma k = -\gamma \tfrac{d-2}{d-1} k, \quad \Rightarrow \quad \delta_\gamma S = 0$$

Symmetries in KK theories

$$\boxed{\begin{array}{c}\text{Higher}\\ \text{dimensional}\\ \text{theory}\end{array}}$$

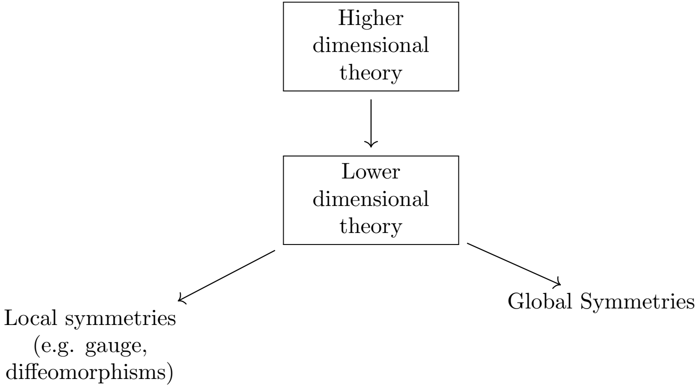
Symmetries in KK theories

Higher dimensional theory

$\downarrow$

Lower dimensional theory

Symmetries in KK theories



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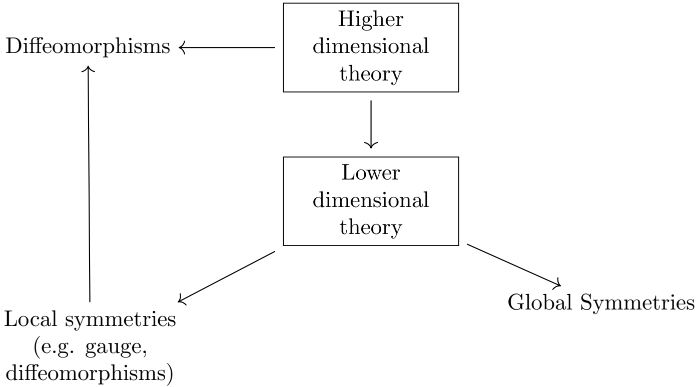

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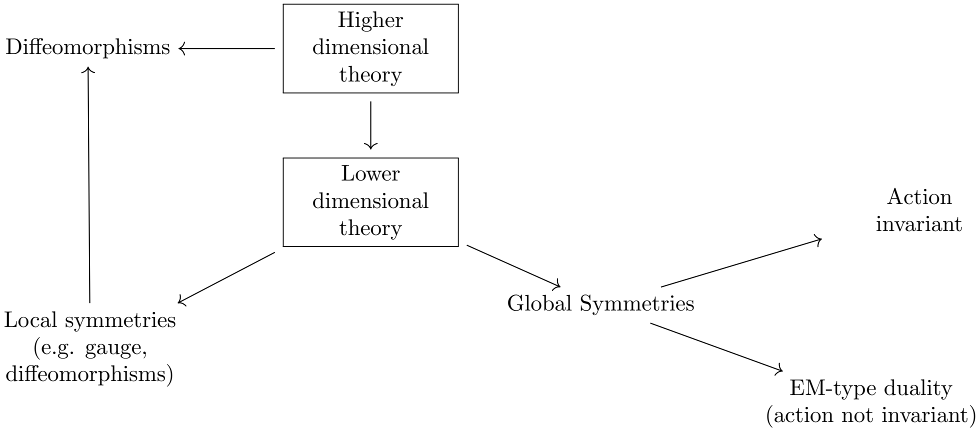

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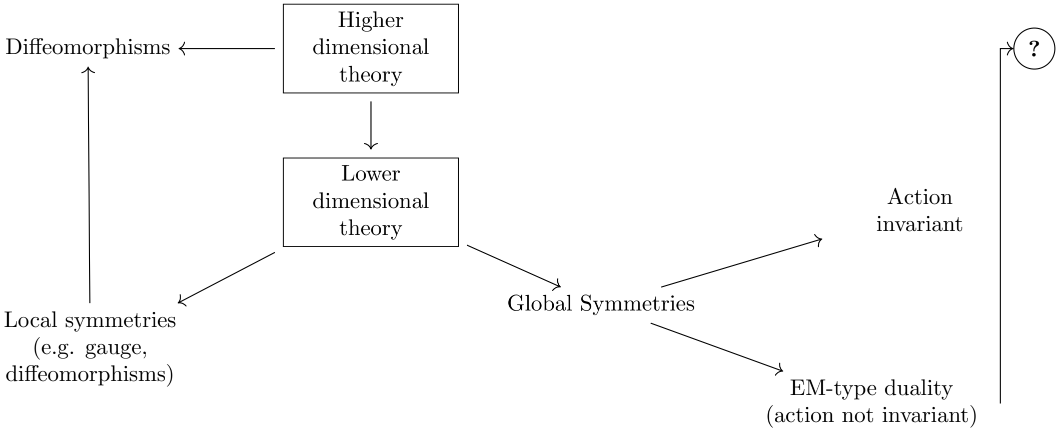

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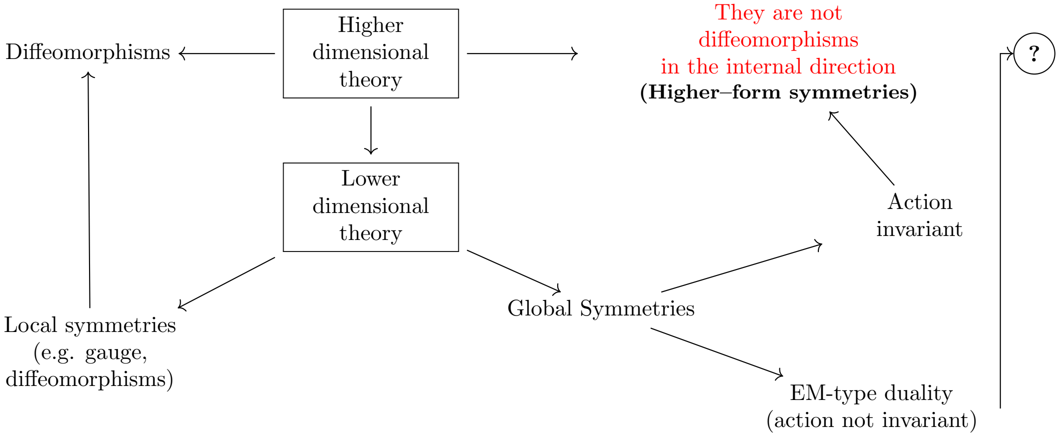

Local Symmetries

Generic Manifold $\rightarrow$

EH action is invariant up to total derivatives under 5-D GCTs $\delta_{\hat{\xi}}\hat{x}^{\hat{\mu}} = \hat{\xi}^{\hat{\mu}}$, that is

$$\delta_{\hat{\xi}}\hat{g}_{\hat{\mu}\hat{\nu}} = -\mathcal{L}_{\hat{\xi}}\hat{g}_{\hat{\mu}\hat{\nu}} = -\left(\hat{\xi}^{\hat{\rho}}\partial_{\hat{\rho}}\hat{g}_{\hat{\mu}\hat{\nu}} + 2\partial_{(\hat{\mu}}\hat{\xi}^{\hat{\rho}}\hat{g}_{\hat{\nu})\hat{\rho}}\right)$$

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KK boundary conditions $\partial_{\underline{z}} \hat{g}_{\hat{\mu}\hat{\nu}} = 0$

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- 4-D GCTs $\xi^{\mu}(x) = \hat{\xi}^{\mu}(x)$
- U(1) $\delta_{\chi} A = \mathrm{d}\chi$, $\chi(x) = -\hat{\xi}^{\underline{z}}(x)$
- U(1) acts on $\mathrm{d}z$ as $\delta_{\chi} \mathrm{d}z = -\mathrm{d}\chi$

Global Symmetries [C. Gómez-Fayrén, P. Meessen, T. Ortín, M. Zatti]

Non-trivial topology

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Isometry - Killing $\hat{k}$

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- **Global Part: it's not a 5-D diffeomorphism**

$$\delta_\gamma \hat{g}_{\hat{\mu}\hat{\nu}} = -2\gamma \delta^{\underline{z}}_{(\hat{\mu}} \hat{g}_{\hat{\nu})\underline{z}} + \frac{2}{(\hat{d}-2)}\gamma \hat{g}_{\hat{\mu}\hat{\nu}}$$

and, translating the 5-D metric to 4-D EF fields, we get the **4-D global symmetries**

$$\delta_\gamma g_{E\mu\nu} = 0 \qquad \delta_\gamma A_{E\mu} = \gamma A_{E\mu} \qquad \delta_\gamma k = -\gamma \tfrac{d-2}{d-1} k$$

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- **Exact Part: gives a 5-D diffeomorphism generated by** $\Lambda(x)\partial_{\underline{z}}$

$$\delta_\Lambda \hat{g}_{\hat{\mu}\hat{\nu}} = -2\partial_{(\hat{\mu}}\Lambda(x)\hat{g}_{\hat{\nu})\underline{z}}$$

and, translating the 5-D metric to 4-D fields, obtains the gauge transformation of the KK vector

$$\delta_\Lambda g_{\mu\nu} = 0, \qquad \delta_\Lambda A_\mu = -\partial_\mu \Lambda \qquad \delta_\Lambda k = 0$$

Index

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2. Introduction and motivation
3. 5-D GR with KK boundary conditions
   - Local Symmetries
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4. The 5-D generalized Komar charge
5. Conclusion and future directions

The 5-D generalized Komar charge

The idea is the following

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1. Write $\hat{\mathbf{L}} \doteq \mathrm{d}\hat{\mathbf{J}}^h \doteq 0$ and $0 = \delta_{\hat{l}}\hat{\mathbf{J}}^h \doteq -\mathrm{d}\hat{\mathbf{Q}}^h_{\hat{l}}$

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2. Build a 1-parameter family of conserved Komar charges $\hat{\mathbf{K}}_\alpha[\hat{l}] \equiv \hat{\mathbf{K}}[\hat{l}] + \alpha\hat{\mathbf{Q}}^h_{\hat{l}}$

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3. Reducing from 5-D to 4-D and choose the proper α

The 5-D generalized Komar charge: Step 1

$$\hat{e}^{\hat{a}} \wedge \hat{\mathbf{E}}_{\hat{a}} = 3\hat{\mathbf{L}} \qquad \Rightarrow \qquad \hat{\mathbf{L}} \doteq 0$$

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- $\delta^h_\epsilon \hat{S} = -\epsilon \hat{S} \qquad \Rightarrow \qquad \hat{\mathbf{L}} \doteq \mathrm{d}\hat{\mathbf{J}}^h \doteq 0, \quad \hat{\mathbf{J}}^h = -\hat{\mathbf{K}}[\hat{k}] \wedge \mathfrak{h}, \quad \hat{\mathbf{K}}[\hat{k}] = \frac{1}{16\pi G_N^{(5)}} \hat{\star}(\hat{e}^{\hat{a}} \wedge \hat{e}^{\hat{b}}) \hat{P}_{\hat{k}\,\hat{a}\hat{b}}$

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$\Rightarrow$ $\hat{\mathbf{J}}^h$ is the Noether current associated to the symmetry $\delta_\epsilon \equiv \delta^h_\epsilon + \delta^s_\epsilon$

$$\hat{\mathbf{L}} \doteq -\mathrm{d}\left\{ \frac{1}{\epsilon} \hat{\mathbf{\Theta}}(\hat{e}, (\delta^h_\epsilon + \delta^s_\epsilon)\hat{e}) \right\} = \mathrm{d}\hat{\mathbf{J}}^h .$$

The 5-D generalized Komar charge: Step 1

Given a $(d-1)$-form current $\mathbf{J}$ which is conserved when evaluated over a solution of the theory with a spacetime symmetry generated by the vector p (*i.e.* δ_p annihilates all the fields of the solution and, therefore, the current), it is always possible to derive from it a $(d-2)$-form charge $\mathbf{Q}_p$ which is also conserved under the same assumptions.

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In our case the Killing vector is $\hat{l}$

$$\boxed{\text{4-form: } 0 = \delta_{\hat{l}}\hat{\mathbf{J}}^h \doteq -\mathrm{d}\,\hat{\mathbf{Q}}^h_{\hat{l}}}$$

$$\downarrow$$

$$\boxed{\text{3-form: } \hat{\mathbf{Q}}^h_{\hat{l}} \equiv \iota_{\hat{l}}\hat{\mathbf{J}}^h + \hat{\mathbf{K}}[\hat{k}]}$$

The 5-D generalized Komar charge: Step 2

Why do we expect $\hat{\mathbf{Q}}^{h}_{\hat{l}}$ to remove the scalar charge contribution?

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 & \boxed{\Sigma = \int_{S^2_\infty} \mathbf{Q}}
\end{tikzcd}
$$

[R. Ballesteros, C. Gómez-Fayrén, T. Ortín, M. Zatti]

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$$\begin{aligned}\hat{\mathbf{K}}_{\alpha}[\hat{l}] &\equiv \hat{\mathbf{K}}[\hat{l}] + \alpha \hat{\mathbf{Q}}^{h}_{\hat{l}} \\ &= \hat{\mathbf{K}}[\hat{l}] - \alpha \imath_{\hat{l}} \hat{\mathbf{K}}[\hat{k}] \wedge \mathfrak{h}\end{aligned}$$

α is an arbitrary coefficient to be determined using some physical criterion

The 5-D generalized Komar charge: Step 3

The pullback of $\hat{\mathbf{K}}_\alpha[\hat{l}]$ over hypersurfaces that include z is (up to a total derivative)

$$\hat{\mathbf{K}}_\alpha[\hat{l}]_{\mathfrak{h}} \doteq \frac{k_\infty}{16\pi G_N^{(5)}}\left\{- \star_E \mathrm{d}\mathbf{l}_E + (1-2\alpha) \star_E (\mathrm{d}\ln k \wedge \mathbf{l}_E) + (1-\alpha) P_{E\,l} k^3 \star_E F_E - \alpha \tilde{P}_{E\,l} F_E\right\} \wedge \mathfrak{h}$$

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$$\hat{\mathbf{K}}_\alpha[\hat{l}]_{\mathfrak{h}} \doteq \frac{k_\infty}{16\pi G_N^{(5)}}\left\{-\star_E \mathrm{d}\mathbf{l}_E + (1-2\alpha)\star_E(\mathrm{d}\ln k \wedge \mathbf{l}_E) + (1-\alpha)P_{E\,l}k^3\star_E F_E - \alpha\tilde{P}_{E\,l}F_E\right\}\wedge\mathfrak{h}$$

The second term gives an unwanted additional contribution which is proportional to the scalar charge of the KK scalar k, so choosing $\alpha = 1/2$ we get

$$\boxed{\hat{\mathbf{K}}_{1/2}[\hat{l}]_{\mathfrak{h}} = \frac{k_\infty}{16\pi G_N^{(5)}}\left\{-\star_E \mathrm{d}\mathbf{l}_E + \tfrac{1}{2}P_{E\,l}k^3\star_E F_E - \tfrac{1}{2}\tilde{P}_{E\,l}F_E\right\}\wedge\mathfrak{h} = \mathbf{K}[l]\wedge\frac{\mathfrak{h}}{2\pi\ell}}$$

which, integrated over the compact direction gives the 4-D generalized Komar 2-form charge.

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- Generalization to coupling with matter: $N = 1$, $d = 5$ SUGRA
  [GB, J. Luis V. Cerdeira, C. Gómez-Fayrén, P. Meessen and T. Ortín, in progress]

Thank you so much for the attention!