# Brane world cosmology and inflation based on a regular black hole predicted from pure gravity theory

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# Introduction to regular black holes

**Singularity theorem with Einstein equations**

- **Black holes generally have a singularity**

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Regular black hole: Black hole without curvature singularities

# How are regular black holes realized ?

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**At this time, there are some possibilities;**

- **Einstein gravity + Nonlinear electromagnetism Lagrangian**

  **(Fine-tuning is needed.)** H. Maeda, J. High Energ. Phys. **2022**, 108 (2022).

- **Asymptotically safe gravity** T. Harada et al, Phys. Rev. D **111**, 126017 (2025).

- **Quasi-topological gravity** P. Bueno, et al. Physics Letters B **861**, 139260 (2025).

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# Quasi-topological gravity

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To define the Quasi-topological gravity, we first consider the general theory of gravity constructed from the metric and the Riemann tensor;

$$S = \frac{1}{16\pi G_N}\int d^D x\sqrt{-g}\mathscr{L}(g^{ab}, R_{abcd}).$$

▼ **Equations of motion**

$$\mathcal{E}_{ab} := P_{acde}R_b{}^{cde} - \frac{1}{2}g_{ab}\mathscr{L} + 2\nabla^c\nabla^d P_{acbd} = 0 \qquad \left(P^{abcd} := \frac{\partial \mathscr{L}}{\partial R_{abcd}}\right).$$

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**Origin of a higher derivative term**

$\nabla^d P_{acbd} = 0$ for arbitraly spacetime $\Longleftrightarrow$ Lovelock gravity

The allowed curvature order of Lovelock gravity $n$ is restricted as

$$n \le \frac{D-1}{2}.$$

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$\nabla^d P_{acbd} = 0$ for <u>arbitraly spacetime</u> $\Longleftrightarrow$ Lovelock gravity

▼ (Birkhoff)

$\nabla^d P_{acbd} = 0$ for <u>spherically symmetric spacetime</u> $\Longleftrightarrow$ Quasi-topological gravity

- ✓ **Equations of motion become a second-derivative equation**
- ✓ **No restrictions for the curvature order** $\mathscr{L} = R + \sum_{n=2}^{n_{\max}} a_n \mathcal{Z}_n$
- ✓ **Black holes do not have curvature singularities if** $n_{\max} \to \infty$
- ✓ **Satisfying the Birkhoff theorem**
- ✓ **Defined only** $D \geq 5$

# Two-dimensional reduced action

In the general spherically symmetric spacetime $ds^2 = \gamma_{AB}dx^A dx^B + r^2 d\Omega^2$, the action of Birkhoff QT gravity can be reduced as

$$I_{2d} = \frac{(D-2)\Omega_{D-2}}{16\pi G}\int d^2x\sqrt{-\gamma}\mathscr{L}_{2d}(\gamma_{AB},\varphi),$$

$$\mathscr{L}_{2d} = G_2(\varphi,X) - \Box\varphi G_3(\varphi,X) + G_4(\varphi,X)R - 2G_{4,X}(\varphi,X)\left[(\Box\varphi)^2 - \nabla_A\nabla_B\varphi\nabla^A\nabla^B\varphi\right],$$

$$\begin{cases} h(\psi) & := \displaystyle\sum_{n=1}^{n_{\max}} \alpha_n\psi^n,\ \psi := \frac{1-X}{\varphi^2},\ X := \nabla_A\varphi\nabla^A\varphi, \\ G_2(\varphi,X) & = \varphi^{D-2}((D-1)h(\psi) - 2\psi h'(\psi)), \\ G_3(\varphi,X) & = 2\varphi^{D-3}h'(\psi), \\ G_4(\varphi,X) & = -\dfrac{\varphi^{D-2}}{2}\psi^{\frac{D-2}{2}}\displaystyle\int d\psi\frac{h'(\psi)}{\psi^{\frac{D}{2}}}. \end{cases}$$

▶ **This theory takes the Horndeski form!**

# Construction regular black holes

Putting a metric as $ds^2 = -N^2(t,r)f(t,r)dt^2 + f^{-1}(t,r)dr^2 + r^2 d\Omega^2$.
If the spaceitm is a vacuum, Birkhoff QT gravity predicts

$$N = N(t),\ f = f(r),\ h(\psi) = \frac{2M}{r^{D-1}},$$

where,

$$h(\psi) := \sum_{n=1}^{n_{\max}} \alpha_n \psi^n, \quad \psi := \frac{1-f}{r^2}.$$

**Lemma 1** *A sufficient condition that spacetimes do not have curvature singularities is* $n_{\max} = \infty,\ \alpha_n \geq 0,$ *and* $\lim_{n\to\infty} (\alpha_n)^{1/n} = C > 0.$

- $h(\psi)$ has a radius of convergence $\psi_0 = 1/C$.
- In the vicinity of $r = 0$, $f$ behaves as $\underline{f \cong 1 - \psi_0 r^2}$

**Regular black holes behave like de Sitter spacetime!!**

# Example of regular black hole

If the coupling constants satisfy $\alpha_n = \dfrac{a^{n-1}}{n}$, the corresponding function $f(r)$ is

$$f(r) = 1 - \frac{r^2}{\alpha}\left(1 - \exp\left(-\frac{2M\alpha}{r^{D-1}}\right)\right).$$

We refer to this as a Dymnikova-like regular black hole.
*Proof.*

$$h(\psi) = \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{n}\psi^n = -\frac{\log(1-\alpha\psi)}{\alpha}$$

$$f = 1 - \psi r^2 = 1 - \frac{r^2}{\alpha}\left(1 - \exp\left(-\frac{2M\alpha}{r^{D-1}}\right)\right)$$

$\square$

# Motivation of my research

**(Birkhoff) QT gravity**

1. **predicts regular black holes naturally,**
2. **is defined only for** $D \geq 5.$

**Our universe is effectively described as** $D = 4.$

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**To describe our universe with Birkhoff QT gravity,**

**We consider the brane world cosmology!**

# Brane world scenario

- Inspired by the Hořava–Witten scenario in M-theory, the brane world picture treats our universe as a 3+1-dimensional hypersurface embedded in a higher-dimensional bulk.
- This provides a geometric framework to unify gravity and gauge interactions.
- The $Z_2$ symmetry is imposed to ensure the brane acts as a physical boundary of the bulk.

P. Horˇava and E. Witten, Nucl. Phys. B460, 506 (1996); Nucl. Phys. B475, 94 (1996).

Consider a $D$-dimensional spherically symmetric spacetime

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2_{(D-2)},$$

and a spherically symmetric brane $r = a(\tau),\ t = T(\tau)$.
The induced metric of the brane is

$$ds^2 = \underline{-d\tau^2 + a^2(\tau)d\Omega^2_{(D-2)}}.$$

**The brane behaves as the closed FLRW universe!!**

# Brane world cosmology on Einstein gravity

1. Einstein gravity $\left(f = 1 - \frac{2M}{r^{D-3}} + \frac{r^2}{l^2}\right)$,
2. $S_{AB} = \rho u_A u_B + P(h_{AB} + u_A u_B) - \sigma h_{AB}$,
3. $\mathbb{Z}_2$ symmetry,

▼ Israel junction condition $(D = 5)$

D. Ida, J. High Energy Phys. **2000**, 014 (2000).

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{1}{a^2} + \underbrace{\frac{32\pi^2 G_N^2 \sigma}{9}}_{\text{Matter}}\rho + \underbrace{\left(-\frac{1}{l^2} + \frac{16\pi^2 G_N^2}{9}\sigma^2\right)}_{\text{Cosmological constant}} + \underbrace{\frac{16\pi^2 G_N^2}{9}}_{\text{Modified matter}}\rho^2 + \underbrace{\frac{2M}{a^4}}_{\text{Dark radiation}}$$

**The brane world picture reproduces the Friedmann equation.**

# Brane world cosmology on QT gravity

1. QT gravity,
2. $S_{AB} = \rho u_A u_B + P(h_{AB} + u_A u_B) - \sigma h_{AB}$,
3. $\mathbb{Z}_2$ symmetry,

▼ Israel junction condition

$$\frac{D-2}{a}\int_0^{\sqrt{f+\dot{a}^2}} dz h' \left(\frac{1+\dot{a}^2-z^2}{a^2}\right) = 4\pi G_N(\rho+\sigma). \quad \left(h(\psi) := \sum_{n=1}^{\infty}\alpha_n \psi^n, \ \psi := \frac{1-f}{r^2}\right)$$

1. **Examining the modified Friedmann equation, taking an example of the coupling constant**
2. **Deriving some universal properties.**

# Example - Dymnikova-like black hole

Taking coupling constants $\alpha_n = \dfrac{a^{n-1}}{n}$,

i.e., $f(r) = 1 - \dfrac{r^2}{\alpha}\left[1 - \exp\left(-\dfrac{2M\alpha}{r^{D-1}} - \alpha\Lambda\right)\right]$ leads to a modified Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2} - \frac{1}{\alpha} = -\frac{1}{\alpha}\exp\left[-\frac{2M\alpha}{a^{D-1}} - \alpha\Lambda\right]\cos^2\left[\sqrt{-\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2} + \frac{1}{\alpha}} \cdot \frac{4\pi G_N(\rho+\sigma)\alpha}{D-2}\right].$$

★ It has an Einstein gravity limit.

★ The brane behaves as de Sitter spacetime near $a = 0$ and depend only $\alpha$.

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2} - \frac{1}{\alpha} = 0 \quad \rightarrow \quad a = \sqrt{\alpha}\cosh\left(\frac{\tau - \tau_{min}}{\sqrt{\alpha}}\right)$$

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**We have shown that this de Sitter inflation happens generally!!**

# $Z_2$ symmetry predicts the de Sitter inflation

In a brane world cosmology based on quasi-topological gravity,
$Z_2$ symmetry induces a de Sitter inflation in the vicinity of $a = 0$.

$$\frac{D-2}{a}\int_0^{\sqrt{f+\dot{a}^2}} dz h'\left(\frac{1+\dot{a}^2-z^2}{a^2}\right) = 4\pi G_N(\rho+\sigma).$$

[Proof]
Since $h(\psi)$ has a radius of convergence $\psi_0 = 1/\alpha$, we obtain

$$[z=0] \;\rightarrow\; \frac{1+\dot{a}^2}{a^2} \le \frac{1}{\alpha}, \quad \left[z=\sqrt{f+\dot{a}^2}\right] \;\rightarrow\; \frac{1-f}{a^2} \le \frac{1}{\alpha}.$$

$$\varphi^2 := f + \dot{a}^2$$

$$\frac{\varphi^2}{a^2} = \left[\frac{1}{\alpha} - \frac{1-f}{a^2}\right] - \left[\frac{1}{\alpha} - \frac{1}{a^2} - \left(\frac{\dot{a}}{a}\right)^2\right]$$

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▶ **de Sitter inflation**

# Summary

- ✓ (Birkhoff) Quasi-Topological gravity generally predicts regular black holes. This theory is defined only by $D \geq 5$.
- ✓ To describe our universe, we considered a brane world cosmology.
- ✓ In brane world cosmology based on QT-gravity, $\mathbb{Z}_2$ symmetry induces de Sitter inflation.
- ✓ As future work, we wish to investigate whether this brane world model describes the actual universe.