Non-closed scalar charge in 4-dimensional Einstein-scalar-Gauss-Bonnet black hole thermodynamics

Romina Ballesteros

Universidad San Sebastián, Santiago, Chile

GRASS-SYMBHOL Meeting
2025

Based on

1. Ballesteros, R., Cárdenas, M., Lescano, E. *Non-closed scalar charge in 4-dimensional EsGB black hole thermodynamics*. arXiv:2510.08686 [hep-th].
2. Ballesteros, R. & Ortín, T. (2024). *Hairy black holes, scalar charges and extended thermodynamics*. *Class. Quantum Grav.*, 41(5), 055007.
3. Ballesteros, R., Gómez-Fayrén, C., Ortín, T. Zatti, M. *On scalar charges and black hole thermodynamics. JHEP 05 (2023) 158.* arXiv:2302.11630 [hep-th].

Presentation Overview

1. Background and motivations
2. Black hole thermodynamics: First law & Smarr formula
3. Previous results: closed scalar charge and its role in the Smarr formula & the first law
4. New results: non-closed scalar charge and its role in the Smarr formula & the first law
5. Conclusions and future directions

Part I: Background and motivations

Why black hole thermodynamics?

A black hole is a thermodynamic object: it has temperature and entropy.

A black hole is a thermodynamic object: it has temperature and entropy.

Hawking temperature [Hawking, 1975]

$$T_H = \frac{\hbar}{k_B c} \frac{\kappa}{2\pi},$$

where

$$\kappa \overset{\mathcal{H}}{=} -\frac{1}{2} \left(\nabla^\mu k^\nu\right) \left(\nabla_\mu k_\nu\right)$$

is the surface gravity.

Part I: Background and motivations

Why black hole thermodynamics?

A black hole is a thermodynamic object: it has temperature and entropy.

Hawking temperature [Hawking, 1975]

$$T_H = \frac{\hbar}{k_B c}\frac{\kappa}{2\pi},$$

where

$$\kappa \overset{\mathcal{H}}{=} -\frac{1}{2}\left(\nabla^\mu k^\nu\right)\left(\nabla_\mu k_\nu\right)$$

is the surface gravity.

Bekenstein's Entropy [Bekenstein, 1973]

$$S = \frac{k_B c^3}{G\hbar}\frac{A}{4}$$

where A is the area of the event horizon.

Part I: Background and motivations

Why black hole thermodynamics?

A black hole is a thermodynamic object: it has temperature and entropy.

- Connect geometry with thermodynamics through known relations (first law & Smarr formula)

Part I: Background and motivations

Why black hole thermodynamics?

A black hole is a thermodynamic object: it has temperature and entropy.

- Connect geometry with thermodynamics through known relations (first law & Smarr formula)
- Works as a lab for study effects of gravity beyond GR: gravity plus scalar fields, higher-order corrections, etc.

Part I: Background and motivations

Why black hole thermodynamics?

A black hole is a thermodynamic object: it has temperature and entropy.

- Connect geometry with thermodynamics through known relations (first law & Smarr formula)
- Works as a lab for study effects of gravity beyond GR: gravity plus scalar fields, higher-order corrections, etc.
- In particular, I will focus in the role role of *scalar charges* in black hole thermodynamics.

- The study of scalar charges is relevant since they are understood as “hair” in gravity theories evolving scalar fields.

- The study of scalar charges is relevant since they are understood as "hair" in gravity theories evolving scalar fields.
- For hairy black holes, scalar charges are secondary hair.

- The study of scalar charges is relevant since they are understood as “hair” in gravity theories evolving scalar fields.
- For hairy black holes, scalar charges are secondary hair.
- *Just as bodies in thermal equilibrium are normally characterized by a small number of ”state parameters” (such as E and P), a stationary black hole is uniquely characterized by (M, Q, J). This is known as ”No-hair theorem”.*

Part I: Background and motivations

Why scalar charges?

- The study of scalar charges is relevant since they are understood as “hair” in gravity theories evolving scalar fields.
- For hairy black holes, scalar charges are secondary hair.
- *Just as bodies in thermal equilibrium are normally characterized by a small number of ”state parameters” (such as E and P), a stationary black hole is uniquely characterized by (M, Q, J). This is known as ”No-hair theorem”.*
- Scalar charges are not protected by any conservation law.

Part I: Background and motivations

Why scalar charges?

- The study of scalar charges is relevant since they are understood as “hair” in gravity theories evolving scalar fields.
- For hairy black holes, scalar charges are secondary hair.
- *Just as bodies in thermal equilibrium are normally characterized by a small number of ”state parameters” (such as E and P), a stationary black hole is uniquely characterized by (M, Q, J). This is known as ”No-hair theorem”.*
- Scalar charges are not protected by any conservation law.
- The role they play in black hole thermodynamics is still not clear and finding a covariant definition can give us some insights.

Part I: Background and motivations

Why scalar charges?

It is in the asymptotic expansion of the scalar field $\phi(r)$ for asymptotically flat spacetimes:

$$\phi(r) = \phi_\infty + \frac{\Sigma}{r} + \mathcal{O}(r^{-2})$$

where Σ is the scalar charge and r is the standard radial coordinate.

Part I: Background and motivations

Why scalar charges?

It is in the asymptotic expansion of the scalar field $\phi(r)$ for asymptotically flat spacetimes:

$$\phi(r) = \phi_\infty + \frac{\Sigma}{r} + \mathcal{O}(r^{-2})$$

where Σ is the scalar charge and r is the standard radial coordinate.

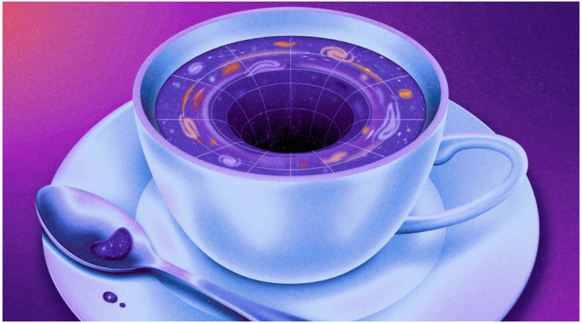
GOAL

Find a covariant definition of the scalar charge that recover this result. Then, see how this *charge* enters into the thermodynamical relations and study its role.

Warnings

1. The black holes solutions considered here, are all static, asymptotically flat, spherically symmetric and by regular we means that exists a bifurcation surface.
2. I will define the charges using Wald's formalism.

Part II: BH thermodynamics: First law & Smarr formula



Source: The #1 Clue to Quantum Gravity Sits on the Surfaces of Black Holes. Quanta Magazine.

1. **0th law**: The surface gravity κ (then, its temperature T_H) is constant over the bifurcate horizon of a stationary black hole.

1. **0th law**: The surface gravity κ (then, its temperature T_H) is constant over the bifurcate horizon of a stationary black hole.
2. **1st law** : When the black hole switches from one stationary state to another, its mass (or energy) changes by

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J + \Phi_H \delta Q$$

where Ω_H is the angular velocity of the horizon, δJ is the variation of the angular momentum, Φ_H is the electrostatic potential (defined as the difference of the electric potential at infinity and at the event horizon) and δQ is the variation of the electric charge. The conjugate chemical potentials Ω_H and Φ_H are constant over the event horizon according to the generalized 0th law.

1. **0th law**: The surface gravity κ (then, its temperature T_H) is constant over the bifurcate horizon of a stationary black hole.
2. **1st law** : When the black hole switches from one stationary state to another, its mass (or energy) changes by

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J + \Phi_H \delta Q$$

   where Ω_H is the angular velocity of the horizon, δJ is the variation of the angular momentum, Φ_H is the electrostatic potential (defined as the difference of the electric potential at infinity and at the event horizon) and δQ is the variation of the electric charge. The conjugate chemical potentials Ω_H and Φ_H are constant over the event horizon according to the generalized 0th law.
3. **2nd law**: In any process, the area of a black hole (so its entropy), do not decrease

$$\delta S \geq 0.$$

Part II: BH thermodynamics: First law & Smarr formula

Comment about the first law for hairy black holes [Gibbons et. al 1996]

1. **0th law**: The surface gravity κ (then, its temperature T_H) is constant over the bifurcate horizon of a stationary black hole.
2. **1st law** : When the black hole switches from one stationary state to another, its mass (or energy) changes by

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J + \Phi_H \delta Q - \Sigma \delta \phi_\infty$$

   where Ω_H is the angular velocity of the horizon, δJ is the variation of the angular momentum, Φ_H is the electrostatic potential (defined as the difference of the electric potential at infinity and at the event horizon) and δQ is the variation of the electric charge. The conjugate chemical potentials Ω_H and Φ_H are constant over the event horizon according to the generalized 0th law. ϕ_∞ is the value of the scalar field at infinity.
3. **2nd law**: In any process, the area of a black hole (so its entropy), do not decrease

$$\delta S \geq 0.$$

- A thermodynamic relationship between quantities measured at spatial infinity and those defined at the horizon
- $$M = 2T_H S + 2\Omega_H J.$$

- A thermodynamic relationship between quantities measured at spatial infinity and those defined at the horizon
- $$M = 2T_H S + 2\Omega_H J.$$
- Quantities measured at infinity, such as the energy M and angular momentum J, represent the global properties of the spacetime.

- A thermodynamic relationship between quantities measured at spatial infinity and those defined at the horizon
- $$M = 2T_H S + 2\Omega_H J.$$
- Quantities measured at infinity, such as the energy M and angular momentum J, represent the global properties of the spacetime.
- In contrast, the thermodynamic quantities at the horizon are directly linked to the geometric characteristics of the black hole: the entropy S is proportional to the area of the event horizon and the temperature T depends on the surface gravity κ, which is constant across the horizon by virtue of the zeroth law.

- Wald and his collaborators, developed a covariant framework that can be applied to black holes for computing charges [Wald et al].

- Wald and his collaborators, developed a covariant framework that can be applied to black holes for computing charges [Wald et al]. Within the Lie-Lorentz derivative, it was shown that this formalism can be used for gravity theories with matter fields [Ortín et al.].

- Wald and his collaborators, developed a covariant framework that can be applied to black holes for computing charges [Wald et al]. Within the Lie-Lorentz derivative, it was shown that this formalism can be used for gravity theories with matter fields [Ortín et al.].
- The formalism is based on generally covariant theories.
  1. Described by a $d-$form Lagrangian $\mathbf{L}(\varphi)$ such that $S = \int_{\mathcal{M}} \mathbf{L}(\varphi)$

- Wald and his collaborators, developed a covariant framework that can be applied to black holes for computing charges [Wald et al]. Within the Lie-Lorentz derivative, it was shown that this formalism can be used for gravity theories with matter fields [Ortín et al.].
- The formalism is based on generally covariant theories.
  1. Described by a $d-$form Lagrangian $\mathbf{L}(\varphi)$ such that $S = \int_{\mathcal{M}} \mathbf{L}(\varphi)$
  2. Under a general variation of the fields $\delta S = \int_{\mathcal{M}} \mathbf{E}_{\varphi} \wedge \delta\varphi + d\mathbf{\Theta}(\varphi, \delta\varphi)$

The first law and the Smarr formula can be found integrating the following on-shell identities over a spacelike hypersurface with boundaries at spatial infinity S^2_∞ and at the bifurcation sphere $\mathcal{BH}$ and using Stokes theorem

1. The first law

$$d\mathbf{W}[k] \doteq 0 \Leftrightarrow \int_{S^2_\infty} \mathbf{W}[k] = \int_{\mathcal{BH}} \mathbf{W}[k]$$

where

$$\mathbf{W}[k] = \delta\mathbf{Q}[k] + \imath_k \Theta(\varphi, \delta\varphi) - \varpi_k , \qquad \delta_{\lambda_k} \Theta(\varphi, \delta\varphi) = d\varpi_k .$$

The first law and the Smarr formula can be found integrating the following on-shell identities over a spacelike hypersurface with boundaries at spatial infinity S^2_∞ and at the bifurcation sphere $\mathcal{BH}$ and using Stokes theorem

1. The first law

$$d\mathbf{W}[k]\doteq 0 \Leftrightarrow \int_{S^2_\infty} \mathbf{W}[k] = \int_{\mathcal{BH}} \mathbf{W}[k]$$

   where

$$\mathbf{W}[k] = \delta \mathbf{Q}[k] + \imath_k \Theta(\varphi, \delta\varphi) - \varpi_k , \qquad \delta_{\lambda_k} \Theta(\varphi, \delta\varphi) = d\varpi_k .$$

2. The Smarr formula

$$d\mathbf{K}[k]\doteq 0 \Leftrightarrow \int_{S^2_\infty} \mathbf{K}[k] = \int_{\mathcal{BH}} \mathbf{K}[k] \tag{1}$$

   where $\mathbf{K}[k] = -\mathbf{Q}[k] + \imath_k \mathbf{L}_{\text{on-shell}}$ is the *generalized Komar charge*.

1. The first law and the Smarr formula are given by surface integrals.

1. The first law and the Smarr formula are given by surface integrals.
2. The surface integrals are consequence of the closedness of a $2-$form charge.

1. The first law and the Smarr formula are given by surface integrals.
2. The surface integrals are consequence of the closedness of a $2-$form charge.
3. For black hole spacetimes, the charges that characterize them are necessarily a result of surface integrals.

Part III: Previous results: closed scalar charge and its role in the Smarr formula & the first law

Theories with global symmetries

Consider a 4-dimensional Einstein-Maxwell-Dilaton theory given by the action

$$S[e^a, \phi] = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star \left(e^a \wedge e^b \right) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \frac{1}{2} e^{-a\phi} F \wedge \star F \right]$$

where ϕ is a real scalar field.

Part III: Previous results: closed scalar charge and its role in the Smarr formula & the first law

Theories with global symmetries

Consider a 4-dimensional Einstein-Maxwell-Dilaton theory given by the action

$$S[e^a,\phi] = \frac{1}{16\pi G_N^{(4)}} \int \left[-\star\left(e^a \wedge e^b\right) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \frac{1}{2} e^{-a\phi} F \wedge \star F \right]$$

where ϕ is a real scalar field. A general variation of the action is

$$\delta S[e^a,\phi] = \int \left[\mathbf{E}_a \wedge \delta e^a + \mathbf{E}\delta\phi + d\boldsymbol{\Theta}(e^a,\phi,\delta e^a,\delta\phi)\right]$$

Part III: Previous results: closed scalar charge and its role in the Smarr formula & the first law

Theories with global symmetries

Consider a 4-dimensional Einstein-Maxwell-Dilaton theory given by the action

$$S[e^a,\phi] = \frac{1}{16\pi G_N^{(4)}}\int\left[-\star\left(e^a\wedge e^b\right)\wedge R_{ab} + \frac{1}{2}d\phi\wedge\star d\phi + \frac{1}{2}e^{-a\phi}F\wedge\star F\right]$$

where ϕ is a real scalar field. A general variation of the action is

$$\delta S[e^a,\phi] = \int\left[\mathbf{E}_a\wedge\delta e^a + \mathbf{E}\delta\phi + d\mathbf{\Theta}(e^a,\phi,\delta e^a,\delta\phi)\right]$$

where $\mathbf{E}$ is the scalar equation of motion

$$\mathbf{E} = -d\star d\phi - \frac{a}{2}e^{-a\phi}F\wedge\star F.$$

Part III: Previous results: closed scalar charge and its role in the Smarr formula & the first law

Theories with global symmetries

Consider a 4-dimensional Einstein-Maxwell-Dilaton theory given by the action

$$S[e^a, \phi] = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star \left(e^a \wedge e^b \right) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \frac{1}{2} e^{-a\phi} F \wedge \star F \right].$$

From the scalar eom we are going to construct a closed-form scalar charge i.e., $\imath_k \mathbf{E} = d\mathbf{Q}_\phi[k]$ Assuming:

1. $\delta_k \phi = -\mathcal{L}_k \phi = -\imath_k d\phi = 0$.
2. By hypothesis $\delta_k \star d\phi = -\imath_k d \star d\phi - d\imath_k \star d\phi = 0$.
3. And $\delta_k A = -\left(dP_k + \imath_k F\right) = 0$ (field with gauge-freedom).

Part III: Previous results: closed scalar charge and its role in the Smarr formula & the first law

Theories with global symmetries

Consider a 4-dimensional Einstein-Maxwell-Dilaton theory given by the action

$$S[e^a, \phi] = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star \left(e^a \wedge e^b \right) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \frac{1}{2} e^{-a\phi} F \wedge \star F \right].$$

The closed $2-$form scalar charge is

$$\mathbf{Q}_\phi[k] = -\frac{1}{16\pi} \left[\imath_k \star d\phi + \frac{a}{2} P_k e^{-a\phi} \star F \right]$$

after integrating

$$\frac{a\,h}{2\,(a^2+1)} \;=\; \frac{1}{4}\,\Sigma\,, \qquad h \;=\; -\frac{a^2+1}{a^2-1}\left[M \;-\; \sqrt{M^2 + 4\,(a^2-1)\,e^{a\phi_\infty}\,q^2}\,\right].$$

Part III: Previous results: closed scalar charge and its role in the Smarr formula & the first law

Theories with global symmetries

1. We found that the existence of a bifurcation surface enforces $\Sigma \neq 0$.
2. The scalar charge depends on the other conserved quantities, hence is a secondary-hair.
3. We computed the first law ($d\mathbf{W}[k]\doteq 0$) and obtained

$$\delta M = T_H \delta S + \Phi_H \delta Q - \Sigma \delta \phi_\infty ,$$

recovering the result of Gibbons et al.

Part III: Previous results: closed scalar charge and its role in the Smarr formula & the first law

Theories with no-global symmetries

We also constructed closed scalar charges $\imath_k \mathbf{E} = d\mathbf{Q}_\phi[k]$ for the theory given by

$$S[e,\phi] = \frac{1}{16\pi G_N^{(4)}} \int \left\{ - \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \star V(\phi) \right\}.$$

Under a general variation of the fields

$$\delta S[e^a,\phi] = \int \left[\mathbf{E}_a \wedge \delta e^a + \mathbf{E}\delta\phi + d\mathbf{\Theta}(e^a,\phi,\delta e^a,\delta\phi) \right],$$

where $\mathbf{E}$ is the scalar eom

$$\mathbf{E} \;=\; -\,d \star d\phi \;+\; \star V', \qquad V' = dV/d\phi.$$

Part III: Previous results: closed scalar charge and its role in the Smarr formula & the first law

Theories with no-global symmetries

We also constructed closed scalar charges $\imath_k \mathbf{E} = d\mathbf{Q}_\phi[k]$ for the theory given by

$$S[e,\phi] = \frac{1}{16\pi G_N^{(4)}} \int \left\{ - \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \star V(\phi) \right\}.$$

where $\mathbf{E}$ is the scalar eom

$$\mathbf{E} \; = \; -d \star d\phi \; + \; \star V' \, , \qquad V' = dV/d\phi.$$

We assumed $\delta_k \phi = 0$, and for the scalar potential $\delta_k \star V' = d\imath_k \star V' = 0$ implies the local existence of $d\mathcal{W}_k = \iota_k \star V'$.

Part III: Previous results: closed scalar charge and its role in the Smarr formula & the first law

Theories with no-global symmetries

We also constructed on-shell closed scalar charges $\imath_k \mathbf{E} = d\mathbf{Q}_\phi[k]$ for the theory given by

$$S[e,\phi] = \frac{1}{16\pi G_N^{(4)}} \int \left\{ - \star\left(e^a \wedge e^b\right) \wedge R_{ab} + \frac{1}{2}\, d\phi \wedge \star d\phi + \star V(\phi) \right\}.$$

Thus we defined the closed $2-$form scalar charge as

$$\mathbf{Q}_\phi[k] \;\equiv\; -\frac{1}{4\pi G_N^{(4)}} \left[\imath_k \star d\phi \;+\; \mathcal{W}_k\right], \qquad d\mathcal{W}_k = \iota_k \star V'\,.$$

Part III: Previous results: closed scalar charge and its role in the Smarr formula & the first law

Theories with no-global symmetries

From the closed $2-$form scalar charge

$$\mathbf{Q}_\phi[k] \equiv -\frac{1}{4\pi G_N^{(4)}}\left[\imath_k \star d\phi + \mathcal{W}_k\right], \qquad d\mathcal{W}_k = \iota_k \star V'.$$

- After integrating $\mathbf{Q}_\phi[k]$ at spatial infinity and at the bifurcation surface, one can find conditions that the scalar potential $V(\phi)$ must satisfy. These are no-hair theorems:

$$V(\phi_\infty) = V'[\phi_\infty] = V''[\phi_\infty] = V^{(3)}[\phi_\infty] = V^{(4)}[\phi_\infty] = 0\,.$$

- The Smarr formula ($d\mathbf{K}[k] \doteq 0$)

$$M = 2ST + 2\alpha\Phi_\alpha\,, \qquad \text{where} \quad \Phi_\alpha \equiv -\frac{1}{16\pi G_N^{(4)}}\int_{\Sigma^3} \iota_k \star \frac{\partial V}{\partial \alpha}$$

Part III: Previous results: closed scalar charge and its role in the Smarr formula & the first law

Theories with no-global symmetries

From the closed $2-$form scalar charge

$$\mathbf{Q}_{\phi}[k] \equiv -\frac{1}{4\pi G_N^{(4)}}\left[\imath_k \star d\phi + \mathcal{W}_k\right], \qquad d\mathcal{W}_k = \iota_k \star V' .$$

- The first law ($d\mathbf{W}[k]\doteq 0$)[1]

$$\delta M = T\,\delta S - \frac{1}{4}\Sigma\,\delta\phi_{\infty} + \Phi_{\alpha}\,\delta\alpha .$$

[1] α' is a dimensionful constant.

Part III: Previous results: closed scalar charge and its role in the Smarr formula & the first law

1. We recover again the result of Gibbons et al.
2. The covariant definition of scalar charges enables the formulation of no-hair theorems.
3. The closedness of the charge, ensures that it will satisfy a Gauss law.
4. The assumption

   $$\delta_k V = 0$$

   was the key condition that ensured its closedness.

New results: non-closed scalar charge and its role in the Smarr formula & the first law

From closed to non-closed scalar charges: what changes and why it matters

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories

For the theories

$$S = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2}\, d\phi \wedge \star d\phi + \alpha' f(\phi)\, \mathcal{G} \right],$$

where $\mathcal{G}$ is the Gauss-Bonnet term written in differential form as

$$\mathcal{G} = \frac{1}{2}\, \epsilon_{abcd}\, R^{ab} \wedge R^{cd}.$$

The eom for the scalar field

$$\mathbf{E}_\phi = -d \star d\phi + \alpha' \partial_\phi f(\phi) \mathcal{G}$$

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories

For the theories

$$S = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \alpha' f(\phi) \mathcal{G} \right],$$

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories

For the theories

$$S = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \alpha' f(\phi) \mathcal{G} \right] ,$$

The eom for the scalar field

$$\mathbf{E}_\phi = -d \star d\phi + \alpha' \partial_\phi f(\phi) \mathcal{G} .$$

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories

For the theories

$$S = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \alpha' f(\phi) \mathcal{G} \right],$$

The eom for the scalar field

$$\mathbf{E}_\phi = -d \star d\phi + \alpha' \partial_\phi f(\phi) \mathcal{G}.$$

Constructing a form scalar charge

$$\imath_k \mathbf{E}_\phi = -\imath_k d \star d\phi + \alpha' \partial_\phi f(\phi) \imath_k \mathcal{G}$$

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories

For the theories

$$S = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2}\, d\phi \wedge \star d\phi + \alpha' \, f(\phi)\, \mathcal{G}\right],$$

The eom for the scalar field

$$\mathbf{E}_\phi = -d \star d\phi + \alpha' \partial_\phi f(\phi) \mathcal{G}.$$

Constructing a form scalar charge

$$\imath_k \mathbf{E}_\phi = -\imath_k d \star d\phi + \alpha' \partial_\phi f(\phi) \imath_k \mathcal{G}$$

Using

$$\imath_k \mathcal{G} = d\mathcal{X}_k, \qquad \mathcal{X}_k = -2 P_k^{ab} \widetilde{R}_{ab}$$

one obtains

$$\imath_k \mathbf{E}_\phi = -\imath_k d \star d\phi + \alpha' \partial_\phi f(\phi) d\mathcal{X}_k$$

New results: non-closed scalar charge and its role in the Smarr formula & the first law

For the theories

$$S = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \alpha' f(\phi) \mathcal{G} \right],$$

one obtains

$$\imath_k \mathbf{E}_\phi = -\imath_k d \star d\phi + \alpha' \partial_\phi f(\phi) d\mathcal{X}_k.$$

1. For the first term

$$\delta_k \star d\phi = 0 \Rightarrow -\imath_k d \star d\phi = d\imath_k \star d\phi$$

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories

For the theories

$$S = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \alpha' f(\phi) \mathcal{G}\right],$$

one obtains

$$\imath_k \mathbf{E}_\phi = -\imath_k d \star d\phi + \alpha' \partial_\phi f(\phi) d\mathcal{X}_k .$$

1. For the first term

$$\delta_k \star d\phi = 0 \Rightarrow -\imath_k d \star d\phi = d\imath_k \star d\phi$$

2. For the second term, we use the chain-rule

$$\partial_\phi f(\phi)\, d\mathcal{X}_k = d\big(\partial_\phi f(\phi)\, \mathcal{X}_k\big) - d\big(\partial_\phi f(\phi)\big) \wedge \mathcal{X}_k\, .$$

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories: the scalar charge

For the theories

$$S = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \alpha' f(\phi) \mathcal{G} \right],$$

The form scalar charge $\mathbf{Q}_\phi$ is non-closed

$$\imath_k \mathbf{E}_\phi = d\mathbf{Q}_\phi = \mathcal{W}_k, \qquad \mathcal{W}_k = d\big(\partial_\phi f(\phi)\big) \wedge \mathcal{X}_k, \qquad \mathbf{Q}_\phi = -\frac{1}{4\pi} \left[\imath_k \star d\phi + \alpha' \partial_\phi f(\phi) \mathcal{X}_k \right].$$

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories: the scalar charge

For the theories

$$S = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2}\, d\phi \wedge \star d\phi + \alpha' \, f(\phi)\, \mathcal{G} \right],$$

The form scalar charge $\mathbf{Q}_\phi$ is non-closed

$$\imath_k \mathbf{E}_\phi = d\mathbf{Q}_\phi = \mathcal{W}_k, \qquad \mathcal{W}_k = d\big(\partial_\phi f(\phi)\big) \wedge \mathcal{X}_k\,, \qquad \mathbf{Q}_\phi = -\frac{1}{4\pi}\left[\imath_k \star d\phi \; + \; \alpha' \, \partial_\phi f(\phi)\, \mathcal{X}_k \right].$$

After using Stoke’s theorem

$$\int_{S^2_\infty} \mathbf{Q}_\phi \;=\; \int_{\mathcal{B}_H} \mathbf{Q}_\phi \;-\; \frac{\alpha'}{4\pi} \int_{\Sigma^3} \mathcal{W}_k\,.$$

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories: the scalar charge

For the theories

$$S = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2}\, d\phi \wedge \star d\phi + \alpha'\, f(\phi)\, \mathcal{G}\right] ,$$

After using Stoke's theorem, the balance eq. reads

$$\int_{S^2_\infty} \mathbf{Q}_\phi = \int_{\mathcal{BH}} \mathbf{Q}_\phi - \frac{\alpha'}{4\pi} \int_{\Sigma^3} \mathcal{W}_k \, .$$

where

$$\int_{S^2_\infty} \mathbf{Q}_\phi = \Sigma,$$

$$\int_{\mathcal{BH}} \mathbf{Q}_\phi = 2\alpha' \kappa \partial_\phi f(\phi_H),$$

$$\int_{\Sigma^3} \mathcal{W}_k = \partial^2_\phi f(\phi) d\phi P \widetilde{R}.$$

For the theories

$$S=\frac{1}{16\pi G_N^{(4)}}\int\left[-\star(e^a\wedge e^b)\wedge R_{ab}+\frac{1}{2}\,d\phi\wedge\star d\phi+\alpha'\,f(\phi)\,\mathcal{G}\right],$$

The no-hair theorem comes from the volume integral. After the asymptotic expansion,

$$-\frac{\alpha'}{4\pi}\int_{\Sigma}\mathcal{W}_k=-\,\alpha'\int_{r_H}^{\infty}\partial_\phi^2 f(\phi_\infty)\cdot\mathcal{O}(r^{-5})\,dr\,.$$

Which ensures absolute convergence for any coupling $f(\phi)$ that is smooth near ϕ_∞, and explains the wide range of black hole solutions reported for these theories.

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories: the scalar charge

For the theories

$$S = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \alpha' f(\phi) \mathcal{G} \right] ,$$

The generalized Komar charge is non-closed ($\mathbf{K}[k] = -\mathbf{Q}[k] + \imath_k \mathbf{L}_{\text{on-shell}}$)

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories: the scalar charge

For the theories

$$S = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \alpha' f(\phi)\, \mathcal{G} \right] ,$$

The generalized Komar charge is non-closed ($\mathbf{K}[k] = -\mathbf{Q}[k] + \imath_k \mathbf{L}_{\text{on-shell}}$)

$$d\mathbf{K}[k] = \alpha' \mathcal{Z}_k , \quad \mathcal{Z}_k = \mathcal{W}_k + \mathcal{Y}_k , \qquad \mathcal{Y}_k \equiv df(\phi) \wedge \mathcal{X}_k .$$

The Smarr formula reads

$$M = 2TS - \frac{\Sigma}{4} + 2\alpha' \Phi_{\alpha'} ,$$

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories: the scalar charge

For the theories

$$S = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \alpha' f(\phi) \mathcal{G} \right],$$

The generalized Komar charge is non-closed ($\mathbf{K}[k] = -\mathbf{Q}[k] + \imath_k \mathbf{L}_{\text{on-shell}}$)

$$d\mathbf{K}[k] = \alpha' \mathcal{Z}_k, \quad \mathcal{Z}_k = \mathcal{W}_k + \mathcal{Y}_k, \qquad \mathcal{Y}_k \equiv df(\phi) \wedge \mathcal{X}_k.$$

The Smarr formula reads

$$M = 2TS - \frac{\Sigma}{4} + 2\alpha' \Phi_{\alpha'},$$

where $S = \frac{1}{4G_N^{(4)}} \Big(\mathcal{A} + 2\alpha' f(\phi_H) \Big)$ and $\Phi_{\alpha'} \equiv \frac{1}{8\pi G_N^{(4)}} \left[\kappa \left(\partial_\phi f(\phi_H) - 2 f(\phi_H) \right) - \frac{1}{2} \int_\Sigma \mathcal{Z}_k \right].$

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories: the scalar charge

For the theories

$$S = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \alpha' f(\phi) \mathcal{G} \right],$$

Varying the entropy with respect to M, ϕ_∞ and α' is found the first law

$$\delta M = T\delta S - \frac{\Sigma}{4} \delta\phi_\infty + 2\Phi_{\alpha'}\delta\alpha'.$$

Recovering again the result of [Gibbons et al.]

Consistency check

Linear coupling: $f(\phi) = \phi$

$$\Sigma = 2\alpha' \kappa$$

$$S = \frac{1}{4G_N}\left(\mathcal{A} + 2\alpha' \phi_H\right)$$

$$\Phi_{\alpha'} = \frac{1}{8\pi G_N}\left[\kappa(1 - 2\phi_H) - \frac{1}{2}\int_\Sigma \mathcal{Y}_k\right]$$

The scalar charge is closed on-shell.

Consistency check

Linear coupling: $f(\phi) = \phi$

$$\Sigma = 2\alpha'\kappa$$

$$S = \frac{1}{4G_N}\left(\mathcal{A} + 2\alpha'\phi_H\right)$$

$$\Phi_{\alpha'} = \frac{1}{8\pi G_N}\left[\kappa(1-2\phi_H) - \frac{1}{2}\int_\Sigma \mathcal{Y}_k\right]$$

The scalar charge is closed on-shell.

Dilatonic coupling: $f(\phi) = e^{-\phi}$

$$\Sigma = -2\alpha'\kappa e^{-\phi_H} - \frac{\alpha'}{4\pi}\int_\Sigma \mathcal{W}_k$$

$$S = \frac{1}{4G_N}\left(\mathcal{A} + 2\alpha' e^{-\phi_H}\right)$$

$$\Phi_{\alpha'} = -\frac{3\kappa}{8\pi G_N}e^{-\phi_H}$$

The generalized Komar charge is closed on-shell.

New results: non-closed scalar charge and its role in the Smarr formula & the first law

For the theories

$$S = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \alpha' f(\phi) \mathcal{G} \right],$$

1. In general couplings, the scalar charge and the generalized Komar charge are non-closed forms and an explicit volume integral will contribute.

New results: non-closed scalar charge and its role in the Smarr formula & the first law

For the theories

$$S = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \alpha' f(\phi)\, \mathcal{G}\right],$$

1. In general couplings, the scalar charge and the generalized Komar charge are non-closed forms and an explicit volume integral will contribute.
2. It has a direct interpretation within the spontaneous scalarization mechanism.

New results: non-closed scalar charge and its role in the Smarr formula & the first law

For the theories

$$S = \frac{1}{16\pi G_N^{(4)}} \int \left[- \star (e^a \wedge e^b) \wedge R_{ab} + \frac{1}{2} d\phi \wedge \star d\phi + \alpha' f(\phi) \mathcal{G} \right],$$

1. In general couplings, the scalar charge and the generalized Komar charge are non-closed forms and an explicit volume integral will contribute.
2. It has a direct interpretation within the spontaneous scalarization mechanism.
3. The finiteness of the scalar charge clarifies why exists black hole solutions for a wide class of coupling functions, beyond the shift-symmetric case.

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories: spontaneous scalarization

Letting small perturbations around ϕ_∞ of the form

$$\phi = \phi_\infty + \epsilon\, \delta\phi^{(1)} + \mathcal{O}(\epsilon^2), \qquad \text{for} \qquad \epsilon \ll 1.$$

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories: spontaneous scalarization

Letting small perturbations around ϕ_∞ of the form

$$\phi = \phi_\infty + \epsilon\,\delta\phi^{(1)} + \mathcal{O}(\epsilon^2), \qquad \text{for} \qquad \epsilon \ll 1.$$

A Taylor expansion of the coupling function and its derivative yields

$$f(\phi) = f(\phi_\infty) + \epsilon\,\delta\phi^{(1)}\partial_\phi f(\phi_\infty) + \mathcal{O}(\epsilon^2),$$
$$\partial_\phi f(\phi) = \partial_\phi f(\phi_\infty) + \epsilon\,\delta\phi^{(1)}\partial_\phi^2 f(\phi_\infty) + \mathcal{O}(\epsilon^2).$$

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Letting small perturbations around ϕ_∞ of the form

$$\phi = \phi_\infty + \epsilon\,\delta\phi^{(1)} + \mathcal{O}(\epsilon^2), \qquad \text{for} \qquad \epsilon \ll 1.$$

A Taylor expansion of the coupling function and its derivative yields

$$\begin{aligned} f(\phi) &= f(\phi_\infty) + \epsilon\,\delta\phi^{(1)}\partial_\phi f(\phi_\infty) + \mathcal{O}(\epsilon^2), \\ \partial_\phi f(\phi) &= \partial_\phi f(\phi_\infty) + \epsilon\,\delta\phi^{(1)}\partial_\phi^2 f(\phi_\infty) + \mathcal{O}(\epsilon^2). \end{aligned}$$

Plugging these expansions into the scalar equation of motion and keeping terms up to $\mathcal{O}(\epsilon)$, we obtain

$$\mathbf{E}_\phi = -\epsilon\, d\star d\,\delta\phi^{(1)} + \partial_\phi f(\phi_\infty)\,\mathcal{G} + \epsilon\,\delta\phi^{(1)}\partial_\phi^2 f(\phi_\infty)\,\mathcal{G} + \mathcal{O}(\epsilon^2).$$

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories: spontaneous scalarization

Imposing that the scalar-free solution $\phi = \phi_\infty$ satisfies the background equation requires $\partial_\phi f(\phi_\infty) = 0$, so that the unperturbed configuration solves the equation at zeroth order.

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories: spontaneous scalarization

Imposing that the scalar-free solution $\phi = \phi_\infty$ satisfies the background equation requires $\partial_\phi f(\phi_\infty) = 0$, so that the unperturbed configuration solves the equation at zeroth order. At linear order in ϵ, the perturbation obeys the equation

$$d \star d\, \delta\phi^{(1)} + m^2_{\text{eff}}(r)\, \delta\phi^{(1)} = 0,$$

where the $m^2_{\text{eff}}(r)$is the effective mass squared and given by $m^2_{\text{eff}}(r) = -\partial^2_\phi f(\phi_\infty)\, \mathcal{G}(r)$.

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories: spontaneous scalarization

Imposing that the scalar-free solution $\phi = \phi_\infty$ satisfies the background equation requires $\partial_\phi f(\phi_\infty) = 0$, so that the unperturbed configuration solves the equation at zeroth order. At linear order in ϵ, the perturbation obeys the equation

$$d \star d\, \delta\phi^{(1)} + m^2_{\text{eff}}(r)\, \delta\phi^{(1)} = 0,$$

where the $m^2_{\text{eff}}(r)$is the effective mass squared and given by $m^2_{\text{eff}}(r) = -\partial^2_\phi f(\phi_\infty)\, \mathcal{G}(r)$. Since $\mathcal{G}(r) > 0$ outside the event horizon of a Schwarzschild spacetime, the sign of m^2_{eff} is determined by the second derivative of the coupling function. When this quantity is positive, $\partial^2_\phi f(\phi_\infty) > 0$, the scalar perturbation becomes tachyonic and grows dynamically, indicating the onset of spontaneous scalarization and the formation of a nontrivial scalar configuration.

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories: spontaneous scalarization

A particular coupling in EsGB admits spontaneous scalarization when the following two conditions are simultaneously satisfied:

1. The scalar-free background must solve the field equations,

$$f(\phi_\infty) = \text{const}, \qquad \partial_\phi f(\phi_\infty) = 0. \tag{2}$$

2. The trivial configuration must be unstable under scalar perturbations,

$$\partial_\phi^2 f(\phi_\infty) > 0. \tag{3}$$

New results: non-closed scalar charge and its role in the Smarr formula & the first law

A linearized scalar charge of the form

$$\Sigma^{(1)} = -\frac{\alpha'}{4\pi} \int_{\Sigma^3} \partial_\phi^2 f(\phi_\infty)\, d(\delta\phi^{(1)}) \wedge \mathcal{X}_k \,, \tag{4}$$

where $\Sigma^{(1)} = -\frac{1}{4\pi} \int_{S^2_\infty} \iota_k \star d(\delta\phi^{(1)})$.

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories: spontaneous scalarization

A linearized scalar charge of the form

$$\Sigma^{(1)} = -\frac{\alpha'}{4\pi}\int_{\Sigma^3} \partial_\phi^2 f(\phi_\infty)\, d(\delta\phi^{(1)}) \wedge \mathcal{X}_k \,, \tag{4}$$

where $\Sigma^{(1)} = -\frac{1}{4\pi}\int_{S^2_\infty} \iota_k \star d(\delta\phi^{(1)})$. The last equation, together with the first condition implies that scalarization may occur since the perturbation $\delta\phi^{(1)}$ can generate a non-vanishing perturbed scalar charge proportional to $\partial_\phi^2 f(\phi_\infty)$.

New results: non-closed scalar charge and its role in the Smarr formula & the first law

Einstein-scalar-Gauss-Bonnet theories: spontaneous scalarization

A linearized scalar charge of the form

$$\Sigma^{(1)} = -\frac{\alpha'}{4\pi} \int_{\Sigma^3} \partial_\phi^2 f(\phi_\infty)\, d(\delta\phi^{(1)}) \wedge \mathcal{X}_k \,, \tag{4}$$

where $\Sigma^{(1)} = -\frac{1}{4\pi} \int_{S^2_\infty} \iota_k \star d(\delta\phi^{(1)})$. The last equation, together with the first condition implies that scalarization may occur since the perturbation $\delta\phi^{(1)}$ can generate a non-vanishing perturbed scalar charge proportional to $\partial_\phi^2 f(\phi_\infty)$. Additionally, it gives support to the non-closedness of the scalar charge and the bulk term provides a geometric measure of the instability.

Part V: Conclusions and future directions

- The role of scalar charges is still not clear...

Part V: Conclusions and future directions

- The role of scalar charges is still not clear...
- It is also possible to investigate how the restrictions on the scalar potential affect the existence of hairy black hole solutions in asymptotically Anti-de Sitter spacetimes.

Part V: Conclusions and future directions

- The role of scalar charges is still not clear...
- It is also possible to investigate how the restrictions on the scalar potential affect the existence of hairy black hole solutions in asymptotically Anti-de Sitter spacetimes.
- Extend the study to other *hairs* in black hole spacetimes.

Part V: Conclusions and future directions

- The role of scalar charges is still not clear...
- It is also possible to investigate how the restrictions on the scalar potential affect the existence of hairy black hole solutions in asymptotically Anti-de Sitter spacetimes.
- Extend the study to other *hairs* in black hole spacetimes.
- Extend the Wald's formalism to define charges for extremal black holes.

Part V: Conclusions and future directions

- The role of scalar charges is still not clear...
- It is also possible to investigate how the restrictions on the scalar potential affect the existence of hairy black hole solutions in asymptotically Anti-de Sitter spacetimes.
- Extend the study to other *hairs* in black hole spacetimes.
- Extend the Wald's formalism to define charges for extremal black holes.
- More...

Acknowledgements

- This presentation was supported by Vicerrectoría de Investigación y Doctorados from the Universidad San Sebastián — Proyecto USS-FIN-25-PDOC-05.