A Conformal Approach to Aristotelian Symmetries

Eric Bergshoeff
Groningen University

presentation given at the

IV GRASS-SYMBHOL Meeting

work done in collaboration with
G. Giorgi, J. Rosseel and P. Salgado-Rebolledo

November 12 2025

Toledo, Spain

Motivation

In recent years several non-Lorentzian symmetries have received special attention, in particular:

$$\begin{array}{ccccc} \text{Poincare} & \Rightarrow & \text{Carroll/Galilei} & \Rightarrow & \text{Aristotelian} \\ \text{(rotations)} & & \text{(boosts)} & & \text{(no boosts)} \end{array}$$

We wish to investigate the conformal extensions of these symmetries having in mind the application of the 'conformal technique' to construct different matter couplings that are relevant to

- suggested relation between Carroll-like geometry and de Sitter Holography

  Blair, Obers, Yan (2025); Fontanella, Payne (2025)
- fractons : objects with restricted mobility

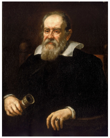
  Haah (2011); Vijay, Haah, Fu (2016); Pretko (2017); Gromov (2019)

Note: the symmetries of 'daily life' non-relativistic gravity are given by

$$\text{Bargmann} \quad = \quad \text{centrally extended Galilei}$$

see talks by Romano and Simon

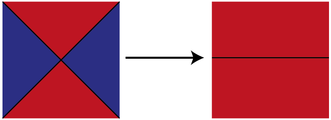
Galilei Symmetries



Galilei symmetries can be obtained as the non-relativistic or $c \to \infty$ limit of the Poincare symmetries

Galilei symmetries are known to relate inertial frames

They are not sufficient to describe the symmetries of the Schrödinger equation



Souriau investigated also representations of the un-extended Galilei symmetries

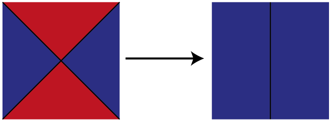
Question: can they describe yet to be discovered exotic Galilei particles ?

Carroll Symmetries

Lévy-Leblond (1965), Sen Gupta (1966)

Carroll symmetries are the opposite of Galilei symmetries in the sense that they can be obtained as the $c \to 0$ limit of the Poincare symmetries

It has been shown that there are two kinds of possible Carroll particles : ones that have non-zero energy but they cannot move (which leads to their name) and ones that can move but they have zero energy



Carroll symmetries have recently re-occurred in several areas of theoretical physics such as at the boundary of flat spacetime, at black hole horizons and even around black hole singularities

Aristotelian Symmetries

Penrose (1968)

There are many physical systems without boost symmetries :

- non-relativistic naturalness

  Horava (2016); Yan (2017)

- hydrodynamics

  . Novak, Sonner, Withers(2020); de Boer, Hartong, Have, Obers, Sybesma (2020); Armas, Jain (2021); Marotta, Szabo (2023)

- Fractional Quantum Hall Effect and semimetals

  Gromov, Son (2017); Ghosh, Peña-Benítez, Salgado-Rebolledo (2025)

- fractons in curved spacetime

  Chamon (2005); Haah (2011); Pretko (2017-2019)

Aristotelian cannot be obtained as a limit of GR: it is the intersection of Galilei with Carroll

Foliated Geometries

Extending particles to p-branes requires working with generalized so-called p-brane limits

time and space directions with clock τ_μ and ruler $e_\mu{}^a$ $\Rightarrow$
longitudinal and transverse directions with Vielbeine $\tau_\mu{}^A$ and $e_\mu{}^a$

Outline

The Conformal Technique

Outline

Outline

The Conformal Technique

A Conformal Approach to Carroll Gravity

From Carroll to Aristotelian Gravity

Outline

Outline

Outline

A Conformal Approach to General Relativity

There is a relationship between the Einstein-Hilbert term $R(\Omega(E))$ and the kinetic term of a real scalar field ϕ:

$$\mathcal{L} \sim R\big(\Omega(E)\big) \quad \rightarrow \quad \mathcal{L} \sim \phi\big[\Box + R(\Omega(E))\big]\phi \quad \rightarrow \quad \mathcal{L} \sim \phi\Box\phi$$

"$\Rightarrow$": first redefine $E_\mu^{(\mathrm{P})A} = \phi\, E_\mu^{(\mathrm{C})A}$ and then take a flat spacetime geometry $E_\mu^{(\mathrm{C})A} = \delta_\mu{}^A$

"$\Leftarrow$": first couple the real scalar field ϕ to conformal gravity obtained by 'gauging' the conformal algebra and next impose the gauge-fixing condition $\phi =$ constant

Note: (1) the dilatation gauge field B_μ is absent and the special conformal gauge field $F_\mu{}^A$ is dependent: $F_\mu{}^A \sim R\big(\Omega(E)\big)$ and (2) The conformal gauge fields are not true connections under the homogeneous conformal transformations

This conformal technique has been very useful in constructing matter couplings in supergravity theories

Gauging the Conformal Algebra

generators $\{P_A\,, M_{AB}\,, D\,, K_A\}$ $\Rightarrow$ gauge fields $\{E_\mu{}^A\,, \Omega_\mu{}^{AB}\,, B_\mu\,, F_\mu{}^A\}$

the Vielbein fields $E_\mu{}^A$ are invertible and we do not gauge the P-translations

we use the invertibility of $E_\mu{}^A$ to solve for some of the gauge fields in terms of the other ones by imposing curvature constraints $\rightarrow$

$$\Omega_\mu{}^{AB} = \Omega_\mu{}^{AB}(E,B) \quad \text{and} \quad F_\mu{}^A = F_\mu{}^A(E,B) \sim R'_{\mu B}(M^{AB})$$

Coupling a massless scalar ϕ with scaling weight w

$$\mathcal{L}_{\rm scalar} = -\frac{1}{2}\,\phi\,\partial^A\partial_A\phi$$

to conformal gravity leads to

$$D_{\hat{A}}\phi \equiv E_A{}^\mu\,(\partial_\mu - w\,B_\mu)\,\phi$$

$$\Box^C\phi \equiv E^{\hat{A}\mu}\Big[\partial_\mu\,(D_{\hat{A}}\phi) + \Omega_{\mu\hat{A}}{}^{\hat{B}}(E,B)\,D_{\hat{B}}\phi - (w+1)\,B_\mu\,D_{\hat{A}}\phi + 2\,w\,F_{\mu\hat{A}}(E,B)\,\phi\Big]$$

and an invariant action for $w = \frac{1}{2}(D-2)$

Outline

Intrinsic Torsion

Figueroa-O'Farrill, van Helden, Rosseel, Rotko + E.B. (2023)

In GR a torsion tensor $T_{\mu\nu}^{A}$ is defined by $T_{\mu\nu}^{A} \equiv 2\,\partial_{[\mu} E_{\nu]}^{A} - 2\Omega_{[\mu}{}^{AB} E_{\nu]B}$

In GR all torsion tensors depend on the spin-connection and all spin-connections are dependent torsionful spin-connections: $\Omega_\mu{}^{AB} = \Omega_\mu{}^{AB}(E, T)$

In non-Lorentzian geometries some so-called intrinsic torsion tensor components are independent of the spin-connection and some spin-connection components remain independent

Setting intrinsic torsion tensor components to zero leads to geometric constraints

Example: NC gravity contains the following intrinsic torsion tensor components:

$$T_{\mu\nu}^{0} = 2\partial_{[\mu}\tau_{\nu]}$$

$T_{\mu\nu}^{0} = 0 \quad \Rightarrow \quad$ time is absolute with an integrable foliation

The Particle Case

Baiguera, Oling, Sybesma, and Søgaard (2023); Concha, Fierro, Rodríguez, Rosseel + E.B. (2025)

In the same way that the particle Carroll algebra can be obtained as the particle Carroll limit of the Poincaré algebra with $\hat{A} = (0, A)$, the conformal Carroll extension we need can be obtained as a particle Carroll limit of the relativistic conformal algebra

One can take two different Carroll limits of a relativistic scalar:

$$\mathcal{L}_{\text{electric scalar}} = -\frac{1}{2}\phi\partial_t\partial_t\phi \quad \text{and} \quad \mathcal{L}_{\text{magnetic scalar}} = \pi\partial_t\phi - \frac{1}{2}\partial^a\phi\partial_a\phi$$

magnetic Carroll

(1) the dilatation gauge field components b_a are shifted away by the transverse special conformal transformations

(2) the special conformal gauge fields f_μ and ${g_\mu}^a$ generate curvature terms in the correct boost-invariant combination:

$$\mathcal{L}_{\text{Carroll gravity}} \sim R_{0a}(J^{0a}) + R_{ab}(J^{ab})$$

The underlying Carroll geometry has zero extrinsic curvature

From Particles to p-branes

Taking the p-brane Carroll limit of the EH term leads to the following p-brane Carroll gravity Lagrangians:

$$p > 0: \quad \mathcal{L}_{\text{Carroll gravity}} \quad \sim \quad R_{AB}(J^{AB}),$$

$$p = 0: \quad \mathcal{L}_{\text{Carroll gravity}} \quad \sim \quad R_{Aa}(J^{Aa}) + R_{ab}(J^{ab})$$

This shows that the particle case is special !

Defining the p-brane conformal Carroll algebra relevant for the conformal technique as the p-brane Carroll limit of the relativistic conformal algebra does not work!

Based upon the observation that the generic Carroll gravity Lagrangians do not contain a boost curvature we propose for the generic case the following conformal Carroll algebra:

$$\mathfrak{so}(2, p+1) \times \mathfrak{so}(D-p-1)$$

The conformal technique for $p > 0$ Carroll branes

'Gauging' of the conformal algebra $\mathfrak{so}(2, p+1) \times \mathfrak{so}(D-p-1)$ leads to

generators : $\{P_A\,, M_{AB}\,, K_A\,, D; \quad P_a\,, J_{ab}\} \quad \Rightarrow$

gauge fields : $\{\tau_\mu{}^A\,, \omega_\mu{}^{AB}\,, f_\mu{}^A\,, b_\mu; \quad e_\mu{}^a\,, \omega_\mu{}^{ab}\}$

Applying the conformal technique to the longitudinal part of this algebra gives the following action:

$$S_{\text{Carroll gravity}} \quad \sim \quad \frac{1}{4}\frac{p-1}{p}\int d^D x\, E\, R_{AB}{}^{AB}(J)$$

This action is invariant under the following emerging Carroll boost and anisotropic dilatation symmetries:

$$\delta\tau_\mu{}^A = \lambda^A{}_a e_\mu{}^a + (D-p-1)\lambda_{\rm D}\tau_\mu{}^A\,, \qquad \delta e_\mu{}^a = -(p-1)\lambda_D\, e_\mu{}^a$$

There are two exceptions:

- the particle with $p=0$: requires a different conformal extension
- the string with $p=1$: no conformal technique

A Duality

There exists the following formal duality between foliated Carroll and Galilei geometries:

$$p - \text{brane Carroll geometry} \qquad \longleftrightarrow \qquad (D-p-2) - \text{brane Galilei geometry}$$

$$(\tau_\mu{}^A, e_\mu{}^a) \qquad \longleftrightarrow \qquad (e_\mu{}^b, \tau_\mu{}^B)$$

where range $A =$ range b, range $a =$ range B and

$$\eta_{AA'} \;\leftrightarrow\; \delta_{bb'} \qquad \text{and} \qquad \delta_{aa'} \;\leftrightarrow\; \eta_{BB'}$$

This suggests that in the Galilei case we take the following conformal extension:

$$\mathfrak{so}(1,p) \times \mathfrak{so}(1, D-p)$$

The special cases are the defect brane with $p = D-3$ and the domain wall with $p = D-2$

Outline

p-brane Aristotelian Geometry

The p-brane Aristotelian G-structure on a D-dimensional manifold is given by

$$G = \mathrm{SO}(p,1) \times \mathrm{SO}(D-p-1) \qquad \text{(no boosts!)}$$

generators $\{P_A\,, M_{AB}\,, P_a\,, M_{ab}\}$ $\Rightarrow$ gauge fields $\{\tau_\mu{}^A\,, \omega_\mu{}^{AB}\,, e_\mu{}^a\,, \omega_\mu{}^{ab}\}$

We have two torsion tensors:

$$T_{\mu\nu}{}^A \;\equiv\; 2\partial_{[\mu}\tau_{\nu]}{}^A - 2\omega_{[\mu}{}^{AB}\tau_{\nu]B}\,, \qquad E_{\mu\nu}{}^a \;\equiv\; 2\partial_{[\mu}e_{\nu]}{}^a - 2\omega_{[\mu}{}^{ab}e_{\nu]b}$$

Some components are intrinsic torsion tensors; the remaining components can be used to solve for all components of the spin-connection fields $\omega_\mu{}^{AB}$ and $\omega_\mu{}^{ab}$

Constraining the different intrinsic torsion tensor components leads to 64 different Aristotelian geometries. The special case of particles or domain walls leads to only 16 different geometries

Figueroa-O'Farrill (2019)

The isotropic conformal Aristotelian algebra

We consider the direct sum of a longitudinal and transverse conformal algebra:

$$\mathfrak{so}(2, p+1) \oplus \mathfrak{so}(1, D-p)$$

generators $\{P_A\,, M_{AB}\,, K_A\,, D_1; \quad P_a\,, J_{ab}\,, K_a\,, D_2\} \quad \Rightarrow$

gauge fields $\{\tau_\mu{}^A\,, \omega_\mu{}^{AB}\,, f_\mu{}^A\,, b_\mu; \quad e_\mu{}^a\,, \omega_\mu{}^{ab}\,, f_\mu{}^a\,, c_\mu\}$

We have now two ‘conformal’ torsion tensors:

$$\mathcal{T}_{\mu\nu}{}^A = T_{\mu\nu}{}^A - 2b_{[\mu}\tau_{\nu]}{}^A\,, \qquad \mathcal{E}_{\mu\nu}{}^a = E_{\mu\nu}{}^a - 2c_{[\mu}e_{\nu]}{}^a$$

We can solve for all spin-connection fields and the traces $f_A{}^A\,, f_a{}^a$ of the special conformal gauge fields but not for the dilatation gauge field components b_A and c_a

The isotropic conformal Aristotelian algebra can be used to construct magnetic Aristotelian gravity via the conformal technique

Magnetic Aristotelian gravity

Consider two real scalars ϕ and ψ for $p \neq 0, 1, D-3, D-2$:

$$\mathcal{S}_2 = \int d^D x\, E\left[\alpha\psi^x\phi D^A D_A\phi + \beta\phi^y\psi D^a D_a\psi\right]$$

with real numbers $\alpha, \beta \neq 0$ and $x,\ y \in \mathbb{R}$. Invariance under special conformal transformations K_A and K_a and the two dilatations D_1 nd D_2 requires that the weights w_1 and w_2 of ϕ, ψ and the numbers x, y are given by

$$w_1 = -\frac{(p-1)}{2}\,, \quad w_2 = -\frac{(D-p-3)}{2}\,, \quad x = 2\frac{(D-p-1)}{(D-p-3)}\,, \quad y = 2\frac{(p+1)}{(p-1)}$$

Fixing the conformal symmetries by imposing the gauge-fixing conditions

$$\phi = \psi = 1 \qquad \text{and} \qquad b_A = c_a = 0$$

one obtains the following magnetic Aristotelian gravity action:

$$S_{\text{magnetic}} = \int d^D x\, E\left[\frac{\alpha}{4}\frac{(p-1)}{p}R_{AB}{}^{AB}(M) + \frac{\beta}{4}\frac{(D-p-3)}{(D-p-2)}R_{ab}{}^{ab}(J)\right]$$

The an-isotropic conformal Aristotelian algebra

We consider the following subalgebra of the isotropic algebra:

generators $\{P_A\,, M_{AB}\,, P_a\,, J_{ab}\,, D \equiv zD_1 + D_2\} \quad \Rightarrow$

gauge fields $\{\tau_\mu{}^A\,, \omega_\mu{}^{AB}\,, e_\mu{}^a\,, \omega_\mu{}^{ab}\,, b_\mu\}$

In the an-isotropic case, the two 'conformal' torsion tensors are given by

$$\tilde{\mathcal{T}}_{\mu\nu}{}^A = T_{\mu\nu}{}^A - 2z\, b_{[\mu}\tau_{\nu]}{}^A\,, \qquad \tilde{\mathcal{E}}_{\mu\nu}{}^a = E_{\mu\nu}{}^a - 2b_{[\mu}e_{\nu]}{}^a$$

We can solve for all spin-connection fields $\omega_\mu{}^{AB}\,, \omega_\mu{}^{ab}$ and all dilatation gauge fields components b_μ

Note: in the conformal case we have less 'conformal' intrinsic torsion tensors. This leads to only 16 different conformal Aristotelian geometries for $0 < p < D-2$ and to 4 different geometries for the special case of particles and domain walls

The isotropic conformal Aristotelian algebra can be used to construct Aristotelian matter couplings via the conformal technique

Outline

Quadratic Derivative Aristotelian Scalars

An action for a real quadratic derivative Aristotelian scalar ρ is given by

$$\mathcal{S}_\rho = \frac{1}{2}\int d^D x\, E\Big(c_1\, \partial_A\rho\partial^A\rho + c_2\, \partial_a\rho\partial^a\rho\Big)\,,$$

where $c_1 \neq c_2$ are real and $z = 1\,, w = 1/2(2-D)$

The coupling to Aristotelian conformal gravity requires the use of dependent dilatation gauge fields

$$b_a = \frac{1}{z(p+1)}\, T_a{}^A{}_A\,, \qquad b_A = \frac{1}{D-p-1} E_A{}^a{}_a$$

that are invariant under special conformal transformations. Imposing the gauge-fixing condition $\rho = 1$ leads, with $\alpha, \beta \sim c_1, c_2$, to the following action for electric Aristotelian gravity:

$$\mathcal{S}_{\text{electric}} \sim \int d^D x\, E\Big(\alpha\, T_a{}^A{}_A T^{aB}{}_B + \beta\, E_A{}^a{}_a E^{Ab}{}_b\Big)$$

Going complex, i.e. $\rho \to \Phi = \rho e^{i\theta}$ we have both dilatations and U(1) transformations

Gauge-equivalent Formulations

Suppose $\mathcal{L}(A) \to \mathcal{L}(B, C)$ by the replacement $A \equiv B + C$ with resulting gauge-invariance $\delta B = \epsilon(x)\,, \delta C = -\epsilon(x)$

The field C is called a compensating field. One can re-obtain $\mathcal{L}(A)$ from $\mathcal{L}(B, C)$ by imposing the gauge condition $C = 0$. The two formulations $\mathcal{L}(A)$ and $\mathcal{L}(B, C)$ are called gauge-equivalent

We will work with a complex scalar $\Phi = \rho e^{i\theta}$ where ρ is a compensating field for dilatations and θ compensates for U(1) (and, see next page, dipole symmetries)

Example with U(1) : Starting from the Lagrangian $\mathcal{L} = -\frac{1}{2}\left(\partial_\mu\theta\right)^2$ and gauging the symmetry $\delta\theta = m\Lambda$ leads to the Lagrangian $\mathcal{L} = -\frac{1}{2}\left(D_\mu\theta\right)^2$ with $D_\mu\theta = \partial_\mu\theta - mA_\mu$ or

$$\mathcal{L}(A,\theta) = \frac{1}{2}m^2 W_\mu^2 \qquad \text{with} \qquad W_\mu \equiv A_\mu - \frac{1}{m}\partial_\mu\theta$$

Combined with a kinetic term for the gauge field A_μ this leads to a Proca action for the massive vector field W_μ

Higher Derivative Aristotelian Scalars

Haah (2011); Vijay, Haah, Fu (2016); Pretko (2017); Gromov (2019)

An action for a real higher-derivative Aristotelian scalar ρ is given by

$$\mathcal{S}_\rho^{\mathrm{HD}} = \frac{1}{2}\int d^Dx\Big(c_1\,\partial_A\rho\partial^A\rho + c_2\,X_{ab}X^{ab}\Big)\,,$$

with real c_1 , c_2 , $X_{ab} \equiv \partial_a\rho\partial_b\rho - \rho\partial_a\partial_b\rho$ and

$$z = \frac{p-3-D}{p-3}\,, \qquad w = \frac{p-3+D}{p-3}\,.$$

Going complex we obtain the fracton model with global symmetries:

$$\delta\Phi \equiv \delta\big(\rho e^{i\theta}\big) = \big(i\lambda_{U(1)} + i\lambda^a_{\mathrm{dp}}\,x_a + w\lambda_D\big)\,\Phi$$

The U(1) and dipole symmetries can be gauged by a longitudinal gauge field A_A and a symmetric transverse gauge field A_{ab} with

$$\delta A_A = \partial_A\lambda_{U(1)} - z\lambda_D A_A\,, \qquad \delta A_{ab} = \partial_a\partial_b\lambda_{U(1)} - 2\lambda_D A_{ab}$$

We find that after gauging the U(1) and dipole symmetries the gauge fields A_A and A_{ab} combine together with the compensating field θ into the invariant combinations

$$W_A = A_A - \partial_A\theta\,, \qquad \text{and} \qquad W_{ab} = A_{ab} - \partial_a\partial_b\theta$$

Kinetic Terms for Fracton Gauge Fields

We define the following gauge-invariant curvatures in flat spacetime:

$$F_{Aab} = F_{Aba} = \partial_A A_{ab} - \partial_a \partial_b A_A , \qquad F_{ab,c} = -F_{ba,c} = \partial_a A_{bc} - \partial_b A_{ac}$$

They can be used to construct the following action (with $b_1, b_2 \neq 0$):

$$\mathcal{S}_{\text{kinetic}} = \frac{1}{2} \int d^D x \left(b_1 \, F_{Abc} \, F^{Abc} + b_2 \, F_{ab,c} \, F^{ab,c} \right) ,$$

Problem : when gauging all symmetries, replacing the derivatives in the curvatures by derivatives that are covariant with respect to all symmetries except U(1) one obtains new curvatures that do not transform covariantly under U(1)

Bidussi, Hartong, Have, Musaeus, Prohazka (2022), Jain, Jensen (2022)

Solution : Use the compensating scalar θ to define new tilded curvatures that are invariant under U(1):

$$\tilde{F}_{Aab} \equiv D_A A_{ab} - D_{(a} D_{b)} A_A - D_A D_{(a} \partial_{b)} \theta + D_{(a} D_{b)} \partial_A \theta ,$$
$$\tilde{F}_{ab,c} \equiv D_a A_{bc} - D_b A_{ac} - D_a D_{(b} \partial_{c)} \theta + D_b D_{(a} \partial_{c)} \theta .$$

Different gaugings

(1) gauging dilatations and fixing $\rho = 1$ leads to a higher-derivative electric Aristotelian gravity theory plus a higher-derivative action for the θ scalars

(2) gauging the U(1) and dipole symmetries, adding kinetic terms for the fracton gauge fields in flat spacetime and fixing $\theta = 0$ leads to a Proca-like action for the fractonic gauge fields plus a higher-derivative action for the ρ scalars

(3) gauging dilatations, U(1) and dipole symmetries, adding the new kinetic term for the fracton gauge fields and fixing $\rho = 1, \theta = 0$ leads to the sum of an higher-derivative electric Aristotelian gravity action plus a Lagrangian action for the massive dipole gauge fields

Outline

Conclusions

Unlike Carroll and Galilei gravity, Aristotelian gravity can be constructed with no constraints on the geometry

non-trivial Galilean, Carroll and Aristotelian matter coupled gravity models can be constructed by introducing more conformal matter fields

supersymmetry ?

see talk by Simon

applications to massive spin-2 GMP modes in the Fractional Quantum Hall Effect ?

Aristotelian symmetries in the fracton-elasticity duality ?