Foliated BF theories and SymTFTs

Giacomo Giorgi

In collaboration with: Riccardo Argurio (ULB)

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Setup

- Symmetries play an important role in physics
  - Generalized symmetries [1401.0740 Kapustin, Seiberg; 1412.5148 Gaiotto, Kapustin, Seiberg, Willett]
  - Fractonic topological phases $\longrightarrow$ multipole symmetries (e.g., dipole, quadrupole) [cond-mat/0404182 Chamon; 1101.1962 Haah] giving rise to mobility constraints on the excitations
  - Subsystem symmetries [1803.02369 You, Devakul, Burnell, Sondhi], theory invariant under a symmetry acting on a submanifold
- A suitable framework is needed for fracton topological phases $\longrightarrow$ Foliated BF theories [Slagle, Aasen, Williamson, arXiv:1812.01613; Slagle, arXiv:2008.03852; 2310.06701v2 Ebisu, Honda, Nakanishi]

Motivation

- Fractons are quasiparticles that cannot move in isolation but can move when combined into bound states
- New symmetries $\longrightarrow$ subsystem symmetries and multipole symmetries beyond ordinary global symmetries
- Goals of this talk:
  - Introduce foliated BF theories for subsystem and multipole symmetries
  - Discuss the effect of torsion (or twist) terms on operator genuineness
  - Present ongoing work generalizing discrete anomaly terms

Outline

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Higher-form symmetries

Global symmetries $\Longleftrightarrow$ Topological operators

- A p-form global symmetry $G^{(p)}$ acts on p-dimensional objects (Wilson lines, ...)

$$W_q(\gamma^{(p)})$$

- The associated conserved current is

$$d \star j_{p+1} = 0$$

- The corresponding topological symmetry operator is supported on a codimension-$(p+1)$ manifold $\Sigma^{(d-p-1)}$:

$$U(\Sigma^{(d-p-1)}) = \exp\Big(i\lambda \int_{\Sigma^{(d-p-1)}} \star j_{p+1}\Big)$$

- $U(\Sigma^{(d-p-1)})$ is topological and invertible:

$$U_g(\Sigma^{(d-p-1)})\, U_h(\Sigma^{(d-p-1)}) = U_{gh}(\Sigma^{(d-p-1)}), \qquad g, h \in G^{(p)}$$

Subsystem symmetries

- Subsystem symmetries act not globally but along lower-dimensional submanifolds, such as lines or planes
  - They appear in fracton phases and other models with constrained mobility on the lattice and in the continuum
- Consider a foliation [2112.12735 Rayhaun, Williamson]

$$\mathcal{F} = \{ L^{(d-k)} \}$$

  of space by codimension-k, leaves $L^{(d-k)}$ of spatial dimension $d-k$
- A theory has a foliated form subsystem symmetry $G(k)(\mathcal{F})$ of codimension k if

$$U_g(L^{(d-k)})U_h(L^{(d-k)}) = U_{gh}(L^{(d-k)})$$

  with $U_g(L^{(d-k)})$ supported on $L^{(d-k)}$ for each $g \in G$ and each leaf $L^{(d-k)} \in \mathcal{F}$
- The operators $U_g(L^{(d-k)})$ are supported only on the leaves of the foliation, unlike higher-form symmetries, whose operators can be defined on any submanifold

Review of BF theory

- Standard $\mathbb{Z}_N$ BF theory

$$\mathcal{L}_{\mathrm{BF}} = \frac{iN}{2\pi} B_{d-p} \wedge dA_p$$

where A $U(1)$ p-form gauge field and B_{d-p-1} a $U(1)$ $(d-p-1)$-form gauge field

- The gauge transformations are

$$\begin{aligned} A_p &\rightarrow A_p + d\Lambda_{p-1}, \\ B_{d-p} &\rightarrow B_{d-p} + d\Lambda_{d-p-1} \end{aligned}$$

- The theory is topological
- The theory has a collection of gauge invariant Wilson operators

$$W_m(\Gamma_p) = e^{im \int_\Gamma A}, \quad W_n(\Sigma_{d-p}) = e^{in \int_\Sigma B}.$$

- Linking phase:

$$\langle W_m(\Gamma)\, W_n(\Sigma) \rangle = \exp\left(\tfrac{2\pi i}{N} nm \,\mathrm{Link}(\Gamma, \Sigma)\right).$$

BF theory with torsion

- We consider a 4d BF theory with an additional torsion term

$$S = \int_{M_4} \left(\frac{iN}{2\pi} B \wedge dA + \frac{iNq}{4\pi} B \wedge B \right), \quad q \in \mathbb{Z}$$

where A and B are $U(1)$ one- and two-form gauge fields.

- The gauge transformations are

$$B \to B + d\lambda_B \,, \qquad A \to A + d\lambda_A - q\lambda_B$$

- The equations of motion give

$$dB = 0 \,, \qquad dA + qB = 0$$

- Effect: Modifies linking phases and can render some Wilson operators non-genuine (i.e., must end on a higher-dimensional defect)

Topological operators

- The gauge transformations and the equations of motion imply that the Wilson surface of B on a closed surface Σ_2,

$$W_B(\Sigma_2) = \exp\left(i\int_\Sigma B\right)$$

  is gauge invariant and topological
- The Wilson line of A, $W_A(\gamma) = \exp\left(i\int_\gamma A\right)$, it is not. To be well defined, it needs to stay at the boundary of $(W_B)^q$

$$\mathcal{W}_A(D_2) \equiv W_A(\gamma)[W_B(D_2)]^q = \exp\left(i\int_{D_2} dA + qB\right)$$

  where D_2 is an open surface with $\delta D_2 = \gamma$
- $\mathcal{W}_A(D_2)$ is a non-genuine line operator depends on the surface D_2 attached to the line γ
- We also have from the theory that $[W_B(\Sigma_2)]^N = 1$ and $[\mathcal{W}_A(D_2)]^N = 1$, then

$$[W_A(\gamma)]^{N/g}\,, \qquad \text{with } \; g = \gcd(N, q)$$

  is a genuine line operator. $[W_B(\Sigma)]^g$ are instead endable

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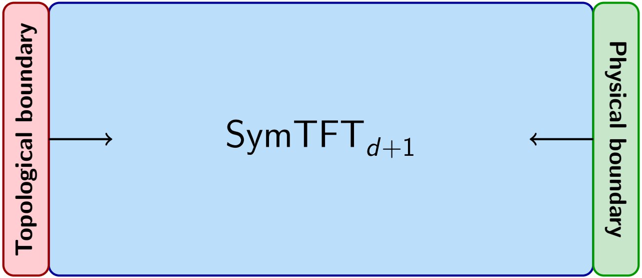
SymTFT Sandwich Construction



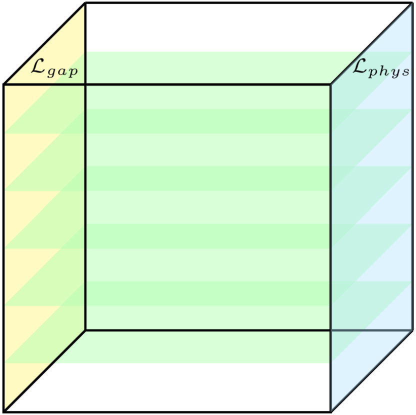
- The $(d+1)$-dimensional SymTFT encodes the anomaly structure and generalized symmetries.
- Operators on the physical boundary extend through the bulk to the topological boundary.
- Gluing the two boundaries defines the full QFT partition function.

SymTFTs for foliated theories

- SymTFTs are a powerful framework to describe the symmetry properties of a QFT
- Some works have started to explore the construction of symmetric topological field theories (SymTFTs) for foliated BF theories [2310.01474v3 Cao, Jia]
- In [2504.11449v2 Apruzzi, Bedogna, Mancani], the construction of SymTFTs for continuous subsystem symmetries was proposed and illustrated through several examples inspired by fractonic field theories

Mille-feuille construction

- The Mille-feuille construction generalizes the sandwich SymTFT idea
- The difference that is the foliated nature due to absence of Lorentz invariance.
- It involves a stack of multiple $(d+1)$-dimensional SymTFT layers along an additional foliation direction
- Couplings between layers capturing fractonic behavior



Taken from [2504.11449v2 Apruzzi, Bedogna, Mancani]

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Example of $\mathbb{Z}_N$ BF construction in 2+1 dimensions

- We consider a theory with global $U(1)$ 0-form symmetry and charge Q

$$Q(V) = \int_V \star j$$

- This charge is global $\longrightarrow$ $[P_I, Q] = 0$, with $I = x, y$ and where P_I generates translations
- We introduce a 1-form $U(1)$ gauge field a coupled with the current j

$$S_C = \int_V a \wedge \star j$$

with $a \to a + d\chi$, and $f = da$

- We introduce the Lagrangian

$$\mathcal{L} = \frac{N}{2\pi} b \wedge f = \frac{N}{2\pi} b \wedge da$$

with b a 1-form $U(1)$ gauge field

Construction of Foliated BF Theory (Dipole case)

- Theory with global $U(1)$ and dipole symmetries, with charges $Q\,, Q_x\,, Q_y$ [2201.10589 Gorantla, Lam, Seiberg, Shao]

$$[P_I, Q] = 0\,, \qquad [P_I, Q_J] = \delta_{IJ} Q$$

associated to conserved currents as

$$Q = \int_V \star j\,, \quad Q_I = \int_V \star K_I\,, \quad \text{with} \;\; \star K_I = \star k_I - x^I \star j$$

- We introduce the three 1-form gauge fields a,A_I coupled to the currents as

$$S_{\mathrm{dip}} = \int_V a \wedge \star j + A_I \wedge \star k_I$$

gauge invariant under ($\Lambda\,, \sigma_I$ the gauge parameters)

$$a \to a + d\Lambda + \sigma_I\, dx^I\,, \qquad A_I \to A_I + d\sigma_I$$

- We define the gauge invariant field strengths

$$f = da - A_I \wedge dx^I\,, \quad F_I = dA_I$$

Foliated dipole BF theory in 2+1 Dimensions

- We write now the foliated BF theory in 2+1 dimensions

$$\mathcal{L}_{\rm dip} = \frac{N}{2\pi}\Big(b \wedge f + \sum_I c_I \wedge F_I\Big)$$

- Introduce the foliation fields e^I. Foliation is defined to be a codimension-one submanifold which is orthogonal to the 1-form foliation field e^I:

$$\mathcal{L}_{\rm dip} = \frac{N}{2\pi}\Big(a \wedge db + \sum_I A_I \wedge dc_I + \sum_I A_I \wedge b \wedge dx^I\Big)$$

  with $e^x = dx\,, e^y = dy$ and

$$b \to b + d\lambda\,, \qquad c \to c^I + \gamma^I + \lambda e^I$$

- There are three layers of the 2+1d BF theories, corresponding to the first two terms
- The last term describes the coupling between the layers

Construction of Foliated BF Theory (Quadrupole case)

- Consider a theory with a global $U(1)$ charge and additional dipole and quadrupole symmetries $\longrightarrow Q, Q_x, Q_y, Qxy$

$$[P_I, Q] = 0\,, \qquad [P_I, Q_J] = \delta_{IJ} Q\,, \qquad [P_I, Q_{xy}] = Q_{\bar{I}}$$

with $I = x, \bar{I} = y$ and $I = y, \bar{I} = x$. The conserved charges are associated with currents

$$Q = \int_V \star j\,, \quad Q_I = \int_V \star K_I\,, \quad Q_{xy} = \int_V \star W$$

with

$$\star K_I = \star k_I - x^I \star j\,, \quad W = \star w - xy \star j - x^I \star k_I$$

- Introduce the 1-form gauge fields a, A_I, a' coupled to the corresponding currents

$$S_{\mathrm{qp}} = \int_V a \wedge \star j + A_I \wedge \star k_I + a' \wedge \star w$$

- The theory is gauge invariant under ($\Lambda\,, \sigma_I\,, \Lambda'$ the gauge parameters)

$$a \to a + d\Lambda + \sigma_I\, dx^I, \qquad A_I \to A_I + d\sigma_I + \Lambda' x^{\bar{I}}\,, \qquad a' \to a' + d\Lambda'$$

Foliated quadrupole BF theory in 2+1 Dimensions

- We define the gauge invariant field strengths

$$f = da - A_I \wedge dx^I , \quad F_I = dA_I - a' \wedge dx^{\bar{I}} , \quad f' = da'$$

- We write now the foliated BF theory in 2+1 dimensions

$$\begin{aligned}\mathcal{L}_{\mathrm{quad}} = \frac{N}{2\pi}\Big(& a \wedge db + a' \wedge db' + \sum_I A_I \wedge dc_I \\ & + \sum_I A_I \wedge b \wedge e^I + \sum_I a' \wedge c_I \wedge e^{\bar{I}} \Big)\end{aligned}$$

- Here a, a', A_I, b, b', c_I are 1-form gauge fields.
- We now have four BF layers (first line), with coupling terms between the layers (second line)
- In [2406.04919v1 Ebisu, Honda. Nakanishi] the construction is generalized to higher-form and subsystem symmetries

Subsystem BF theory model

- In the subsystem version for the dipole, we replace

$$c_I^{(d-p)} \;\to\; B_I^{(d-p-1)} \wedge e^I \,, \quad \text{(with } I \text{ not summed)}$$

leading to the Lagrangian (in d dimensions and $1 \leq p \leq d-1$)

$$\mathcal{L}_{\text{sub}} \;=\; \frac{N}{2\pi}\Big[b^{(d-p)} \wedge \Big(da^{(p)} + (-1)^p \sum_I A_I^{(p)} \wedge e^I\Big) + \sum_I B_I^{(d-p-1)} \wedge dA_I^{(p)} \wedge e^I\Big]$$

The first term in the Lagrangian describes a $(d+1)$-dimensional $\mathbb{Z}_N$ gauge theory. The third is a (continuous) stack of d-dimensional $\mathbb{Z}_N$ gauge theories for each foliation. The third term couples the d layers to the $(d+1)$ gauge theory.

- Invariance under the subsystem symmetry

$$A_I^{(p)} \;\to\; A_I^{(p)} + \gamma_I^{(p-1)} \wedge e^I$$

where $\gamma_I^{(p-1)}$ is an arbitrary $(p-1)$-form field

- Also possible in the quadrupole case

Foliated BF theory with twist term

- Let's consider, then, in 4 dimensions, the foliated BF theory

$$\mathcal{L}_{\text{sub}} = \frac{N}{2\pi}\Big[b \wedge \Big(da - q\sum_I A_I \wedge e^I\Big) + \sum_I B_I \wedge dA_I \wedge e^I\Big]$$

  where b is a 2-form, a a 1-form and B_I, A_I foliated 1-forms
- We have now inserted $q \in \mathbb{Z}$ in front of the twist term, this term produces an effect analogous to a torsion
- The gauge transformations are now

$$\begin{aligned}
\delta b &= d\chi\,,\\
\delta a &= d\lambda - q\sum_I \lambda^I e^I\,,\\
\delta B_I &= d\chi^I - q\chi\,,\\
\delta A_I &= d\lambda^I
\end{aligned}$$

  where χ is a 1-form parameter and $\lambda, \lambda^I, \chi^I$ are 0-form parameters
- For $q = 1$, this reduces to the standard foliated BF theory describing dipole symmetries

Equations of Motion and Topological Operators

- From the action, the equations of motion (with I not summed over) are:

$$
\begin{aligned}
(dB_I + q\, b) \wedge e^I &= 0, \\
db &= 0, \\
dA_I \wedge e^I &= 0, \\
q \sum_I A_I \wedge e^I + da &= 0
\end{aligned}
$$

  These relations encode the topological constraints between the foliated layers and the coupling parameter q

- We now construct the gauge-invariant topological operators of the model:

$$
W_b(\Sigma) = \exp\left(i \int_\Sigma b\right),
$$

  a surface operator, and

$$
W_A(\gamma) = \exp\left(i \oint_\gamma A_I\right),
$$

  line operators supported on curves γ orthogonal to e^I.

Non-Genuine Operators

- There are also two examples of partially topological line operators

$$W_a(\gamma) = \exp\left(i \oint_\gamma a\right), \qquad W_B(\gamma) = \exp\left(i \oint_\gamma B^I\right)$$

- We can open the surface Σ and restore gauge-invariance by adding the foliated operators defining the strip σ^I

$$\exp\left(i \oint_{\gamma(x_1^I)} a - i \oint_{\gamma(x_2^I)} a\right) = \exp\left(iq \int_{x_1^I}^{x_2^I} \oint_\gamma A^I \wedge e^I dx^I\right)$$

- The result are the non-genuine operators

$$\mathcal{W}(\sigma^I(x_1^I, x_2^I)) = \exp\left(i \int_{x_1^I}^{x_2^I} \oint_\gamma (da + qA^I \wedge e^I) dx^I\right),$$

$$\mathcal{W}(\sigma^I(x_1^I, x_2^I)) = \exp\left(i \int_{x_1^I}^{x_2^I} \oint_\gamma (dB^I + qb) dx^I\right)$$

- For $q = 1$ all W_a , W_B are non-enuine

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Conclusions and Outlook

- We have reviewed some approaches to constructing topological foliated BF theories
- These theories can serve as the foundation for SymTFT constructions, encoding the multipole and subsystem symmetries of fracton theories
- Future directions:
  - Generalizing the construction by including twist terms or other topological contributions for dipole and quadrupole symmetries
  - Studying the topological and physical boundaries of SymTFTs built from these foliated theories

Thank You!