Quantum Channels, Jones Index, and Entropic Signatures of Symmetry Breaking

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Collaboration w/

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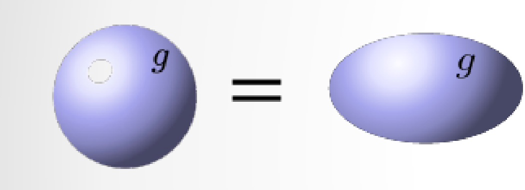
Outline

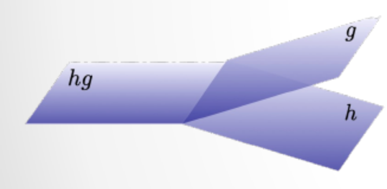
Motivation

In QFT, global symmetries $\equiv$Topological operators $U(\Sigma)$ [Gaiotto et al 2014].



Topological: deformations of Σ do not modify correlators

Linking: Operator crossing the defect $\rightarrow$ symmetry operation.

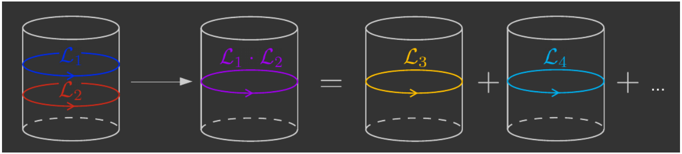


Fusion: Group multiplication law $U_g(\Sigma) \times U_h(\Sigma) = U_{hg}(\Sigma)$

Broken Symmetries

When a global symmetry involves an ordinary group, patterns of symmetry breaking are classified by subgroups of the original symmetry group.

Generalization: Non-Invertible Symmetries



$$\mathcal{L}_a \times \mathcal{L}_b = \sum_c N_{ab}^c \, \mathcal{L}_c$$

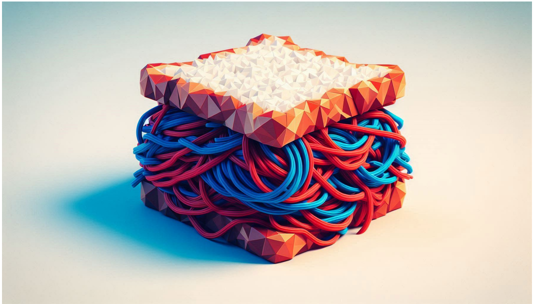
$\mathcal{L}'s$ don't have inverses. No conserved currents (in general) but conserved charges exist.

Broken Symmetries

Pattern of symmetry breaking $\equiv$ Frobenius algebra $\mathbb{A}$
analog of a subgroup, w/ the invariant subspaces under their action $\rightarrow$ different gaugings/ phases.

Generalized Landau Paradigm

- Phases of systems with non-invertible symmetries as patterns of symmetry breaking
- Order parameters for these symmetries
- Progress made w/ SymTFT [Bhardwaj et al 25]



Motivation

Our work

- We propose an algebraic and information-theoretic framework to analyze symmetry-breaking in generalized global symmetries.
- Elegant formulation within subfactor theory of von Neumann algebras.
- Subfactors provide a framework for quantifying quantum information loss which acts as an entropic order parameter.

ANNALS OF MATHEMATICS
Vol. 37, No. 1, January, 1936

ON RINGS OF OPERATORS

BY F. J. MURRAY* AND J. v. NEUMANN

(Received April 3, 1935)

Invent. math. 72, 1–25 (1983)

Index for Subfactors

V.F.R. Jones

Outline

Subfactors in von Neumann algebras

Quantum systems are described by their algebras of observables $\mathcal{M}$, which act on a Hilbert space $\mathcal{H}$.

An operator algebra consists of a set of operators s.t

$$\mathbf{1} \in \mathcal{M}\,, \quad a, b \in \mathcal{M} \implies \alpha a + \beta b \in \mathcal{M}\,, \quad a\,, b \in \mathcal{M}\,, \quad a^{\dagger} \in \mathcal{M}\,,$$

with $\alpha, \beta \in \mathbb{C}$.

$\mathcal{M}$ is a von Neumann algebra if $\mathcal{M} = \mathcal{M}''$, where the prime refers to the commutant,

$$\mathcal{M}' = \{b / \,[\,b,\, a\,] = 0,\, \forall a \in \mathcal{M}\}.$$

$\mathcal{M}$ is called a *factor* if $\mathcal{M} \cap \mathcal{M}' = \mathbb{C} \cdot \mathbf{1}$.

Factors are classified into Type I, II, and III. Here, we focus on Type I factors isomorphic to matrix algebras.

Subfactors in von Neumann algebras

A subfactor is a unital inclusion $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras where both $\mathcal{N}$ and $\mathcal{M}$ are *factors*.

Why we use Subfactors?

- Physically, we are considering a quantum mechanical system with some global symmetry that is described by an algebra $\mathcal{M}$ of operators.
- The full or partial symmetry breaking in the system implies that the superselection sectors describing the broken phase are given by a ”small” subalgebra $\mathcal{N} \subset \mathcal{M}$.

States in von Neumann algebras

Quantum states are described by linear maps $\omega : \mathcal{M} \to \mathbb{C}$

$$\omega(\alpha a + \beta b) = \alpha\,\omega(a) + \beta\,\omega(b)\,, \qquad \omega(aa^\dagger) \geq 0\,, \qquad \omega(\mathbf{1}) = 1\,.$$

In finite-dimensional algebras $\mathcal{M}$, each state ω is represented by a density matrix $\rho_\omega \in \mathcal{M}$ such that

$$\omega(a) = \mathrm{tr}\left(\rho_\omega\, a\right)\,, \;\; \forall a \in \mathcal{M}\,.$$

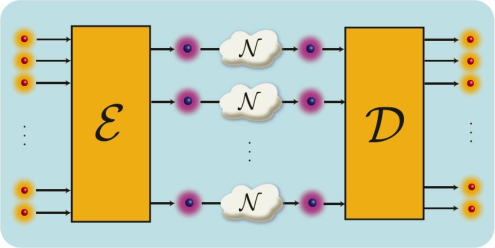
Conditional Expectations/Quantum Channels

Completely positive linear maps $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ that act as a projection from $\mathcal{M}$ onto $\mathcal{N}$.

For systems with finite-dimensional Hilbert space $\mathcal{H}$, $\mathcal{E}$ can be expressed as:

$$\mathcal{E}(m) = \sum_{i=1}^{N} \mathbb{K}_i \, m \, \mathbb{K}_i^\dagger$$

$\mathbb{K}_i$ *Kraus operators* $\implies$ $\mathcal{E}$ a specific example of a quantum channel.



Jones Index

For $\mathcal{N} \subset \mathcal{M}$, with $\mathcal{E} : \mathcal{M} \to \mathcal{N}$, the Jones (or Jones–Kosaki–Longo) index is defined as [Jones 83]

$$\lambda = [\mathcal{M} : \mathcal{N}] := [\sup\{\lambda > 0 : \mathcal{E}(m) \geq \lambda\, m \text{ for all } m \in \mathcal{M}_+\}]^{-1}.$$

that quantifies the relative "size" of $\mathcal{M}$ compared to $\mathcal{N}$, analogous to the index of a subgroup.

Relative entropy $H(\mathcal{M}|\mathcal{N})$ for inclusion $\mathcal{N} \subset \mathcal{M}$ of finite (dimensional) algebras [Connes and Störmer 75], [Pimsner and Popa 86],

$$H(\mathcal{M}|\mathcal{N}) \leq \log \lambda$$

Quantum Relative Entropy

Central quantity in quantum information theory . For Type I algebras,

$$S_{\mathcal{M}}(\rho\,|\,\sigma) = \mathrm{tr}_{\mathcal{M}}[\,\rho\,(\log\rho - \log\sigma)]\,,$$

where ρ, σ are two density operators associated with the two states in comparison.

Measures distinguishability between states. Natural option involving quantum channels is to use it as a measure of information loss as

$$S_{\mathcal{M}}(\rho\,|\,\mathcal{E}(\rho))$$

which in our setting will act as an Entropic Order Parameter [Casini et al 20]

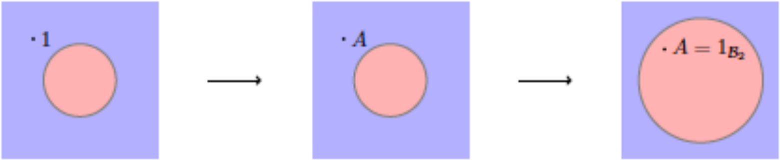
Outline

Phases of Non-Invertible Symmetries and Anyon Condensation

We focus in (2+1)-D where

- System of anyons with a generalized (noninvertible) symmetry
- New phase is formed when an anyon with charge a undergoes condensation, that is, a new vacuum is formed, in which the topological charge a is indistinguishable.
- The condensate breaks the original symmetry encoded in the topological excitation spectrum of the system. [Kong 2014]

Anyon Condensation



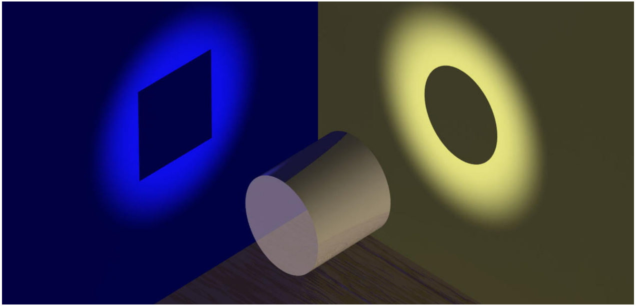
- Before condensation, the superselection sectors (anyon types) described by algebra $\mathcal{A}$.
- Condensation process reduces $\mathcal{A}$ to a smaller subalgebra $\mathcal{T}$.
- Operators of $\mathcal{A}$ become decomposable in terms of elements of $\mathcal{T}$.

$\mathcal{A}$

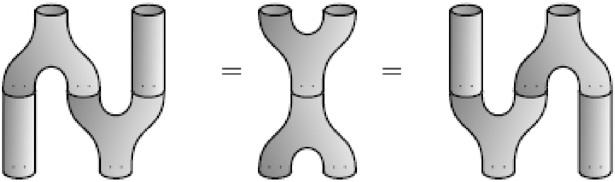
$\mathcal{T}$

Symmetry patterns described by a Frobenius algebra

$$\mathbb{A} = \sum_{a \in \mathcal{A}} w_{\mathbb{A}}^{a}\, a$$



Analog of a subgroup whose invariant subspaces $\equiv$ different gaugings/phases



$$\mathbb{A} \implies \mathcal{E}_{\mathbb{A}} : \mathcal{A} \to \mathcal{T} \text{ and a subfactor } \mathcal{T} \subset \mathcal{A}.$$

Subfactors $\cong$ Symmetries

Subfactor $\mathcal{T} \subset \mathcal{A} \longleftrightarrow$ Pattern of symmetry breaking $\equiv \mathbb{A}$

Restriction and Lift maps

Algebra $\mathcal{T}$, with labels $\phi, r, s, t, ..,$ and a set of branching (or restriction) coefficients n_a^t implementing [Bais et al 2009, 2014].

- The Restriction map, characterizing the new phase:

$$a \to \sum_{t \in \mathcal{T}} n_a^t \, t \qquad \forall a \in \mathcal{A} \qquad \text{(restriction)}$$

  where $n_a^t \in \mathbb{Z}_{\geq 0}$.

- The same branching coefficients also define the adjoint of the restriction map, referred to as the Lift map

$$t \to \sum_{a \in \mathcal{A}} n_a^t \, a \qquad \forall t \in \mathcal{T} \qquad \text{(lift)}$$

Consistency Conditions

We assume that the restriction $\mathcal{A} \to \mathcal{T}$ commutes with fusion

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} & \xrightarrow{f} & \mathcal{A} \\ \big\downarrow{\scriptstyle r\otimes r} & & \big\downarrow{\scriptstyle r} \\ \mathcal{T} \otimes \mathcal{T} & \xrightarrow[f]{} & \mathcal{T} \end{array}$$

Quantum Dimensions

$$d_a = \sum_{t\in\mathcal{T}} n_a^t d_t \quad \forall\, a \in \mathcal{A},$$
$$d_t = \frac{1}{q} \sum_{a\in\mathcal{A}} n_a^t d_a \quad \forall\, t \in \mathcal{T},$$

where q denotes the quantum dimension of the condensed vacuum,

$$q := \sum_{a \in \mathcal{A}} n_a^{\phi} d_a = \frac{D_{\mathcal{A}}^2}{D_{\mathcal{T}}^2}\,, \quad \text{w/} \quad D_{\mathcal{A}}^2 = \sum_{a \in \mathcal{A}} d_a^2\,, \quad D_{\mathcal{T}}^2 = \sum_{t \in \mathcal{T}} d_t^2\,,$$

the total quantum dimensions of $\mathcal{A}$ and $\mathcal{T}$

Subfactor $\mathcal{T} \subset \mathcal{A}$ and anyon condensation

In this language, the Jones index

$$\lambda = [\mathcal{A} : \mathcal{T}] \equiv q$$

Symmetry Quantum Channels (SymQuOp)

Frobenius Algebra $\mathbb{A} \to n_a^t$

$$\mathcal{E}_{\mathbb{A}} : \rho \in \mathcal{A} \xrightarrow{restr} \rho_{\mathbb{A}} \in \mathcal{T}$$

$$\alpha : \rho_{\mathbb{A}} \in \mathcal{T} \xrightarrow[lift]{} \tilde{\rho} \in \mathcal{A}$$

- Information Loss for non-symmetric states:

$$\alpha \circ \mathcal{E}_{\mathbb{A}}(\rho) = \tilde{\rho} \neq \rho \implies S_{\mathcal{A}}(\rho \,|\, \alpha \circ \mathcal{E}_{\mathbb{A}}(\rho)) > 0$$

- Symmetric states

$$\alpha \circ \mathcal{E}_{\mathbb{A}}(\rho) = \tilde{\rho} = \rho \implies S_{\mathcal{A}}(\rho \,|\, \alpha \circ \mathcal{E}_{\mathbb{A}}(\rho)) = 0$$

Quantum Channel $\mathcal{E}_{\mathbb{A}}$ for the Restriction map

Kraus representation

$$\mathcal{E}_{\mathbb{A}}(\rho) = \sum_{a\in\mathcal{A}} \mathbb{K}_a \, \rho \, \mathbb{K}_a^{\dagger} \,,$$

Anyonic superselection rules $\Longrightarrow$ a topological state given by a convex sum with a set of projectors Π_a s.t

$$\rho = \bigoplus_{a\in\mathcal{A}} \rho_a = \sum_{a\in\mathcal{A}} p_a \, \Pi_a \,, \quad \text{s.t} \quad \Pi_a^{\dagger} = \Pi_a, \quad \Pi_a \Pi_b = \delta_{ab} \Pi_a$$

with probabilities $p_a \equiv \mathrm{tr}(\rho \, \Pi_a)$ such that $\sum_a p_a = 1$.

Quantum Channel $\mathcal{E}_{\mathbb{A}}$ for the Restriction map

Kraus operators in terms of branching coefficients n_a^t

$$\mathbb{K}_a = \sum_{t\in\mathcal{T}} \sqrt{n_a^t \left(\frac{d_t}{d_a}\right)}\, \Lambda_{t,a}\,, \quad a \in \mathcal{A}, \quad t \in \mathcal{T}\,,$$

where $\Lambda_{t,a}\, \Pi_b = \delta_{ab}\, \Lambda_{t,a}, \quad \Lambda_{t,a}\, \Lambda_{b,s} = \delta_{ab}\, \delta_{ts}\, \Pi_t$.

$$\mathcal{E}_{\mathbb{A}}(\rho) = \sum_{a\in\mathcal{A}} \mathbb{K}_a\, \rho\, \mathbb{K}_a^\dagger = \sum_{t\in\mathcal{T}} p_t\, \Pi_t$$

where

$$p_t = \sum_{a\in\mathcal{A}} n_a^t \left(\frac{d_t}{d_a}\right) p_a\,.$$

Quantum Channel α for the Lift map

$\alpha : \mathcal{T} \to \mathcal{A}$ Lift map

$$\widetilde{\rho} = \alpha(\rho_{\mathbb{A}}) = \sum_{a\in\mathcal{A}} \mathbb{L}_a \, \mathcal{E}_{\mathbb{A}}(\rho) \, \mathbb{L}_a^\dagger$$

Implemented by the (non-Kraus) operators

$$\mathbb{L}_a = \sum_{t\in\mathcal{T}} \sqrt{\frac{n_a^t}{\lambda}\left(\frac{d_a}{d_t}\right)} \Lambda_{a,t} \quad a \in \mathcal{A} \quad t \in \mathcal{T}\,, \text{with}$$

$$\lambda = [\mathcal{A} : \mathcal{T}] = \sum_{a\in\mathcal{A}} n_a^\phi d_a = q$$

Then

$$\widetilde{\rho} = \alpha(\rho_{\mathbb{A}}) = \sum_{a\in\mathcal{A}} \mathbb{L}_a \, \mathcal{E}_{\mathbb{A}}(\rho) \, \mathbb{L}_a^\dagger = \sum_{a\in\mathcal{A}} \widetilde{p}_a \, \Pi_a \,,$$

Quantum Channel α for the Lift map

where the lifted probabilities $\tilde{p}_a$ are given by

$$\widetilde{p}_a = \frac{1}{\lambda} \sum_{t \in \mathcal{T}} n_a^t \left(\frac{d_a}{d_t} \right) p_t = \frac{1}{\lambda} \sum_{b \in \mathcal{A}} M'_{ab} \left(\frac{d_a}{d_b} \right) p_b \,,$$

and the matrix

$$M'_{ab} = \sum_{t \in \mathcal{T}} n_a^t n_b^t$$

encodes modular invariants governing the extension of superselection sectors and, in our setting, the the probabilities $\tilde{p}_a$ of the lifted states

Properties

1. **Idempotency**

$$\mathcal{E}_{\mathbb{A}}\left(\tilde{\rho}\right) = \mathcal{E}_{\mathbb{A}}\left(\alpha \circ \mathcal{E}_{\mathbb{A}}\left(\rho\right)\right) = \mathcal{E}_{\mathbb{A}}\left(\rho\right)$$

2. **Bimodule property**

$$\mathcal{E}_{\mathbb{A}}(\tilde{t}_1\, a\, \tilde{t}_2) = t_1\, \mathcal{E}_{\mathbb{A}}(a)\, t_2 \quad \forall\, t_1, t_2 \in \mathcal{T}, a \in \mathcal{A},$$

3. **Trace Preserving**

$$\mathrm{tr}_{\mathcal{T}}\, \mathcal{E}_{\mathbb{A}}(\rho) = \mathrm{tr}_{\mathcal{A}}\, \rho$$

Quantum Relative Entropy as Order Parameter

Quantifies the information loss due to symmetry breaking after the condensation process

Entropic Order Parameter

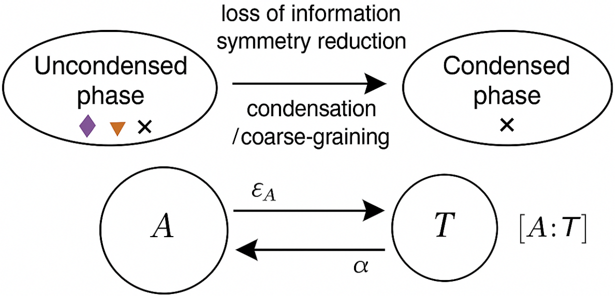
$$S_{\mathcal{A}}(\rho \,|\, \alpha \circ \mathcal{E}_{\mathbb{A}}(\rho)) = \log \lambda - H(p) - \sum_{a \in \mathcal{A}} p_a \log \left[\sum_{b \in \mathcal{A}} M'_{ab} \left(\frac{d_a}{d_b} \right) p_b \right] ,$$

where $H(p) = -\sum_{a \in \mathcal{A}} p_a \log p_a$.

Index Bound on Information Loss

$$S_{\mathcal{A}}(\rho | \, \alpha \circ \mathcal{E}_{\mathbb{A}}(\rho)) \leq \log \lambda$$

Summary



$$S_A\left(\rho \,||\, \alpha \bullet \varepsilon_A(\rho) \leq \log[A:T]\right.$$

Information loss bounded by
quantum dimension of condensate

If anomaly $[\omega] \neq 0$:
only $H \subset G$ with
$[\omega]_H$ 0 dmiss-
sible

Outline

Abelian Group $\mathbb{Z}_N$

Condensable algebra

$$\mathbb{A} = \bigoplus_r X_r \quad \longleftrightarrow \quad n_r^{\phi} = 1, \quad \forall\, r \in \text{irreps}\, G$$

In this case, $\lambda = |G|$ and $\mathcal{T} = \{\phi\} \cong \mathrm{Vec}(\mathbb{C})$

$$\rho = \sum_r p_r \Pi_r \xrightarrow{\mathcal{E}_{\mathbb{A}}} \Pi_{\phi} \xrightarrow{\alpha} \frac{1}{|G|} \sum_r \Pi_r$$

Relative Entropy

$$S_{\mathcal{A}}(\rho |\, \alpha \circ \mathcal{E}_{\mathbb{A}}(\rho)) = \log |G| - H(p)\,.$$

Symmetry Breaking

Symmetric phase

$$\bar{p}_r = \frac{1}{|G|}\,,\forall\, r \implies S_{\mathcal{A}}(\rho|\,\alpha\circ\mathcal{E}_{\mathbb{A}}(\rho)) = 0$$

Perturbations

$\delta\, p_r \ll \bar{p}_r$

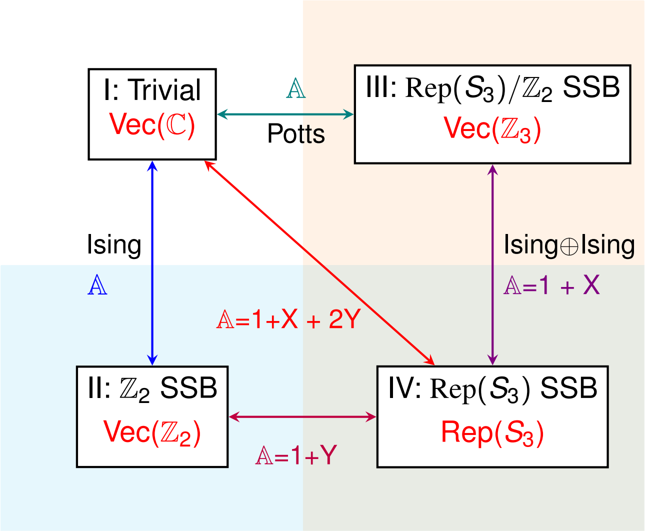
$$p_r = \bar{p}_r + \delta p_r \quad \text{s.t} \quad \sum_r \delta p_r = 0\,, \quad S_{\mathcal{A}}(\rho|\alpha\circ\mathcal{E}_{\mathbb{A}}(\rho)) \sim \sum_r (\delta p_r)^2\,.$$

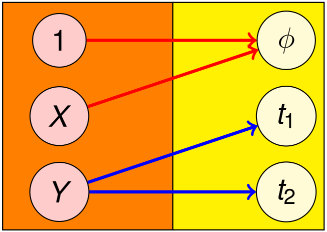
$\delta\, p_r \sim \bar{p}_r$ (Example: $\mathbb{Z}_2$ with $\lambda = |\mathbb{Z}_2| = 2$)

$$\{p\} = \{1/2, 1/2\} \implies S_{\mathcal{A}}(\{p\}|\{\tilde{p}\}) = 0$$
$$\{p\} = \{1, 0\} \implies S_{\mathcal{A}}(\{p\}|\{\tilde{p}\}) = \log 2$$
$$\{p\} = \{0, 1\} \implies S_{\mathcal{A}}(\{p\}|\{\tilde{p}\}) = \log 2$$

Category $\mathrm{Rep}(S_3)$

For this category $\mathcal{A} = \{1, X, Y\}$ with $\{d_1 = 1, d_X = 1, d_Y = 2\}$. The non trivial fusion rules are: $X \otimes X = 1, X \otimes Y = Y, Y \otimes Y = 1 \oplus X \oplus Y$.



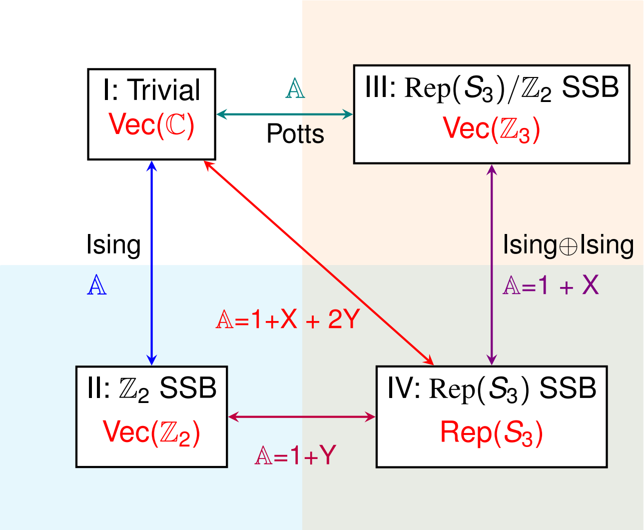


$\mathcal{A} \cong \mathrm{Rep}(S_3) \qquad \mathcal{T} \cong \mathrm{Vec}(\mathbb{Z}_3)$

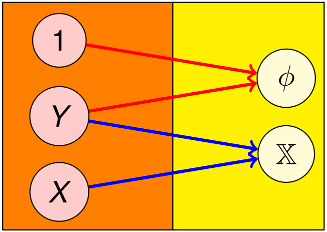
$$\{p\} = \{1, 0, 0\} \implies S_{\mathcal{A}}(\{p\}|\{\tilde{p}\}) = \log 2$$
$$\{p\} = \{1/2, 1/2, 0\} \implies S_{\mathcal{A}}(\{p\}|\{\tilde{p}\}) = 0$$
$$\{p\} = \{1/2, 0, 1/2\} \implies S_{\mathcal{A}}(\{p\}|\{\tilde{p}\}) = \frac{1}{2}\log 2$$
$$\{p\} = \{0, 1/2, 1/2\} \implies S_{\mathcal{A}}(\{p\}|\{\tilde{p}\}) = \frac{1}{2}\log 2$$

Category $\mathrm{Rep}(S_3)$

For this category $\mathcal{A} = \{1, X, Y\}$ with $\{d_1 = 1, d_X = 1, d_Y = 2\}$. The non trivial fusion rules are: $X \otimes X = 1, X \otimes Y = Y, Y \otimes Y = 1 \oplus X \oplus Y$.



$\mathbb{A} = 1 + Y \;/\quad \lambda = n_1^{\phi} d_1 + n_Y^{\phi} d_Y = 3$

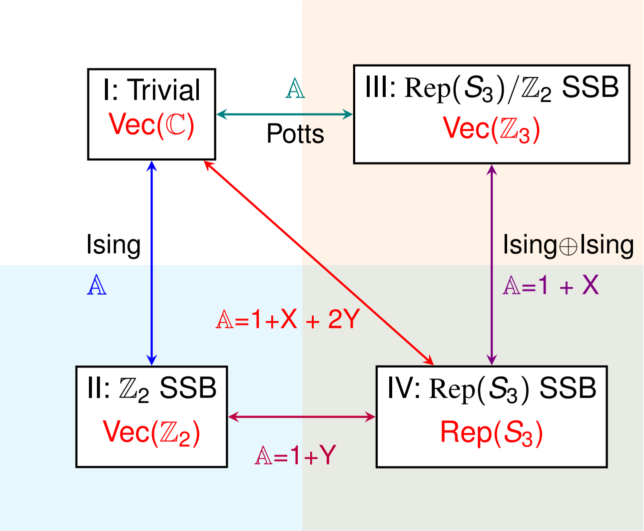


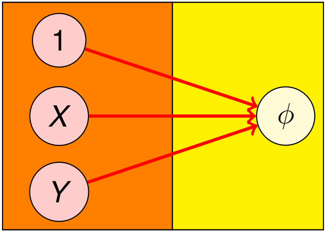
$\mathcal{A} \cong \mathrm{Rep}(S_3) \qquad \mathcal{T} \cong \mathrm{Vec}(\mathbb{Z}_2)$

$$\{p\} = \{1, 0, 0\} \implies S_{\mathcal{A}}(\{p\}|\{\tilde{p}\}) = \log 3$$
$$\{p\} = \{1/2, 1/2, 0\} \implies S_{\mathcal{A}}(\{p\}|\{\tilde{p}\}) = \log 3$$
$$\{p\} = \{1/2, 0, 1/2\} \implies S_{\mathcal{A}}(\{p\}|\{\tilde{p}\}) = \frac{1}{2}\log 3/2$$
$$\{p\} = \{0, 1/2, 1/2\} \implies S_{\mathcal{A}}(\{p\}|\{\tilde{p}\}) = \frac{1}{2}\log 3/2$$

Category $\mathrm{Rep}(S_3)$

For this category $\mathcal{A} = \{1, X, Y\}$ with $\{d_1 = 1, d_X = 1, d_Y = 2\}$. The non trivial fusion rules are: $X \otimes X = 1$, $X \otimes Y = Y$, $Y \otimes Y = 1 \oplus X \oplus Y$.





$\mathcal{A} \cong \mathrm{Rep}(S_3) \qquad \mathcal{T} \cong \mathrm{Vec}(\mathbb{C})$

$$\{p\} = \{1, 0, 0\} \implies S_{\mathcal{A}}(\{p\}|\{\tilde{p}\}) = \log 6$$
$$\{p\} = \{1/2, 1/2, 0\} \implies S_{\mathcal{A}}(\{p\}|\{\tilde{p}\}) = \log 3$$
$$\{p\} = \{1/2, 0, 1/2\} \implies S_{\mathcal{A}}(\{p\}|\{\tilde{p}\}) = \log 3/2$$
$$\{p\} = \{0, 1/2, 1/2\} \implies S_{\mathcal{A}}(\{p\}|\{\tilde{p}\}) = \log 3/2$$

Outline

Summary

- We establish the **SymQuOp** implementing the symmetry breaking patterns of non-invertible symmetries. Basic operations in QFT.
- Entropic order parameters quantify these symmetry-breaking patterns.
- Bounded by the Jones index, the fundamental invariant of the subfactor $\mathcal{T} \subset \mathcal{A}$ that defines the pattern of symmetry breaking.

Outlook

- Extend the present finite–dimensional analysis to type II/III settings, where Tomita-Takesaki modular theory plays a central role.
- How this connects with progress made through the SymTFT?
- The interplay between condensation, anomalies, and entropic quantities deserves further exploration.

Thanks for your attention GRASS-SYMBHOL!

$\mathcal{E}$ as group average

$$\mathcal{E}(\rho) = \sum_{g\in G} \mathbb{K}_g\, \rho\, \mathbb{K}_g^\dagger \quad \text{w/} \quad \mathbb{K}_g = \Lambda_{g,\phi} \equiv \Lambda_g$$
$$= \frac{1}{|G|} \sum_{g\in G} \bar{\Lambda}_g\, \rho\, \bar{\Lambda}_g^\dagger \quad \text{where} \quad \bar{\Lambda}_g = \sqrt{\lambda}\,\Lambda_g\,.$$

Lagrangian algebras and trivial phases

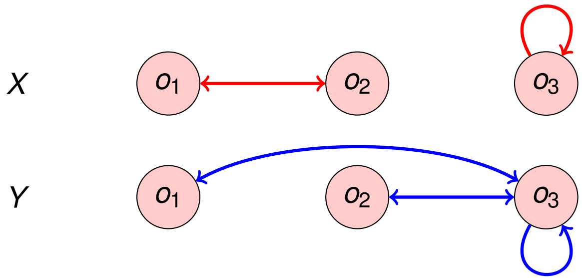
- *i)* $n_a^\phi \neq 0,\ \forall\, a \in \mathcal{A}$,
- *ii)* The quantum dimension $\mathcal{D}_\mathcal{T}^2 = 1 \implies$

$$\lambda = \sum_{a\in\mathcal{A}} n_a^\phi\, d_a = \mathcal{D}_\mathcal{A}^2$$

vacuum ϕ in which all the non-trivial anyonic charges trivialize.

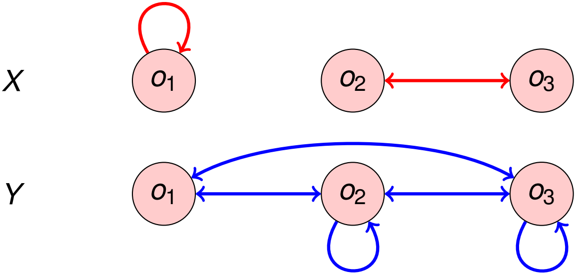
Quiver Diagram Rep(S_3) SSB

$$\phi \equiv o_1 : 1, \quad o_2 : X, \quad o_3 : Y,$$



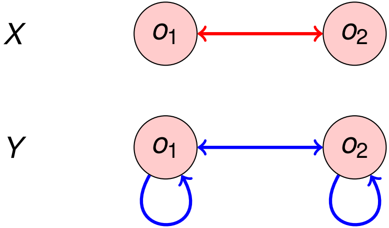
Quiver Diagram Rep(S_3)/$\mathbb{Z}_2$ SSB

$$\phi \equiv o_1 : 1 + X, \quad t_1 \equiv o_2 : Y, \quad t_2 \equiv o_3 : Y,$$



Quiver Diagram $\mathbb{Z}_2$ SSB

$$\phi \equiv o_1 : 1 + Y, \qquad \mathbb{X} \equiv o_2 : X + Y$$



Quiver Diagram: Trivial

$$\phi \equiv o_1 : 1 + X + 2Y,$$

$1, X, Y$