Holographic Rényi $n \to 0$ entropies, and Euclidean Fluids

Based on: arXiv 2503.08773
and arXiv:2309.00044

In collaboration with:

Horacio Casini and Pedro J. Martinez,

Centro Atómico Bariloche and
Universidad de la Plata

Cesar Agon

Universidad
Politécnica
de Cartagena

IV GRASS-SYMBHOL

Toledo, 2025

Organization

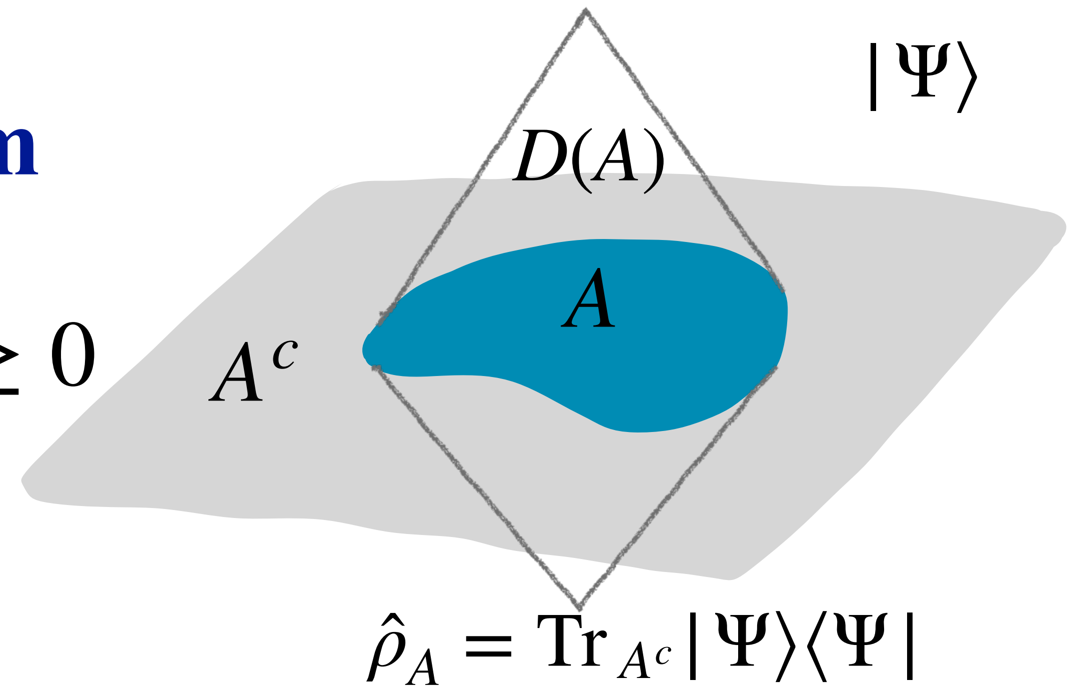
- Motivation

- Refined Rényi Entropy in Holography

- Holographic Derivation: $d > 2$ and $d = 2$

- Summary and Future directions

Motivation



Effective description of subsystem

$$\hat{\rho}_A = \frac{e^{-H_A}}{Z} \quad \text{where} \quad \mathrm{tr}\,\hat{\rho}_A = 1,\ \hat{\rho}_A \geq 0$$

$$H_A = -\log \hat{\rho}_A \qquad U_A \sim e^{i\tau H_A}$$

Thermal system H_A with $\beta = 1$

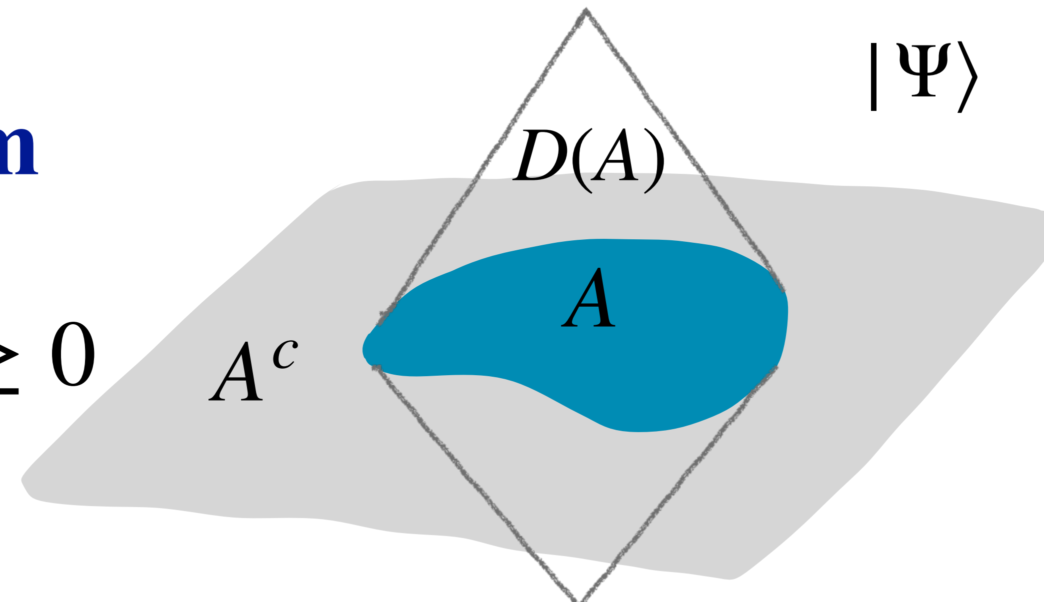
Motivation

Effective description of subsystem

$$\hat{\rho}_A = \frac{e^{-H_A}}{Z} \quad \text{where} \quad \mathrm{tr}\,\hat{\rho}_A = 1, \;\; \hat{\rho}_A \geq 0$$

$$H_A = -\log \hat{\rho}_A \qquad\qquad U_A \sim e^{i\tau H_A}$$

Thermal system H_A with $\beta = 1$



$$\hat{\rho}_A = \mathrm{Tr}_{A^c} |\Psi\rangle\langle\Psi|$$

Moments of $\hat{\rho}_A$ or Rényi entropies

$$S_n(A) = \frac{1}{1-n} \log \left(\mathrm{tr}\, \hat{\rho}_A^n \right)$$

Entanglement entropy

$$S(A) = -\,\mathrm{Tr}_A\, \hat{\rho}_A \log \hat{\rho}_A$$

where $\quad \hat{\rho}_A = \mathrm{Tr}_{A^c} |\Omega\rangle\langle\Omega|$

$$S(A) = \lim_{n\to 1} S_n(A)$$

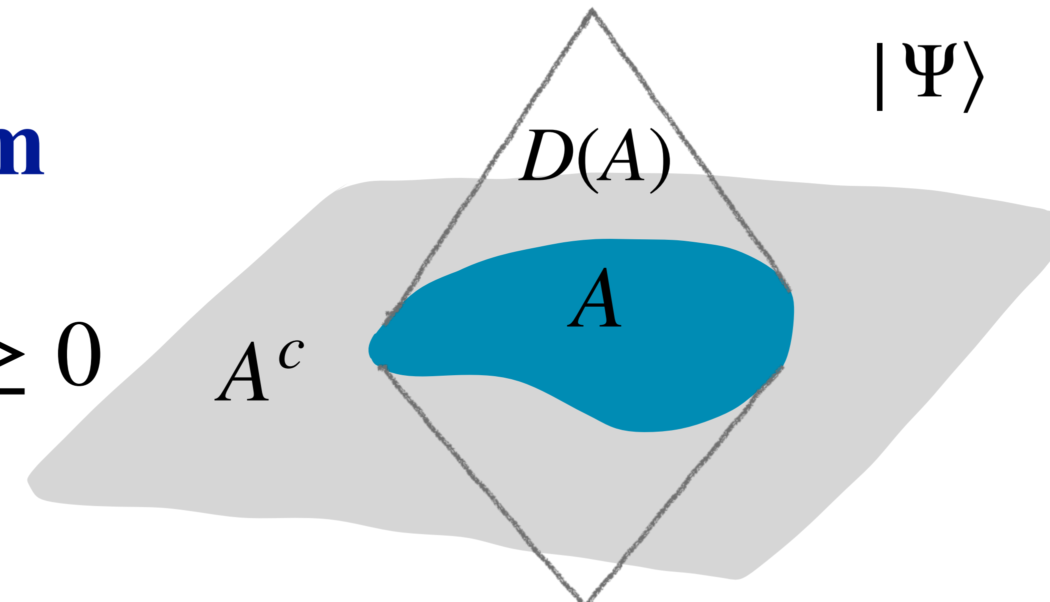
Motivation

Effective description of subsystem

$$\hat{\rho}_A = \frac{e^{-H_A}}{Z} \quad \text{where} \quad \mathrm{tr}\,\hat{\rho}_A = 1, \;\; \hat{\rho}_A \geq 0$$

$$H_A = -\log \hat{\rho}_A \qquad U_A \sim e^{i\tau H_A}$$

Thermal system H_A with $\beta = 1$



$$\hat{\rho}_A = \mathrm{Tr}_{A^c} |\Psi\rangle\langle\Psi|$$

Moments of $\hat{\rho}_A$ or Rényi entropies or Refined Rényi entropies

$$S_n(A) = \frac{1}{1-n} \log \left(\mathrm{tr}\,\hat{\rho}_A^n\right)$$

$$\tilde{S}_n(A) \equiv -n^2 \partial_n \left(\frac{1}{n} \ln \mathrm{tr} \hat{\rho}_A^n\right)$$

$$\tilde{S}_n(A) = -\,\mathrm{tr}\left(\tilde{\rho}_n(A)\, \log \tilde{\rho}_n(A)\right)$$

$$\tilde{\rho}_n(A) := \frac{\hat{\rho}_A^n}{\mathrm{tr}\hat{\rho}_A^n}$$

Entanglement entropy

$$S(A) = -\,\mathrm{Tr}_A\, \hat{\rho}_A \log \hat{\rho}_A$$

where $\quad \hat{\rho}_A = \mathrm{Tr}_{A^c} |\Omega\rangle\langle\Omega|$

$$S(A) = \lim_{n \to 1} S_n(A)$$

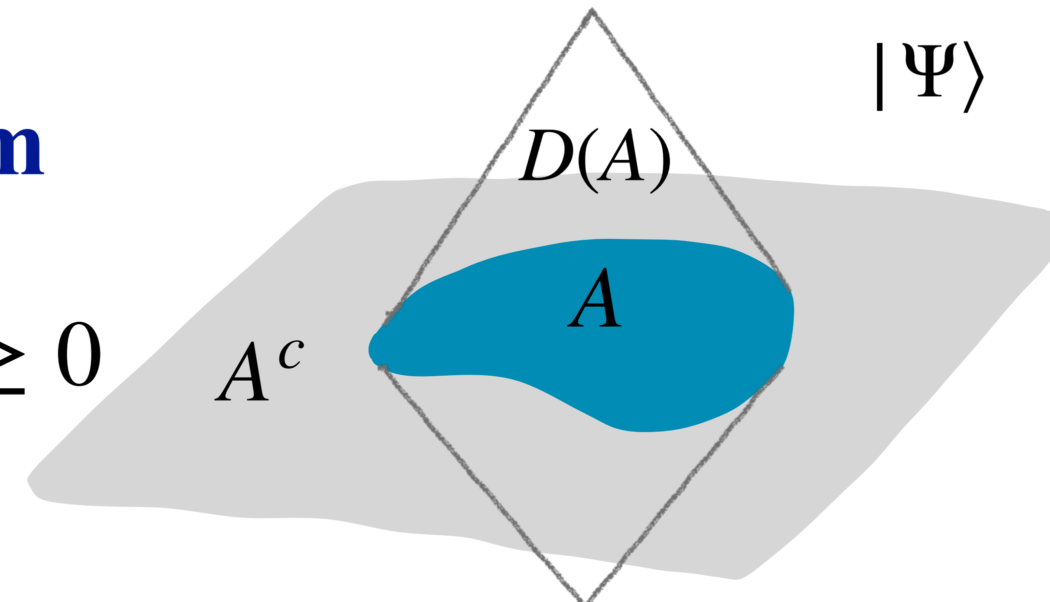
Motivation

Effective description of subsystem

$$\hat{\rho}_A = \frac{e^{-H_A}}{Z} \quad \text{where } \operatorname{tr}\hat{\rho}_A = 1,\ \hat{\rho}_A \geq 0$$

$$H_A = -\log\hat{\rho}_A \qquad U_A \sim e^{i\tau H_A}$$

Thermal system H_A with $\beta = 1$



$$\hat{\rho}_A = \mathrm{Tr}_{A^c}|\Psi\rangle\langle\Psi|$$

Moments of $\hat{\rho}_A$ or Rényi entropies or Refined Rényi entropies

$$S_n(A) = \frac{1}{1-n}\log\left(\operatorname{tr}\hat{\rho}_A^n\right)$$

$$\tilde{S}_n(A) \equiv -n^2\partial_n\left(\frac{1}{n}\ln\operatorname{tr}\hat{\rho}_A^n\right)$$

$$\tilde{S}_n(A) = -\operatorname{tr}\left(\tilde{\rho}_n(A)\,\log\tilde{\rho}_n(A)\right)$$

$$\tilde{\rho}_n(A) := \frac{\hat{\rho}_A^n}{\operatorname{tr}\hat{\rho}_A^n}$$

Density of states ρ_A

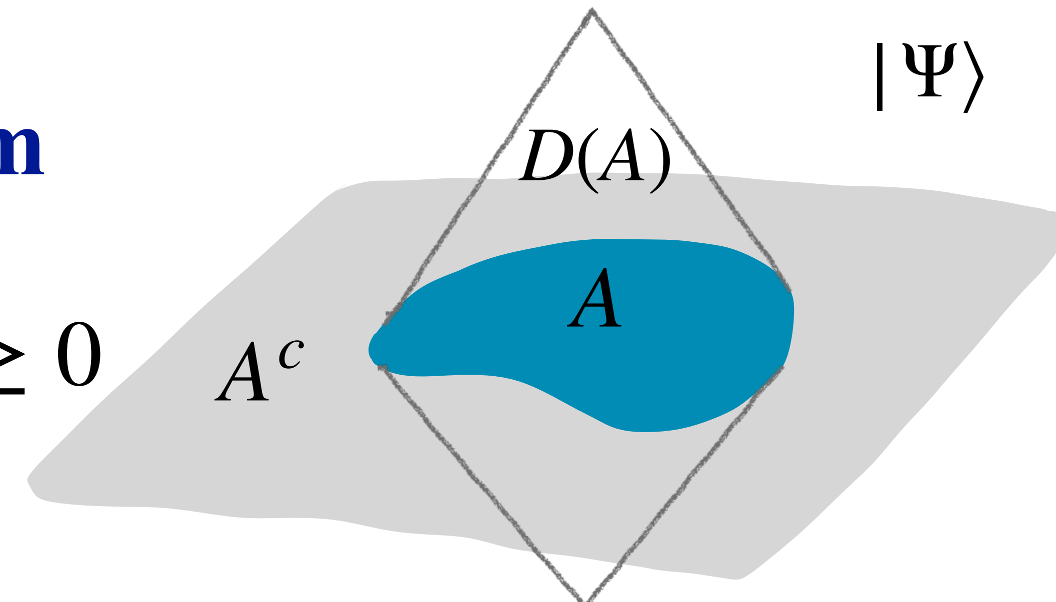
$$e^{(1-n)S_n(A)} = \int_0^\infty dE\, e^{-nE}\rho_A(E)$$

inverse Laplace $\rho_A(E)$

$$\text{“}\beta \text{”} = n$$

Motivation

Effective description of subsystem



$$\hat{\rho}_A = \frac{e^{-H_A}}{Z} \quad \text{where} \quad \mathrm{tr}\,\hat{\rho}_A = 1,\ \hat{\rho}_A \geq 0$$

$$\hat{\rho}_A = \mathrm{Tr}_{A^c}|\Psi\rangle\langle\Psi|$$

$$H_A = -\log\hat{\rho}_A \qquad U_A \sim e^{i\tau H_A}$$

Thermal system H_A with $\beta_{\mathrm{mod}} = 1$

Refined Rényi entropies $n \to 0$
Hartle entropy

$$\tilde{S}_n(A) = -\,\mathrm{tr}\left(\tilde{\rho}_n(A)\,\log\tilde{\rho}_n(A)\right)$$

$$\tilde{S}_{n\to 0}(A) = \log\left(\mathrm{rank}\,\hat{\rho}_A\right)$$

Conjecture [Agon, Casini, Martinez: 2309.00044]

Density of states ρ_A

$$e^{S_{n\to 0}(A)} = \int_0^\infty dE\, e^{-nE}\rho_A(E)$$

$\rho_A(E)$ at high E

$$\tilde{S}_{n\to 0}(A) \sim \frac{\sigma_s\, g(A)}{n^{d-1}}, \qquad s_{\mathrm{th}} \sim \frac{\sigma_s}{\beta^{d-1}} \qquad \tilde{\rho}_n(A) := \frac{\hat{\rho}_A^n}{\mathrm{tr}\hat{\rho}_A^n}$$

$$\beta_{\mathrm{mod}} = n \to 0$$

See [B,C,N,P,S: 2411.00937]

Goal

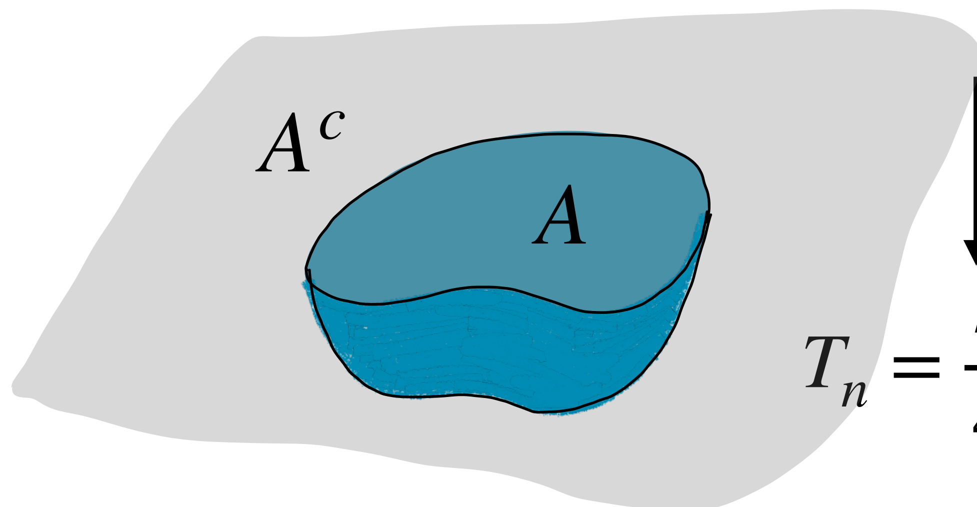
Present a Holographic Proof

$$\tilde{S}_{n\to 0}(A) \sim \frac{\sigma_s\, g(A)}{n^{d-1}},$$

with a prescription of how to compute

$$g(A)$$

Holographic Rényi Entropies



Holographic Rényi entropy

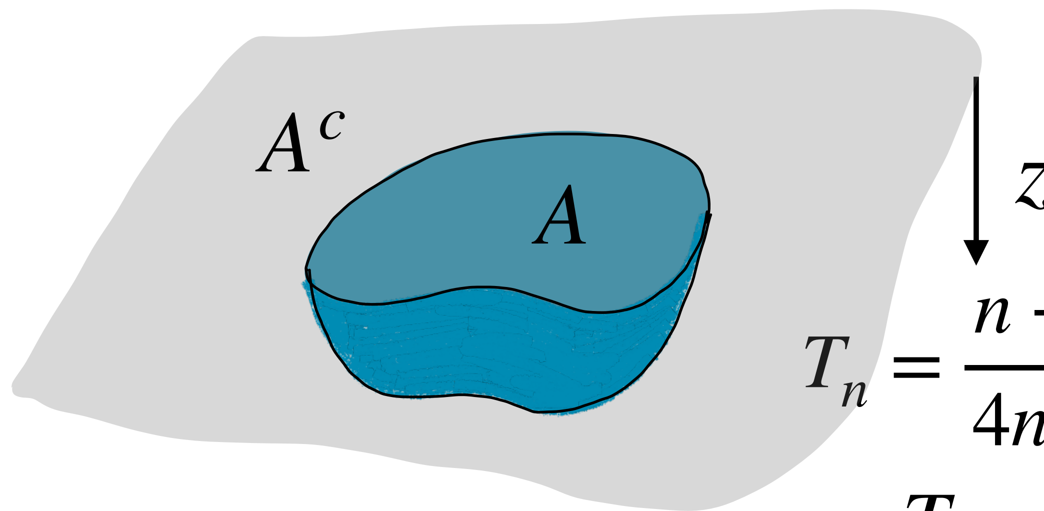
$$I = -\frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{|g|}\left(R + \frac{d(d-1)}{L^2}\right) + T_n\int_{\sim A} d^{d-1}x\sqrt{|h|}$$

Dong's prescription [Dong: 1601.0678]

$$\boxed{\tilde{S}_n(A) = \frac{\text{area(CosmicBrane)}^{\text{A}}{}_n}{4G_N}}$$

Holographic Rényi Entropies

$n \to 0$



$$T_n = \frac{n-1}{4nG_N}$$

$$T_n \to -\infty$$

Holographic Rényi entropy

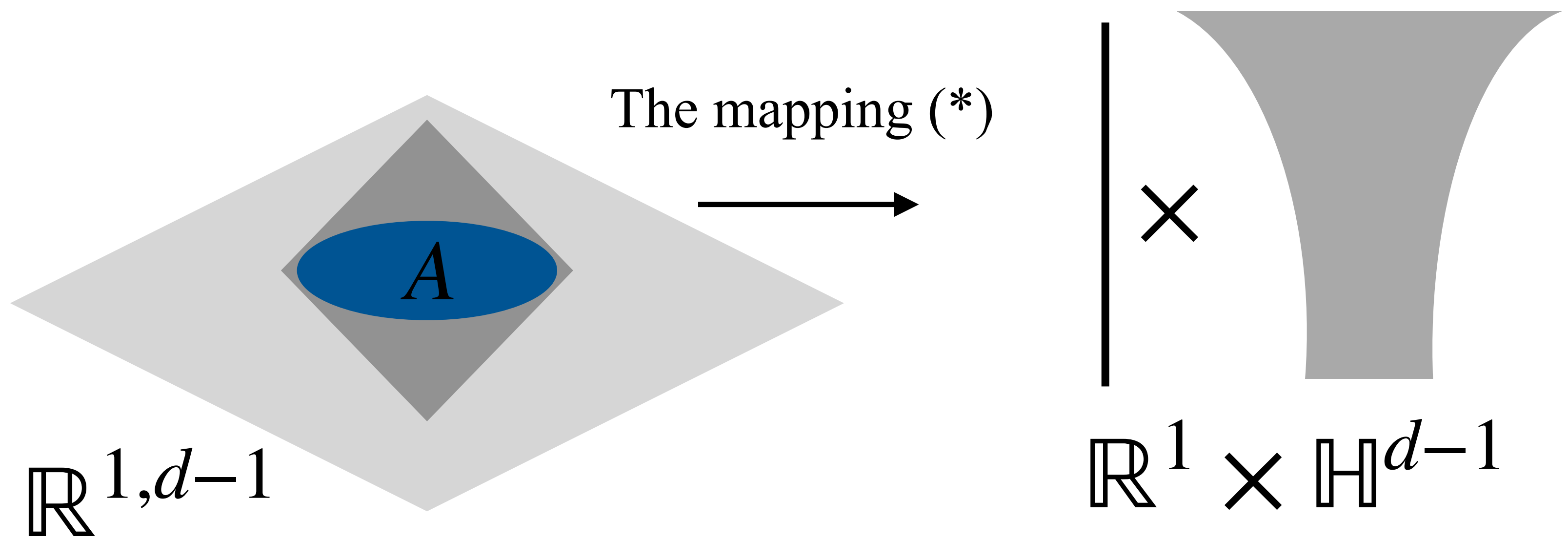
$$I = -\frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{|g|}\left(R + \frac{d(d-1)}{L^2}\right) + T_n\int_{\sim A} d^{d-1}x\sqrt{|h|}$$

Dong's prescription [Dong: 1601.0678]

$$\boxed{\tilde{S}_n(A) = \frac{\mathrm{area(CosmicBrane)}^{\mathrm{A}}{}_{n}}{4G_N}}$$

(*) = [Casini, Huerta, Myers: 1102.0440]

Rindler, the Sphere and the Hyperbolic Black Hole

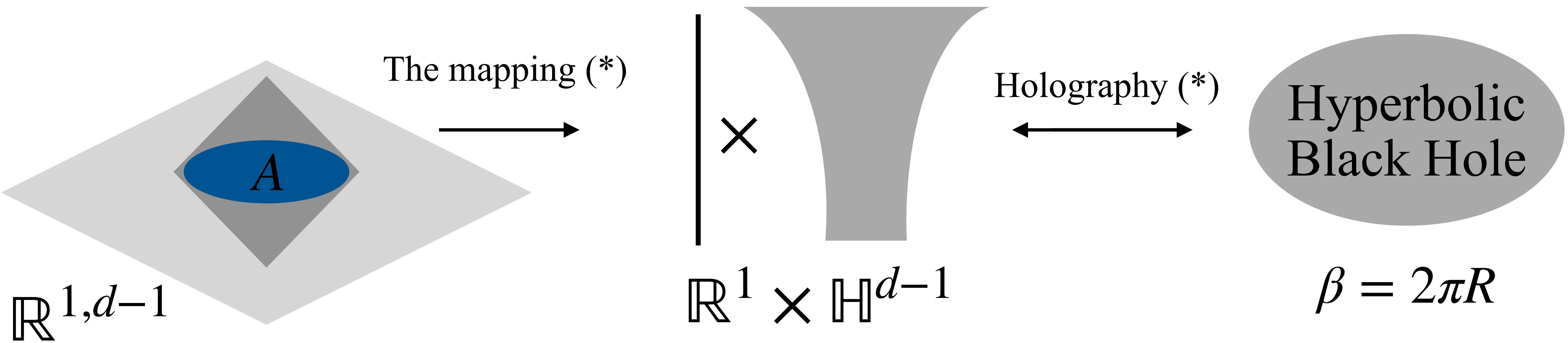


$$\rho_A \longrightarrow \rho_{\text{th}} \quad \text{at} \quad \beta = 2\pi R$$

$$S(\rho_A) \quad = \quad S_{\text{th}}(\rho_{\text{th}})$$

(*) = [Casini, Huerta, Myers: 1102.0440]

Rindler, the Sphere and the Hyperbolic Black Hole

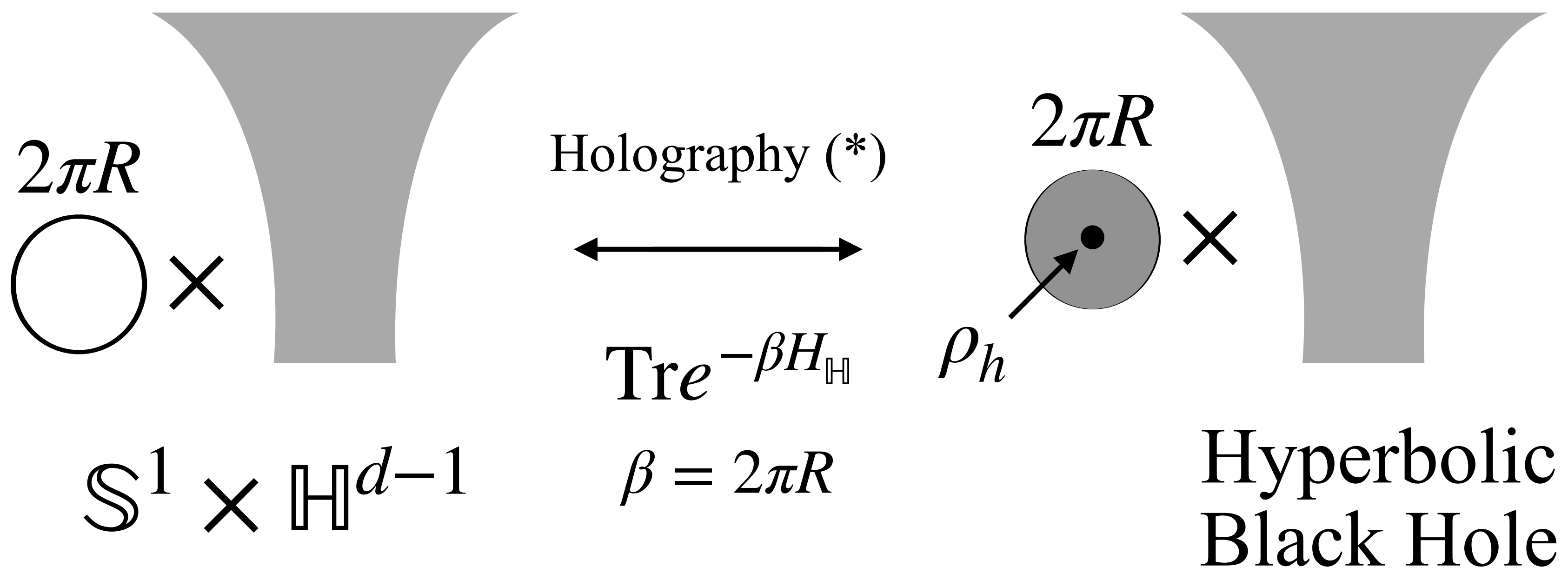


$$\rho_A \longrightarrow \rho_{\text{th}} \quad \text{at} \quad \beta = 2\pi R$$

$$S(\rho_A) \qquad = \qquad S_{\text{th}}(\rho_{\text{th}}) \qquad = \qquad S_{\text{BH}}$$

(*) = [Casini, Huerta, Myers: 1102.0440]

Rindler, the Sphere and the Hyperbolic Black Hole

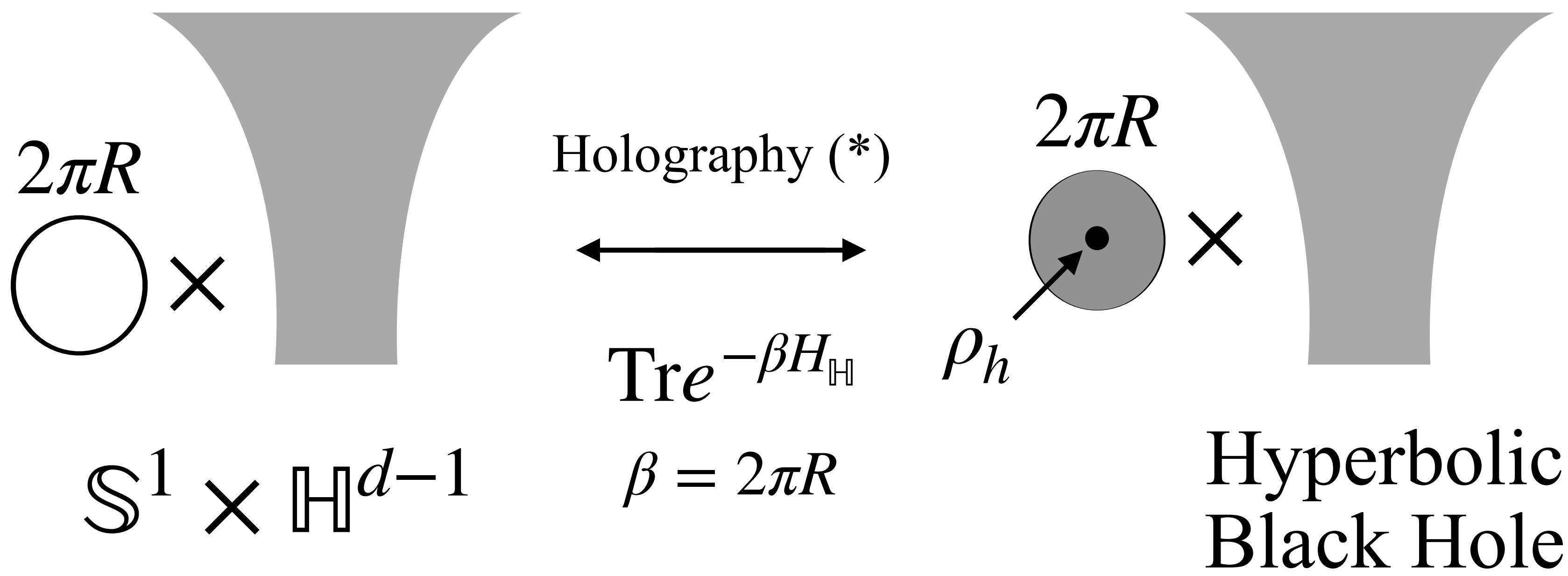


$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H_{\mathbb{H}}}$$

$$S(\rho_A) = S_{\rm th}(\rho_{\rm th}^{\mathbb{H}}(\beta)) = \frac{\mathrm{area}(\mathrm{BH}_{\rm horizon})}{4G_N} = \frac{\rho_h^{d-1}\mathrm{vol}(\mathbb{H}_{d-1})}{4G_N}$$

(*) = [Casini, Huerta, Myers: 1102.0440]

Rindler, the Sphere and the Hyperbolic Black Hole

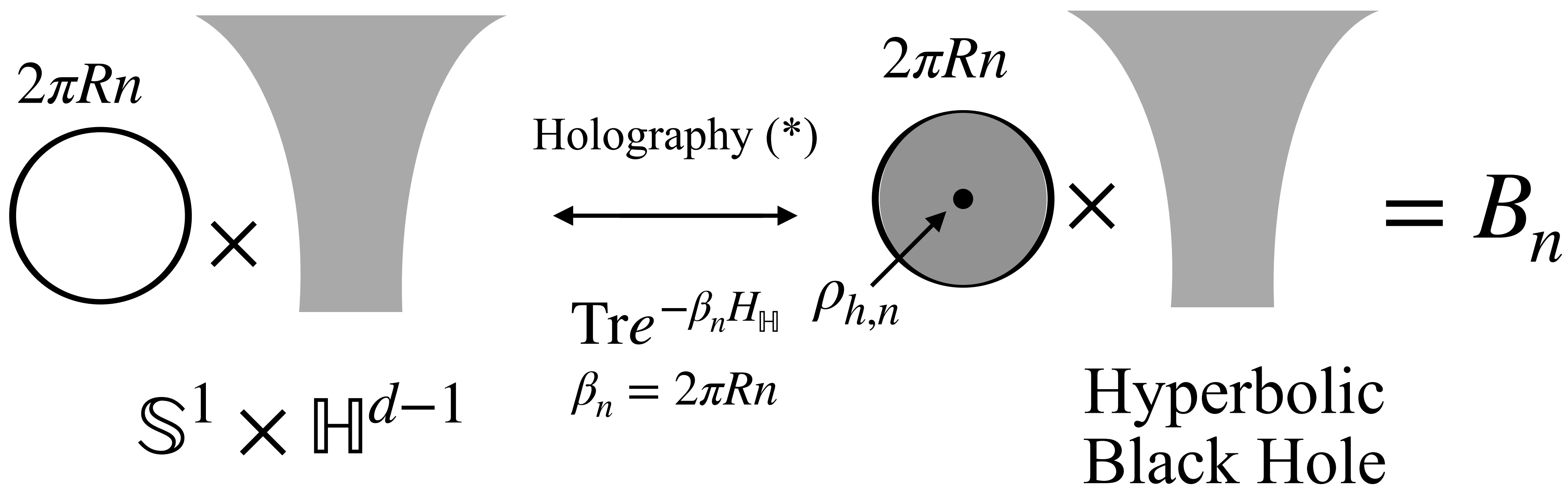


$$Z(\beta) = \mathrm{Tr} e^{-\beta H_{\mathbb{H}}} \xrightarrow[\beta_n \to n\beta]{} Z(n\beta) = \mathrm{Tr} e^{-n\beta H_{\mathbb{H}}}$$

$$S(\rho_A) = S_{\mathrm{th}}(\rho_{\mathrm{th}}^{\mathbb{H}}(\beta)) = \frac{\mathrm{area}(\mathrm{BH}_{\mathrm{horizon}})}{4G_N} = \frac{\rho_h^{d-1}\mathrm{vol}(\mathbb{H}_{d-1})}{4G_N}$$

(*) = [Hung, Myers, Smolkin, Yale: 1102.0440]

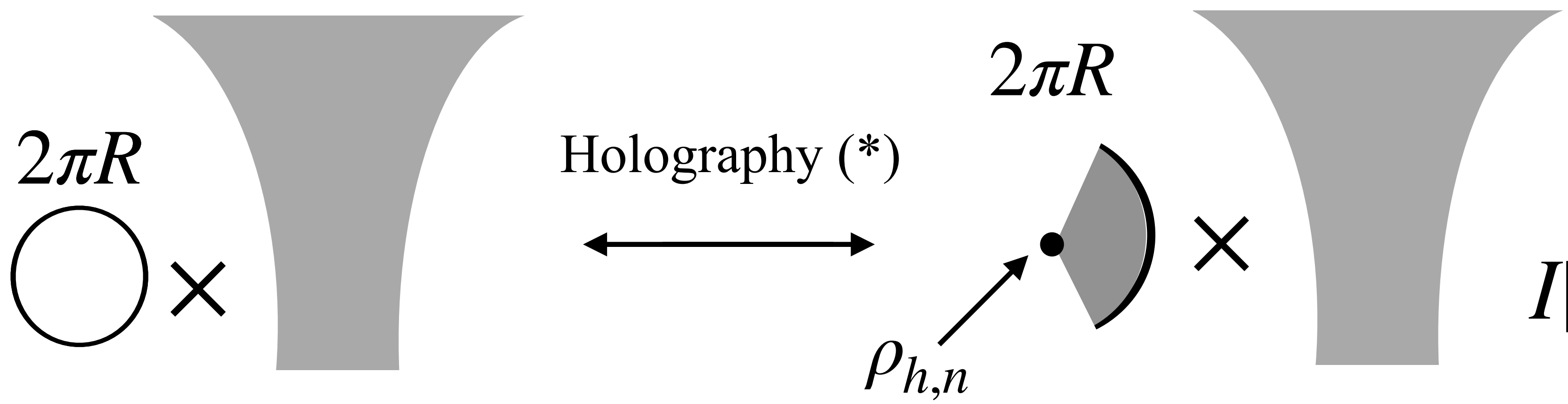
Rindler, the Sphere and the Hyperbolic Black Hole



$$\tilde{S}_n(\rho_A) \quad = \quad S_{\mathrm{th}}(\rho_{\mathrm{th}}(n\beta)) \quad = \quad S_{\mathrm{BH}}(\rho_{h,n})$$

(*)= [Dong: 1601.0678]

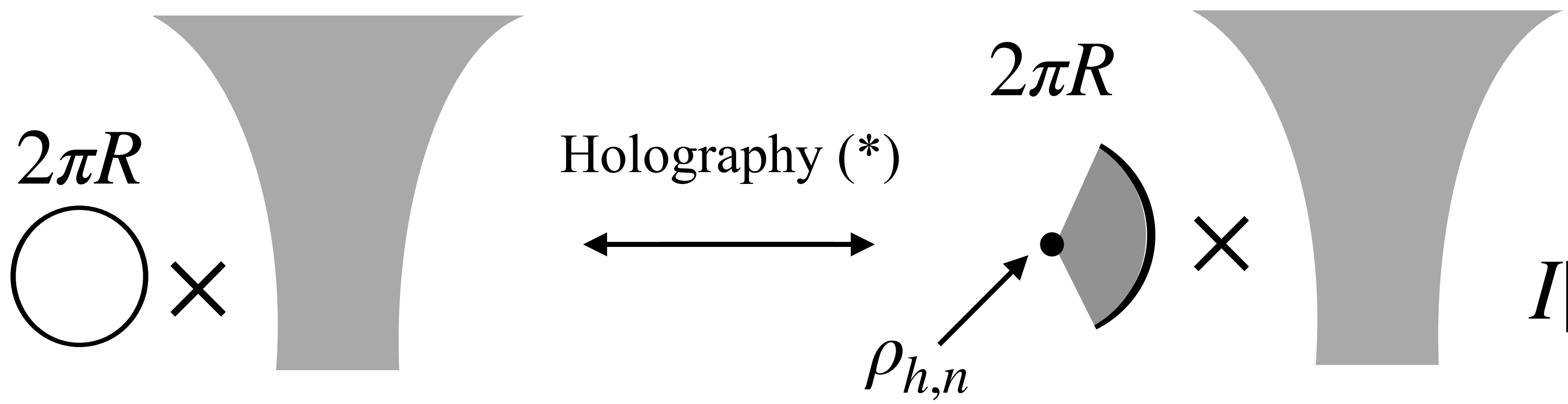
Rindler, the Sphere and the Hyperbolic Black Hole



$$\hat{B}_n = B_n/\mathbb{Z}_n$$
$$I[B_n] = n\, I[\hat{B}_n]$$

(*)= [Dong: 1601.0678]

Rindler, the Sphere and the Hyperbolic Black Hole

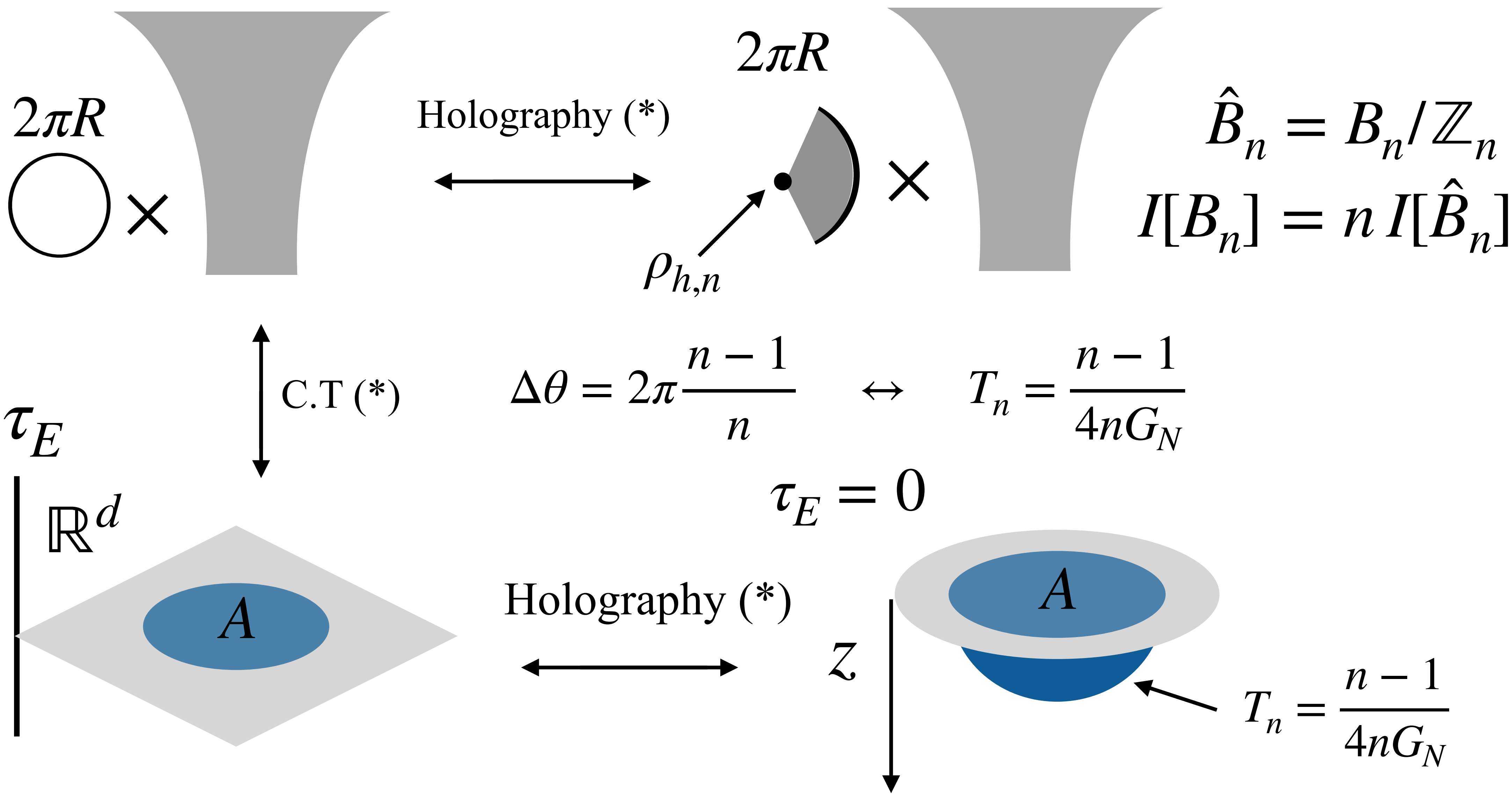


$$\hat{B}_n = B_n/\mathbb{Z}_n$$
$$I[B_n] = n\, I[\hat{B}_n]$$

$$\Delta\theta = 2\pi \frac{n-1}{n} \quad \leftrightarrow \quad T_n = \frac{n-1}{4nG_N}$$

(*)= [Dong: 1601.0678]

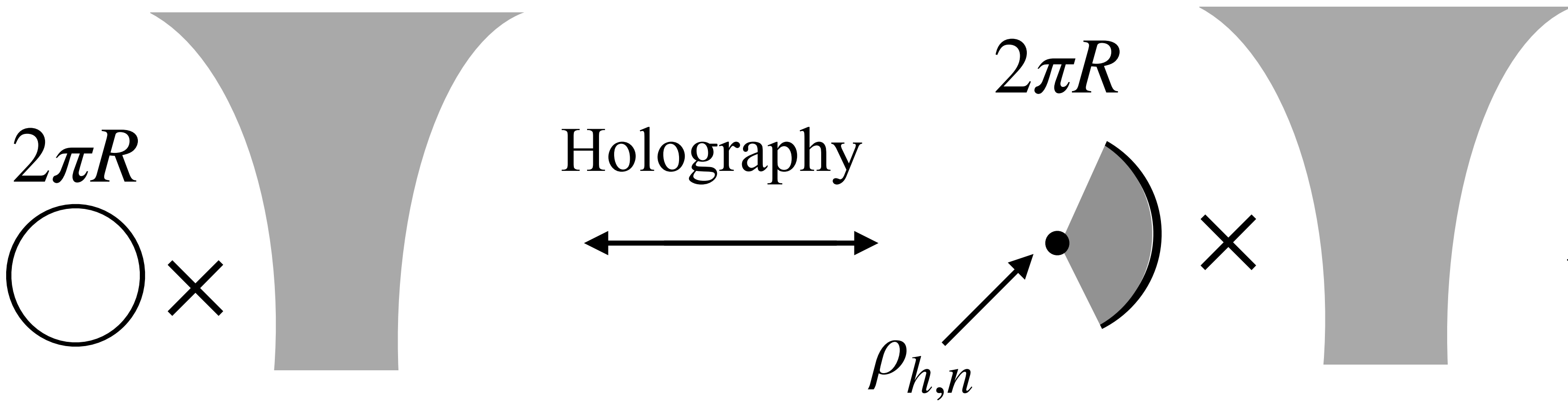
Rindler, the Sphere and the Hyperbolic Black Hole



$$\tilde{\rho}_n(A) = \frac{\rho_A^n}{\mathrm{Tr}\rho_A^n}$$

$$S(\tilde{\rho}_n(A)) = \frac{\mathrm{area}(\mathrm{Brane}_\mathrm{n}^\mathrm{A})}{4G_N}$$

Boundary State



$$\hat{B}_n = B_n/\mathbb{Z}_n$$

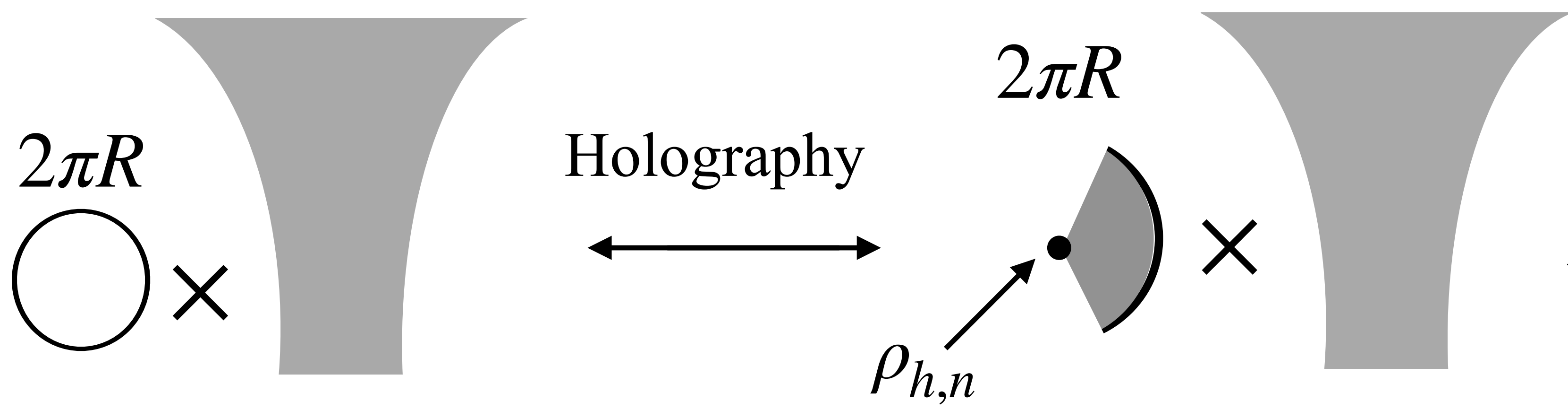
$$\beta_n = 2\pi R n$$

$$ds_B^2 = L^2 g(\rho)\frac{d\tau_H^2}{R^2} + \frac{d\rho^2}{g(\rho)} + \rho^2 d\mathbb{H}_{d-1}^2\,,$$

$$g(\rho) = \frac{\rho^2}{L^2} - \frac{\rho_{h,n}^{d-2}}{\rho^{d-2}}\left(\frac{\rho_{h,n}^2}{L^2} - 1\right) - 1$$

(**) = [Haro, Solodukhin, Skenderis: 0002230], [Balasubramanian, Kraus: 9902121]

Boundary State



$$\hat{B}_n = B_n/\mathbb{Z}_n$$

$$\beta_n = 2\pi R n$$

$$ds_B^2 = L^2 g(\rho)\frac{d\tau_H^2}{R^2} + \frac{d\rho^2}{g(\rho)} + \rho^2 d\mathbb{H}_{d-1}^2\,, \qquad g(\rho) = \frac{\rho^2}{L^2} - \frac{\rho_{h,n}^{d-2}}{\rho^{d-2}}\left(\frac{\rho_{h,n}^2}{L^2} - 1\right) - 1$$

Dual boundary stress tensor is conserved and traceless (**). **Perfect fluid!**

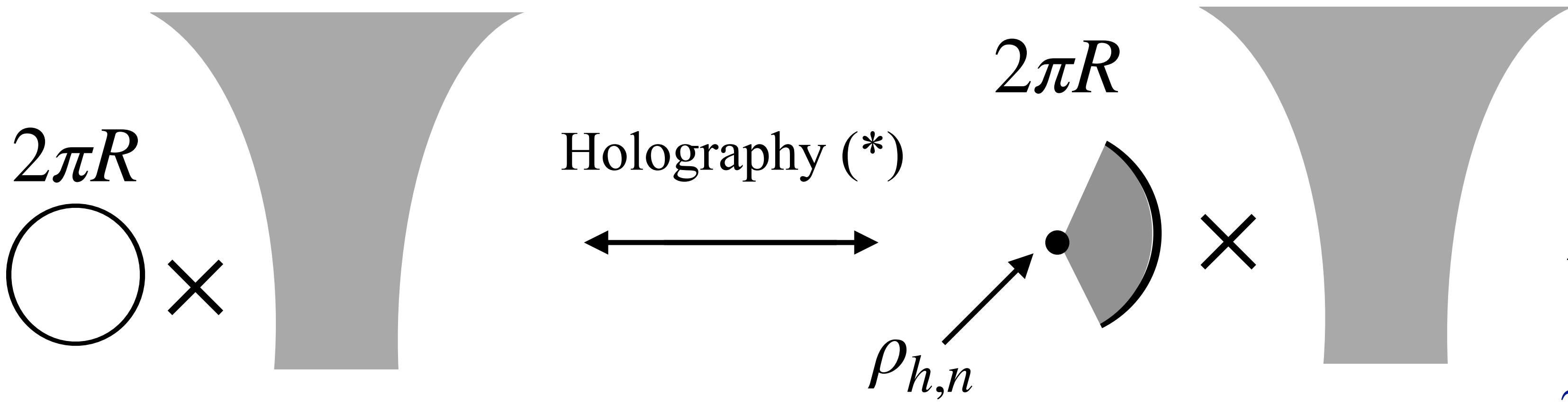
$$\langle T^{\mu\nu}_{\mathbb{H}}\rangle_{\beta_n} = P\left(g^{\mathbb{H}}_{\mu\nu} - d\,u^{\mathbb{H}}_\mu u^{\mathbb{H}}_\nu\right)\,, \qquad \varepsilon = (d-1)P\,, \qquad \varepsilon = \frac{\sigma^{\mathbb{H}}_\varepsilon}{\beta_n^d},$$

$$\langle J^\mu_{\mathbb{H}}\rangle_{\beta_n} = s\,u^{\mathbb{H}}_\mu\,, \qquad s = \frac{\sigma^{\mathbb{H}}_s}{\beta_n^{d-1}}, \qquad \sigma^{\mathbb{H}}_\varepsilon = \sigma_\varepsilon f\left(\frac{\beta_n}{R}, d\right) \qquad \sigma^{\mathbb{H}}_s = \sigma_s g\left(\frac{\beta_n}{R}, d\right)$$

$$\oint \frac{u^{\mathbb{H}}_\mu}{\beta_n} dx^\mu = \frac{1}{n} \qquad S_{\rm th} = \int_\Sigma \langle J^\mu_{\mathbb{H}}\rangle_{\beta_n} n_\mu \qquad \beta_n = 2\pi R n$$

(**) = [Haro, Solodukhin, Skenderis: 0002230], [Balasubramanian, Kraus: 9902121]

Boundary State $n \to 0$



$$\hat{B}_n = B_n / \mathbb{Z}_n$$

$$\beta_n = 2\pi R n$$

$$g(\rho) \approx \frac{\rho^2}{L^2}\left(1 - \frac{\rho_{h,n}^{d-2}}{\rho^{d-2}}\right)$$

$$ds_B^2 = L^2 g(\rho)\frac{d\tau_H^2}{R^2} + \frac{d\rho^2}{g(\rho)} + \rho^2 d\mathbb{H}_{d-1}^2\,, \qquad \rho_{h,n} \approx \frac{2L}{n\,d}$$

Dual boundary stress tensor is conserved and traceless (**). Perfect fluid!

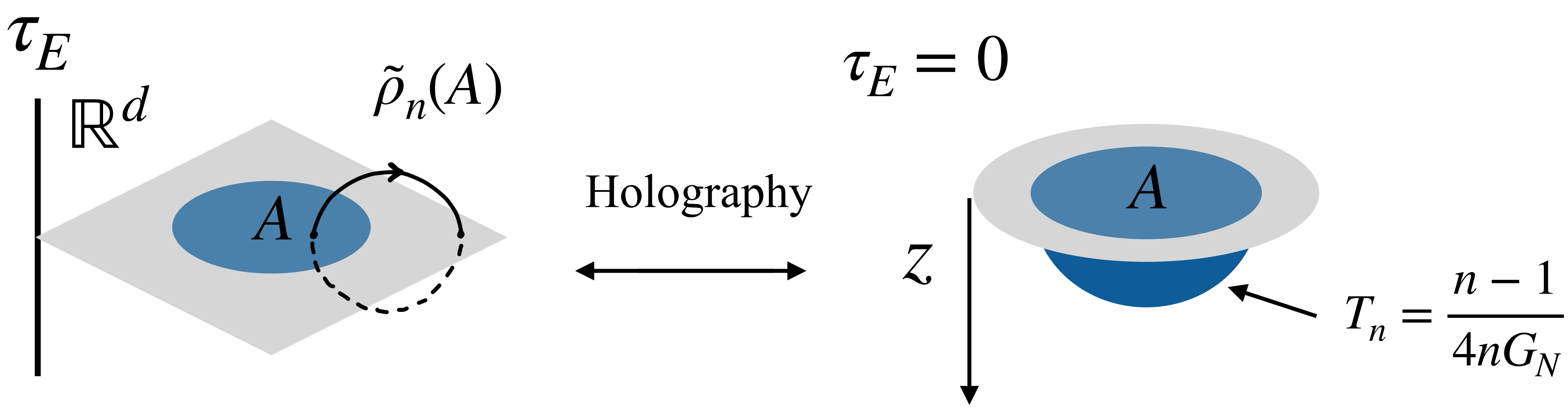
$$\langle T^{\mu\nu}_{\mathbb{H}}\rangle_{\beta_n} = P\left(g^{\mathbb{H}}_{\mu\nu} - d\,u^{\mathbb{H}}_\mu u^{\mathbb{H}}_\nu\right)\,, \qquad \varepsilon = (d-1)P\,, \qquad \varepsilon \approx \frac{\sigma_\varepsilon}{\beta_n^d},$$

$$\langle J^\mu_{\mathbb{H}}\rangle_{\beta_n} = s\,u^{\mathbb{H}}_\mu\,, \quad s \approx \frac{\sigma_s}{\beta_n^{d-1}},$$

$$\sigma^{\mathbb{H}}_\varepsilon \to \sigma_\varepsilon \qquad \sigma^{\mathbb{H}}_s = \sigma_s$$

$$\oint \frac{u^{\mathbb{H}}_\mu}{\beta_n} dx^\mu = \frac{1}{n} \qquad S_{\text{th}} = \int_\Sigma \langle J^\mu_{\mathbb{H}}\rangle_{\beta_n} n_\mu \qquad \beta_n = 2\pi R n$$

Boundary Euclidean Fluid at $n \to 0$



$$\delta_{\mu\nu} dx^\mu dx^\nu = \Omega^2(x)\, g^{\mathbb{H}}_{\mu\nu} dx^\mu_{\mathbb{H}} dx^\nu_{\mathbb{H}}$$

$$\beta_n(x) = \Omega(x)\, \beta^{\mathbb{H}}_n$$

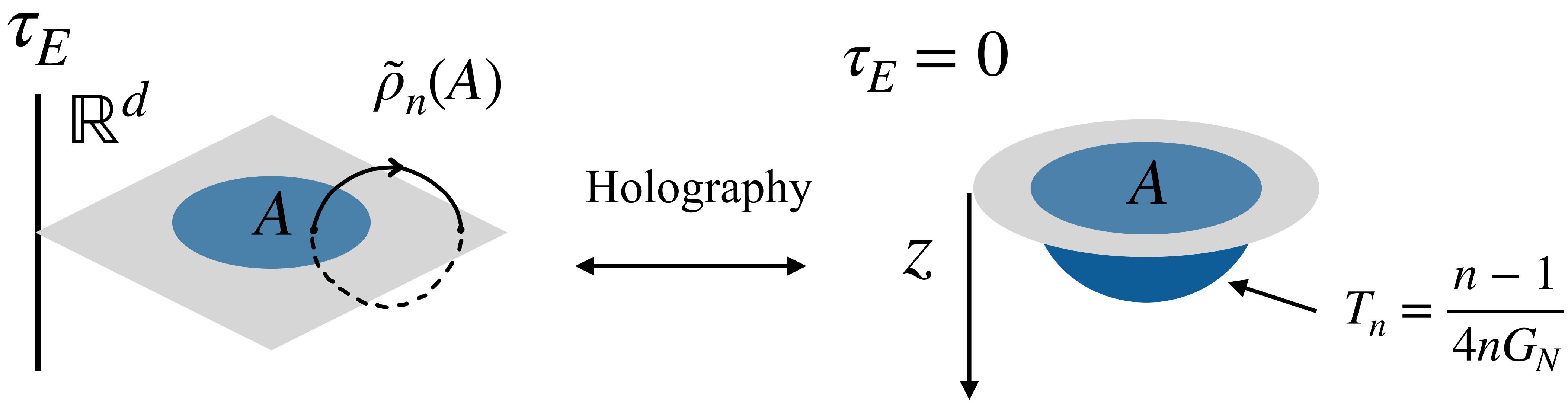
$$u_\mu(x) = \Omega(x) \frac{\partial x^\nu_{\mathbb{H}}}{\partial x^\mu} u^H_\nu$$

$$\langle T^{\mu\nu}_{\mathbb{R}} \rangle_n \approx \frac{(d-1)\,\sigma_\epsilon}{\beta_n(x)^d} \left(\delta_{\mu\nu} - d\, u_\mu(x) u_\nu(x) \right) ,$$

$$\langle J^\mu_{\mathbb{R}} \rangle_n = \frac{\sigma_s}{\beta_n(x)^{d-1}} u_\mu(x) , \qquad \oint_\Gamma \frac{u_\mu}{\beta_n(x)} dx^\mu = \frac{1}{n}$$

$$\langle \cdot \rangle_n = \mathrm{Tr}\left(\rho_n(A) \cdot \right)$$

Boundary Euclidean Fluid at $n \to 0$



$$\delta_{\mu\nu} dx^\mu dx^\nu = \Omega^2(x)\, g^{\mathbb{H}}_{\mu\nu} dx^\mu_{\mathbb{H}} dx^\nu_{\mathbb{H}}$$

$$\beta_n(x) = \Omega(x)\, \beta^{\mathbb{H}}_n$$

$$\langle T^{\mu\nu}_{\mathbb{R}} \rangle_n \approx \frac{(d-1)\,\sigma_\epsilon}{\beta_n(x)^d} \left(\delta_{\mu\nu} - d\, u_\mu(x) u_\nu(x) \right) ,$$

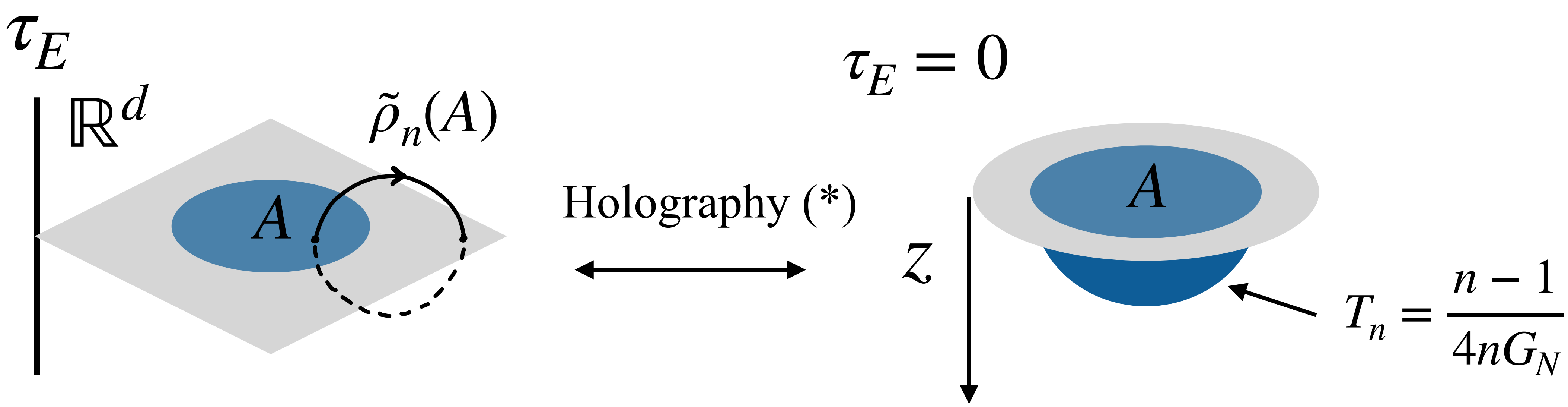
$$u_\mu(x) = \Omega(x) \frac{\partial x^\nu_{\mathbb{H}}}{\partial x^\mu} u^H_\nu$$

$$\langle J^\mu_{\mathbb{R}} \rangle_n = \frac{\sigma_s}{\beta_n(x)^{d-1}} u_\mu(x) , \qquad \oint_\Gamma \frac{u_\mu}{\beta_n(x)} dx^\mu = \frac{1}{n}$$

$$\boxed{\beta_n(r, \tau_E = 0) = \frac{n\pi \sqrt{R^2 - r^2}}{R}}$$

$$\langle \,\cdot\, \rangle_n = \mathrm{Tr} \left(\tilde{\rho}_n(A) \,\cdot\, \right)$$

Dual metric at $n \to 0$



Dual metric is (*)

$$ds^2 = \frac{L^2}{z^2}\Big(f(z,x)\,(dx\cdot u)^2 + (dx^2 - (dx\cdot u)^2)\Big) + \frac{L^2}{f(z,x)}\left(\frac{dz}{z} - \frac{dz_h}{z_h}\right)^2 ,$$

$$f(z,x) = 1 - \left(\frac{z}{z_h(x)}\right)^d , \qquad z_h(x) = \frac{d\beta_n(x)}{4\pi}, \qquad z \in [\epsilon, z_h(x)] \,.$$

Near boundary

$$ds^2_{bdy} = \lim_{z\to 0} \frac{z^2}{L^2} ds^2 = dx^2$$

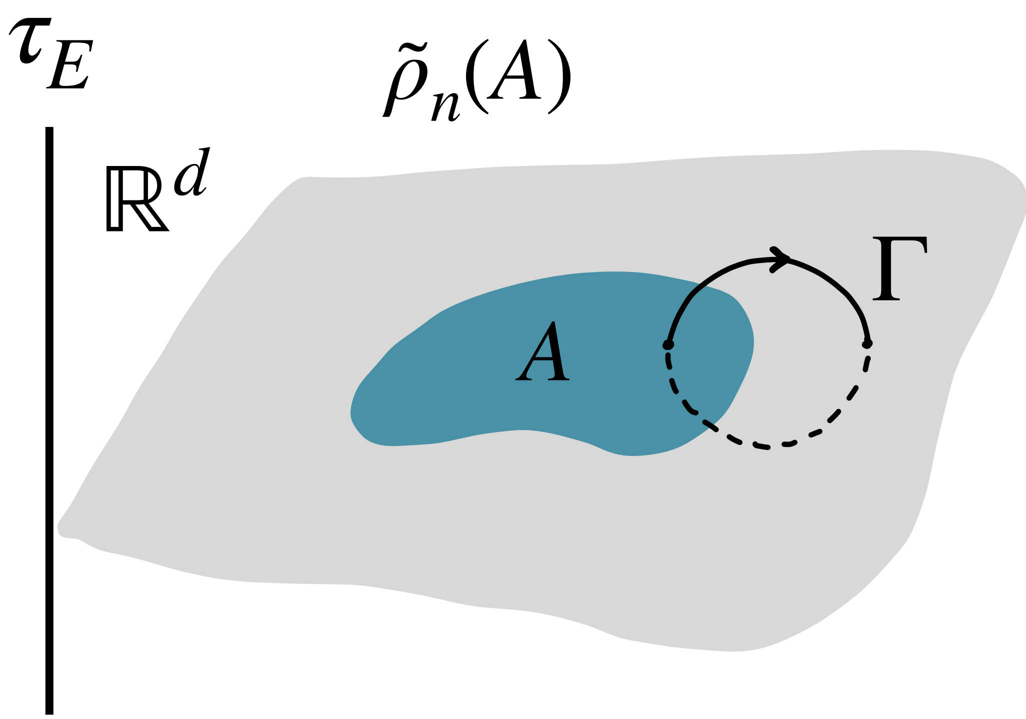
Near brane $\zeta = z/z_h(x)$ conical singularity imposes

$$\frac{d}{4\pi}\oint_\Gamma \frac{dx\cdot u}{z_h(x)} = \frac{1}{n}$$

Conjecture $n \to 0$

$$\tilde{S}_n(\rho_A) = S(\tilde{\rho}_n(A)) = \frac{\sigma_s}{n^{d-1}} \int_{\sim A} \frac{u^\mu}{\beta^{d-1}(x)} n_\mu$$

$$\tilde{\rho}_n(A) = \frac{\rho_A^n}{\mathrm{Tr}\rho_A^n}$$



$$\nabla_\mu \langle T^{\mu\nu} \rangle_n = 0, \qquad \nabla_\mu \langle J^\mu \rangle_n = 0\,,$$

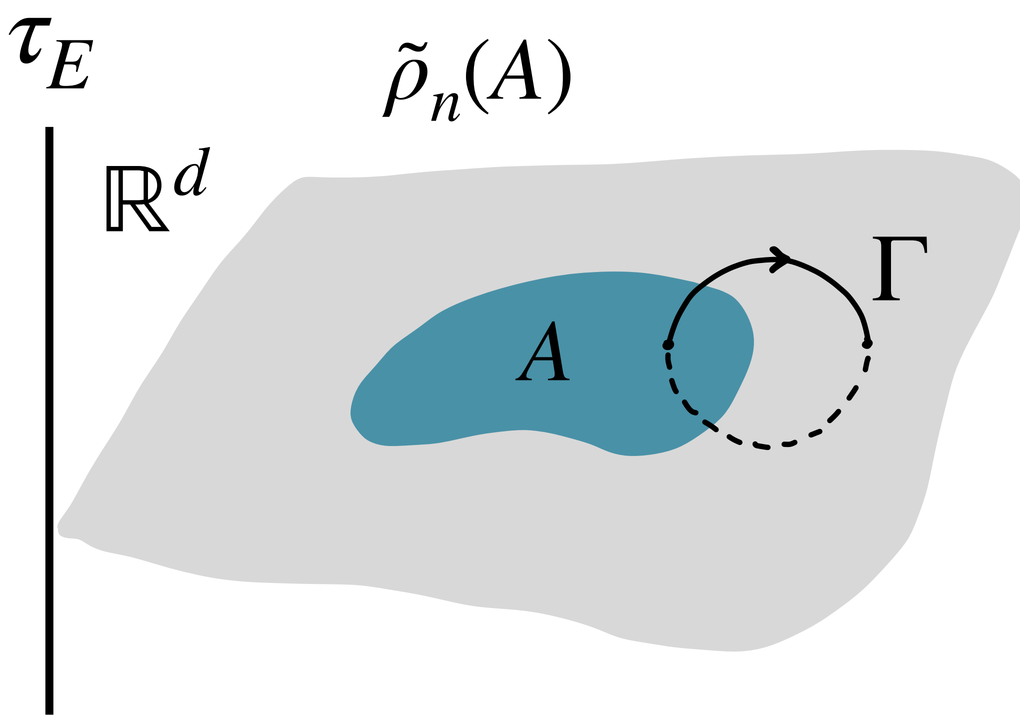
$$\partial_\mu w_\nu - \partial_\nu w_\mu = (\hat{n}^1_\mu \hat{n}^2_\nu - \hat{n}^1_\nu \hat{n}^2_\mu)\delta^2\left(x - x_{\partial A}\right),$$

$$w_\mu = \frac{u^\mu}{\beta}\,, \qquad \oint_\Gamma w_\mu dx^\mu = 1$$

Conjecture $n \to 0$

$$\tilde{S}_n(\rho_A) = S(\tilde{\rho}_n(A)) = \frac{\sigma_s}{n^{d-1}} \int_{\sim A} \frac{u^\mu}{\beta^{d-1}(x)} n_\mu$$

$$\tilde{\rho}_n(A) = \frac{\rho_A^n}{\mathrm{Tr}\rho_A^n}$$



$$\nabla_\mu \langle T^{\mu\nu} \rangle_n = 0, \qquad \nabla_\mu \langle J^\mu \rangle_n = 0\,,$$

$$\partial_\mu w_\nu - \partial_\nu w_\mu = (\hat{n}^1_\mu \hat{n}^2_\nu - \hat{n}^1_\nu \hat{n}^2_\mu)\delta^2\left(x - x_{\partial A}\right),$$

$$w_\mu = \frac{u^\mu}{\beta}, \qquad \oint_\Gamma w_\mu dx^\mu = 1$$

Fluid equations of a **irrotational perfect fluid** with vortices sources at the **entangling surface**

$$\partial^2 \alpha + (d-2)\, \partial_\mu \alpha \partial^\mu \log\left(\,|\,\partial\alpha\,|\,\right) = 0\,,$$

$$u_\mu = \frac{\partial_\mu \alpha}{|\,\partial\alpha\,|}, \qquad \beta = \frac{1}{|\,\partial\alpha\,|}.$$

Holographic Proof of the Conjecture

$$ds^2 = \frac{L^2}{z^2}\left(f(z,x)\,(dx\cdot u)^2 + (dx^2 - (dx\cdot u)^2)\right) + \frac{L^2}{f(z,x)}\left(\frac{dz}{z} - \frac{dz_h}{z_h}\right)^2 ,$$

$$f(z,x) = 1 - \left(\frac{z}{z_h(x)}\right)^d , \qquad z_h(x) = \frac{d\beta_n(x)}{4\pi} , \qquad z \in [\epsilon, z_h(x)] .$$

Holographic Proof of the Conjecture

$$ds^2 = \frac{L^2}{z^2}\left(f(z,x)\,(dx\cdot u)^2 + (dx^2 - (dx\cdot u)^2)\right) + \frac{L^2}{f(z,x)}\left(\frac{dz}{z} - \frac{dz_h}{z_h}\right)^2 ,$$

$$f(z,x) = 1 - \left(\frac{z}{z_h(x)}\right)^d , \qquad z_h(x) = \frac{d\beta_n(x)}{4\pi} , \qquad z \in [\epsilon, z_h(x)] .$$

Vacuum Einstein's equations with negative cosmological constant

$$R_{ab} - \frac{d}{L^2} g_{ab} = 0$$

Put metric in Fefferman Graham form

$$ds^2 = L^2 \frac{d\rho^2}{4\rho^2} + \frac{L^2}{n^2\rho} h_{\mu\nu} dx^\mu dx^\nu \qquad \rho \in \left[\frac{\#\epsilon^2}{n^2\beta^2(x)}, 1\right]$$

$$h_{\mu\nu} = \frac{16\pi^2}{d^2\beta^2(x)}\left(\frac{1+\rho^{\frac{d}{2}}}{2}\right)^{\frac{4}{d}}\left[\delta_{\mu\nu} - \frac{4\rho^{\frac{d}{2}}}{\left(1+\rho^{\frac{d}{2}}\right)^2} u_\mu u_\nu\right] , \qquad \rho = \left(\frac{1-\sqrt{1-(z/z_h(x))^d}}{1+\sqrt{1-(z/z_h(x))^d}}\right)^{\frac{2}{d}}$$

Holographic Proof of the Conjecture

Vacuum Einstein's equations with negative cosmological constant

$$R_{ab} - \frac{d}{L^2} g_{ab} = 0$$

$$ds^2 = L^2 \frac{d\rho^2}{4\rho^2} + \frac{L^2}{n^2 \rho} h_{\mu\nu} dx^\mu dx^\nu$$

Vacuum Einstein's equations for small n

$$g_{\rho\rho} \sim \mathcal{O}(1) \longrightarrow R_{\rho\rho} \sim \mathcal{O}(1)$$

$$g_{\mu\rho} \sim \mathcal{O}(1) \longrightarrow R_{\mu\rho} \sim \mathcal{O}(1)$$

$$g_{\mu\nu} \sim \mathcal{O}\left(\frac{1}{n^2}\right) \longrightarrow R_{\mu\nu} \sim \mathcal{O}\left(\frac{1}{n^2}\right)$$

Holographic Proof of the Conjecture

Vacuum Einstein's equations with negative cosmological constant

$$R_{ab} - \frac{d}{L^2} g_{ab} = 0$$

$$\boxed{ds^2 = L^2 \frac{d\rho^2}{4\rho^2} + \frac{L^2}{n^2 \rho} h_{\mu\nu} dx^\mu dx^\nu}$$

Vacuum Einstein's equations for small n

$$g_{\rho\rho} \sim \mathcal{O}(1) \longrightarrow R_{\rho\rho} \sim \mathcal{O}(1)$$

$$g_{\mu\rho} \sim \mathcal{O}(1) \longrightarrow R_{\mu\rho} \sim \mathcal{O}(1)$$

$$g_{\mu\nu} \sim \mathcal{O}\left(\frac{1}{n^2}\right) \longrightarrow R_{\mu\nu} \sim \mathcal{O}\left(\frac{1}{n^2}\right)$$

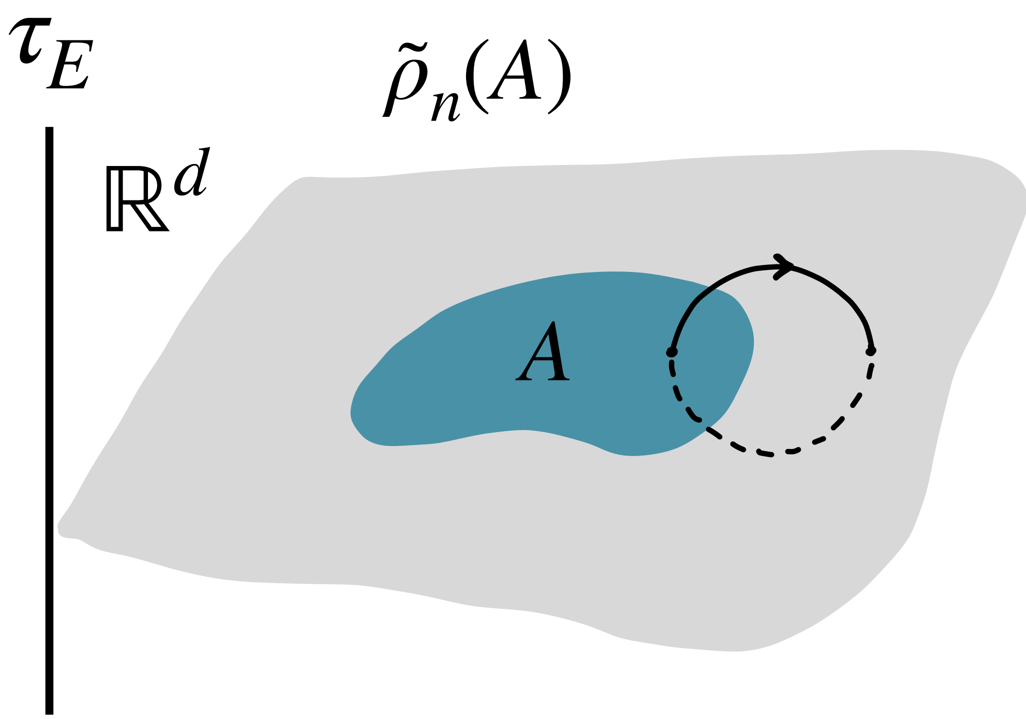
We find:

- $(\rho\rho)$ identity
- $(\mu\rho)$ equivalent to $\longrightarrow$ Fluid equations $\nabla_\mu T^{\mu\nu} = 0$
- $(\mu\nu)$ equivalent to Ricci$(h_{\mu\nu}) = 0$

Holographic $\tilde{S}_{n\to 0}$

$$\tilde{S}_n(\rho_A) = S(\tilde{\rho}_n(A)) = \frac{\sigma_s}{n^{d-1}} \int_{\sim A} \frac{u^\mu}{\beta^{d-1}(x)} n_\mu$$

$$\sigma_s = \frac{L^{d-1}}{4G_N} \left(\frac{4\pi}{d}\right)^{d-1}$$



$$\nabla_\mu \langle T^{\mu\nu} \rangle_n = 0, \qquad \nabla_\mu \langle J^\mu \rangle_n = 0\,,$$

$$\partial_\mu w_\nu - \partial_\nu w_\mu = (\hat{n}^1_\mu \hat{n}^2_\nu - \hat{n}^1_\nu \hat{n}^2_\mu)\delta^2\left(x - x_{\partial A}\right),$$

$$w_\mu = \frac{u^\mu}{\beta}\,, \qquad \oint_\Gamma w_\mu dx^\mu = 1$$

Fluid equations of a **irrotational perfect fluid** with vortices sources at the **entangling surface**

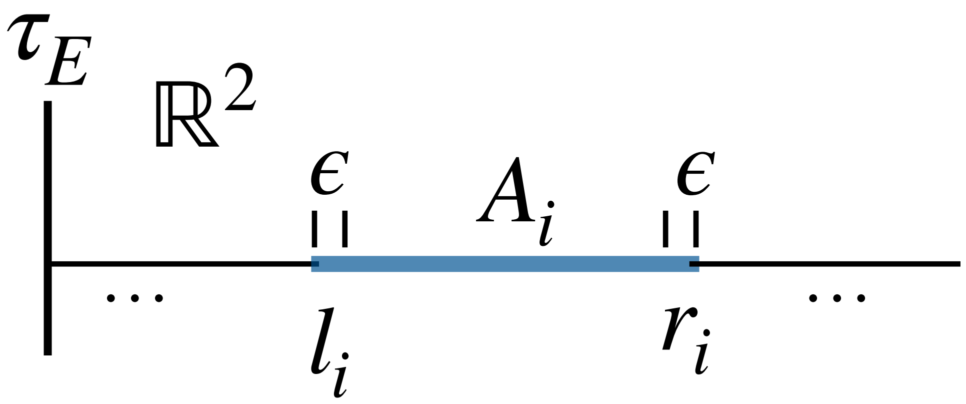
$$\partial^2\alpha + (d-2)\,\partial_\mu\alpha\partial^\mu \log\left(|\partial\alpha|\right) = 0\,,$$

$$u_\mu = \frac{\partial_\mu \alpha}{|\partial\alpha|}\,, \qquad \beta = \frac{1}{|\partial\alpha|}\,.$$

Solutions of $\tilde{S}_{n\to 0}$ in $d = 2$

$$\partial^2 a = 0 \quad \& \quad \oint_\Gamma da = 1$$

$$u_\mu = \frac{\partial_\mu a}{|\partial a|}, \qquad \beta = \frac{1}{|\partial a|}.$$

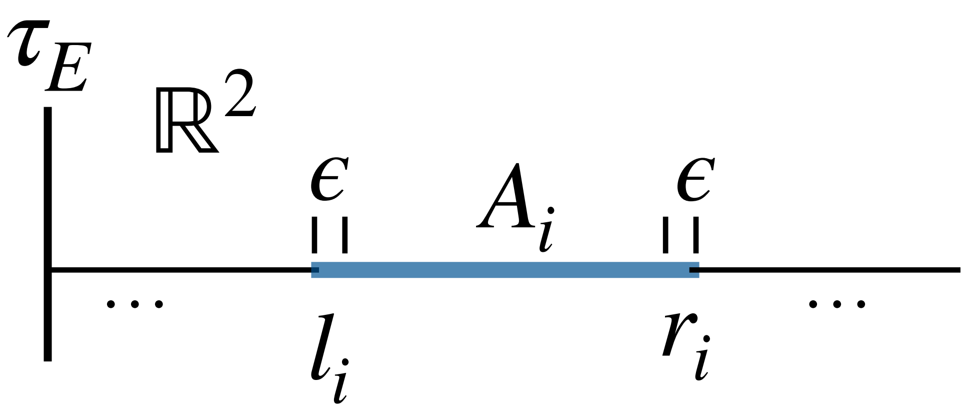


$$\alpha(z) = \frac{i}{2\pi}\log\left(\prod_{i=1}^{N}\frac{z-l_i}{r_i-z}\right)$$

Solutions of $\tilde{S}_{n\to 0}$ in $d=2$

$$\partial^2 a = 0 \quad \& \quad \oint_\Gamma da = 1$$

$$u_\mu = \frac{\partial_\mu a}{|\partial a|}, \qquad \beta = \frac{1}{|\partial a|}.$$



$$\alpha(z) = \frac{i}{2\pi}\log\left(\prod_{i=1}^{N}\frac{z-l_i}{r_i-z}\right)$$

$$\beta(x,y) = 2\pi\left|\sum_{i=1}^{N}\left[\frac{1}{z-l_i}+\frac{1}{r_i-z}\right]\right|^{-1}$$

$$\tilde{S}_{n\to 0} = \frac{\sigma_s}{n}\int_{\sim A}\frac{u^\mu}{\beta(x)}n_\mu$$

$$\tilde{S}_{n\to 0} = \frac{\sigma_s}{n}\left(\sum_{i,j}\log|r_j-l_i| - \sum_{i<j}\log|r_j-r_i| - \sum_{i<j}\log|l_j-l_i| - N\log\epsilon\right)$$

Same as free fermion in two dimensions [Casini, Huerta: 0905.2562]

Holographic geometries of $\tilde{\rho}_{n\to 0}(A)$ in $d=2$

$$ds^2 = L^2\frac{d\rho^2}{4\rho^2} + \frac{L^2}{n^2\rho}h_{\mu\nu}dx^\mu dx^\nu$$

$$\mathrm{Ricci}(h_{\mu\nu}) = 0$$

$$\text{if} \quad \partial^2 a = 0$$

Anstaz becomes exact!

Holographic geometries of $\tilde{\rho}_{n\to 0}(A)$ in $d=2$

$$ds^2 = L^2\frac{d\rho^2}{4\rho^2} + \frac{L^2}{n^2\rho}h_{\mu\nu}dx^\mu dx^\nu$$

$$\mathrm{Ricci}(h_{\mu\nu}) = 0$$
$$\text{if} \quad \partial^2 a = 0$$

Anstaz becomes exact!

Ansatz does not describe $\tilde{\rho}_n(A)$ for all n

Bdy Ansatz $_{n\to 0}$

$T_{\mu\nu}^{\mathrm{PF}}$ and J_μ^{PF}

$\Updownarrow$

Bulk Ansatz $_{n\to 0}$

$g_{\mu\nu}(\rho, \beta(x), u^\mu)$

Holographic geometries of $\tilde{\rho}_{n\to 0}(A)$ in $d=2$

$$ds^2 = L^2\frac{d\rho^2}{4\rho^2} + \frac{L^2}{n^2\rho}h_{\mu\nu}dx^\mu dx^\nu$$

$\mathrm{Ricci}(h_{\mu\nu}) = 0$

if $\quad \partial^2 a = 0$

Anstaz becomes exact!

Ansatz does not describe $\tilde{\rho}_n(A)$ for all n

Bdy Ansatz $_{n\to 0}$

$T^{\mathrm{PF}}_{\mu\nu}$ and J^{PF}_{μ}

$T_{\mu\nu}$ and J_μ ??

Additional divergences!!

Bulk Ansatz $_{n\to 0}$

$g_{\mu\nu}(\rho, \beta(x), u^\mu)$

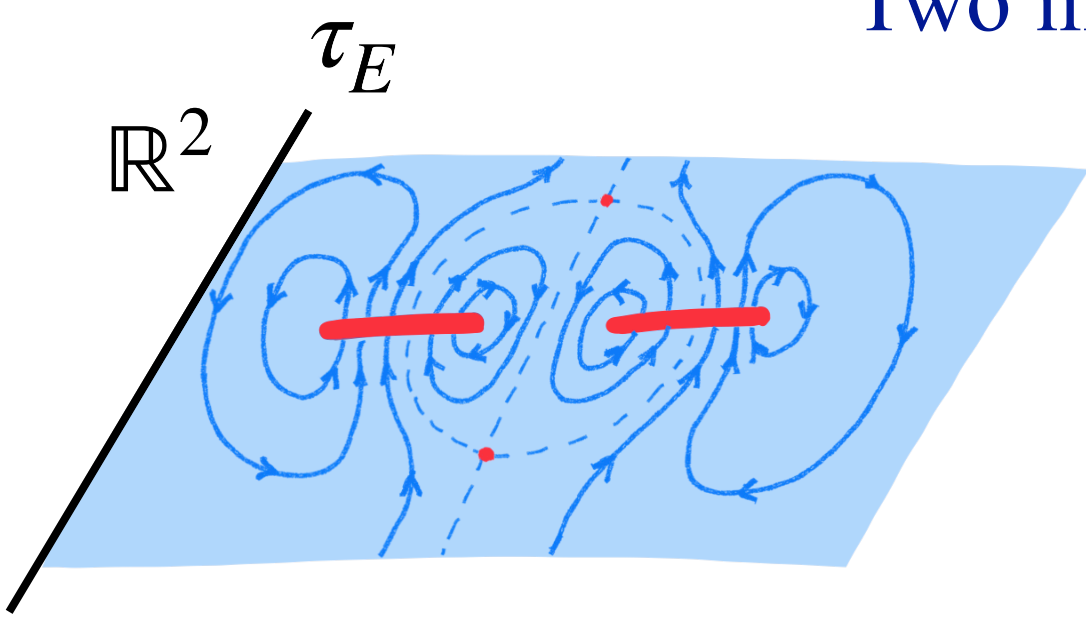
Finite n

Bulk Ansatz n

$g_{\mu\nu}(\rho, \beta(x), u^\mu)$

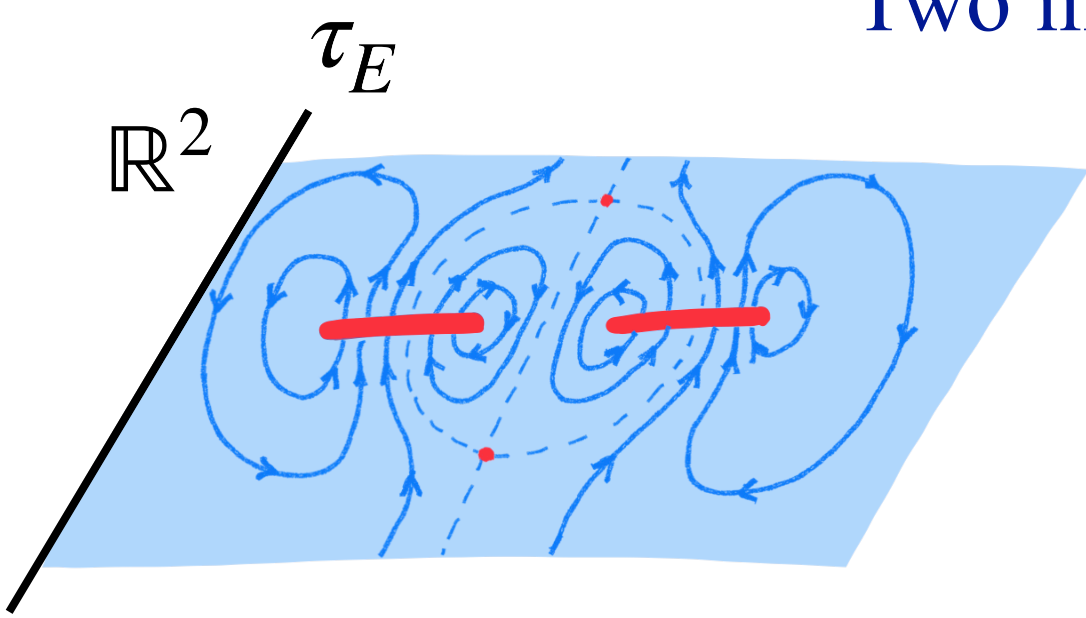
Boundary to bulk mapping in $d = 2$

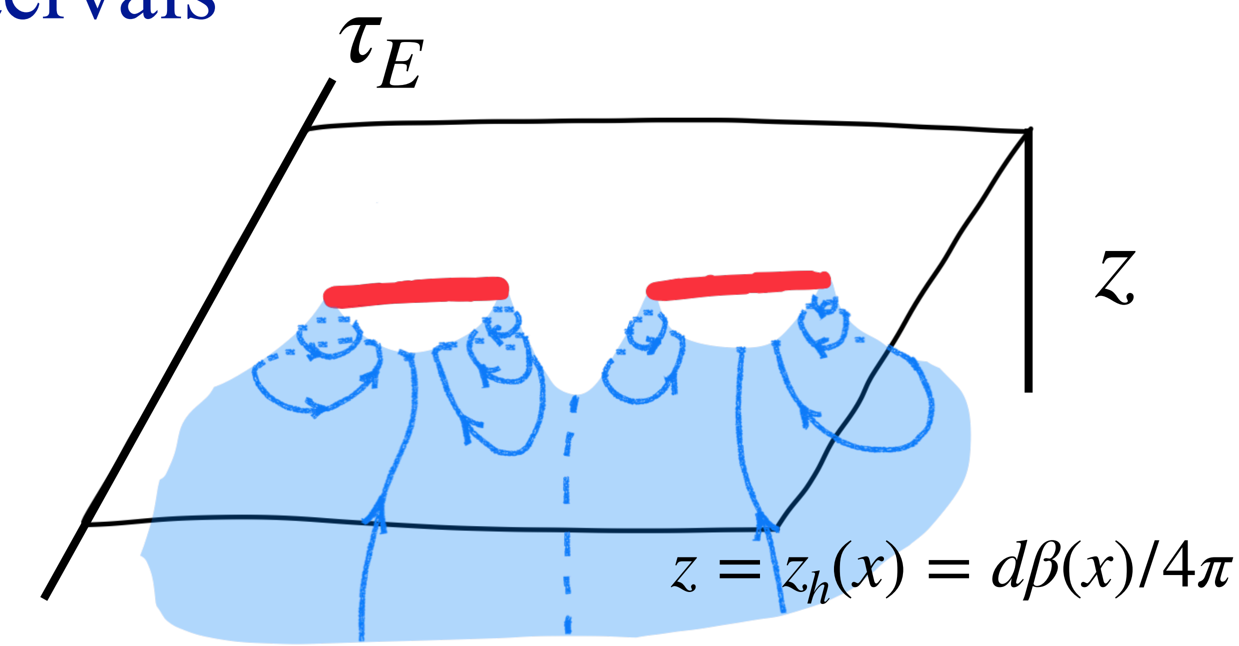
Two intervals



Boundary to bulk mapping in $d = 2$

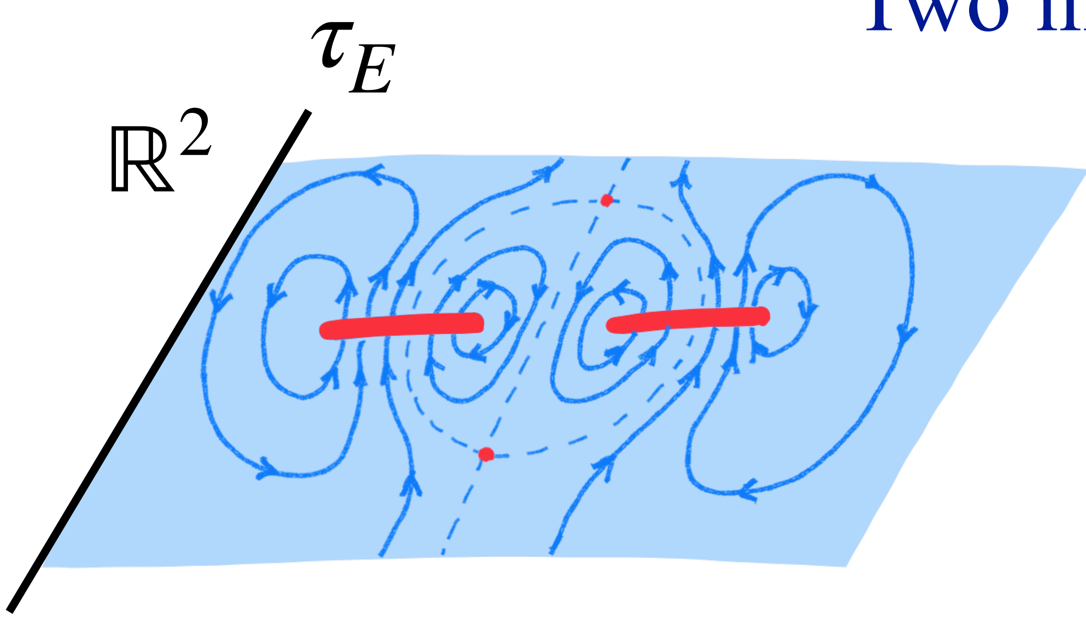
Two intervals

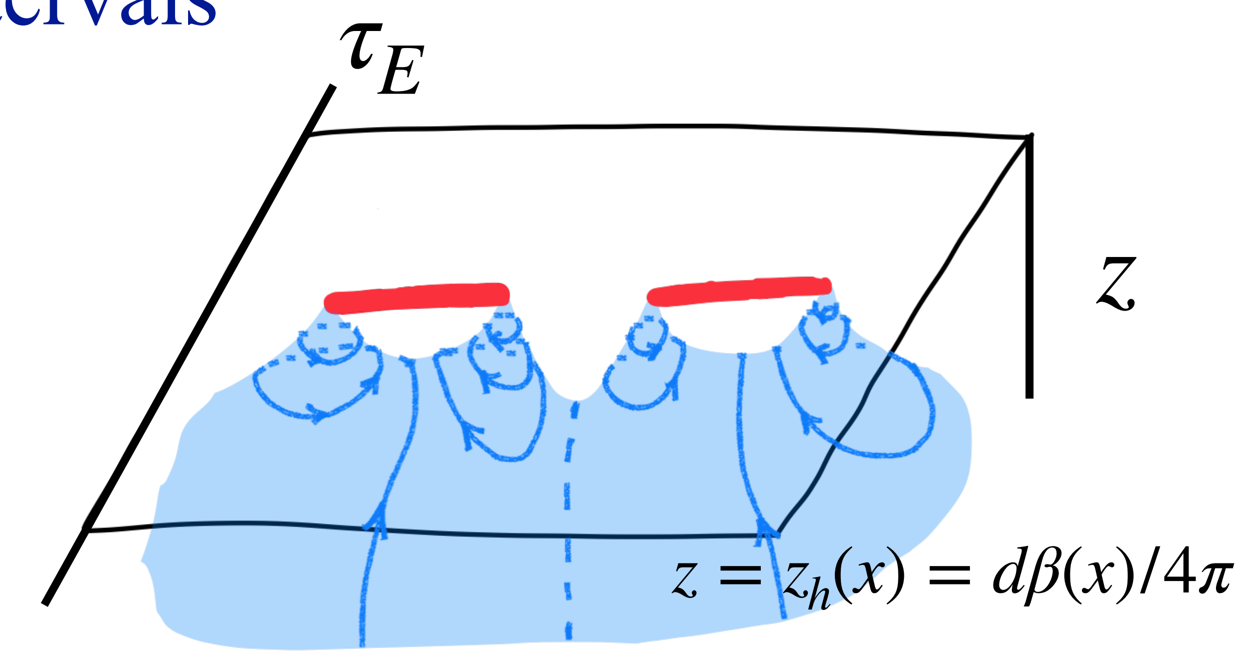


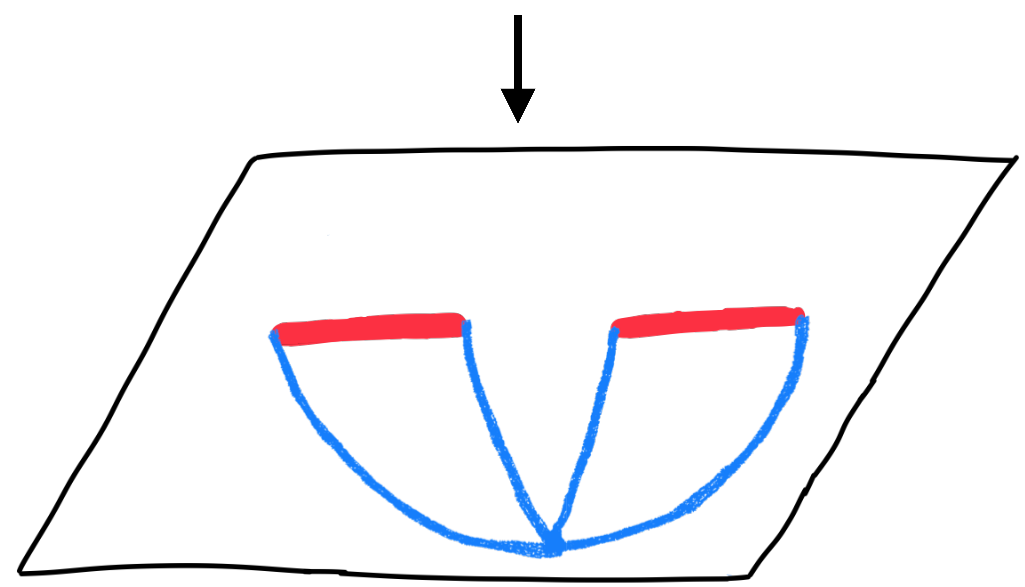


Boundary to bulk mapping in $d = 2$

Two intervals





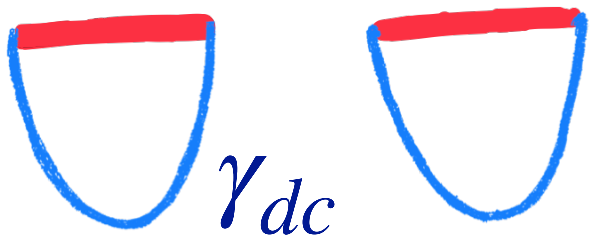


What about phase transitions in the holographic Rényis??

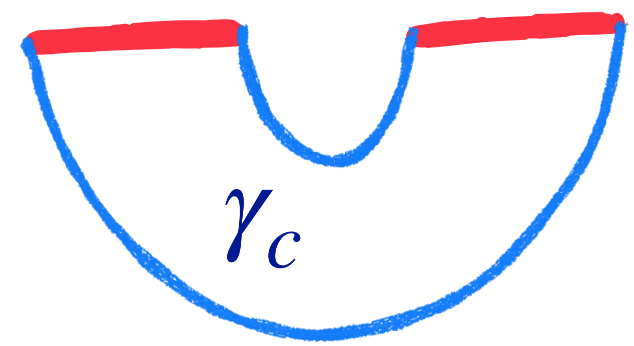
Phase transitions in Rényi entropies

$n \in \mathbb{Z}, \quad n > 1$

Disconnected



Connected

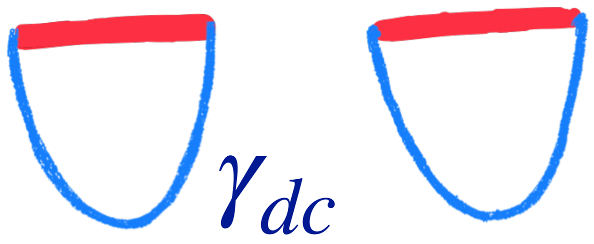


Works by [Headrick:1006.0047], [Faulkner: 1303.7221], [Hartman:1303.6955]

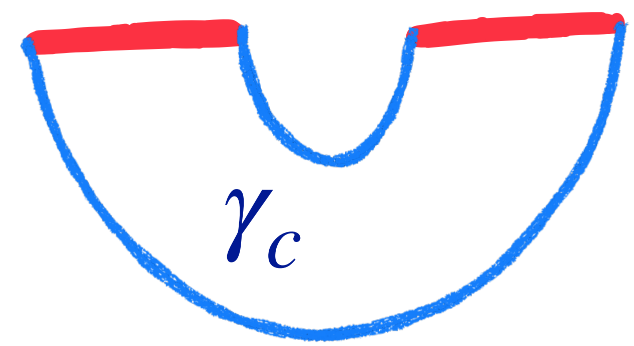
Phase transitions in Rényi entropies

$n \in \mathbb{Z}, \quad n > 1$

Disconnected



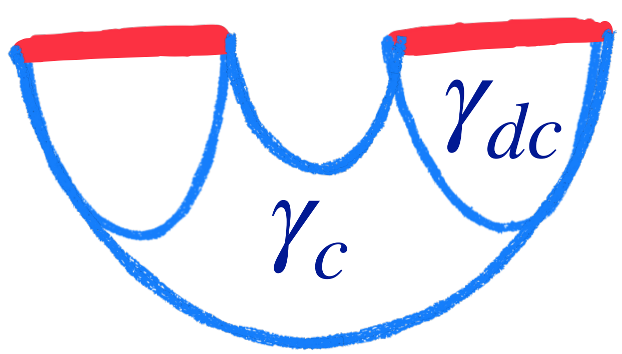
Connected



Works by [Headrick:1006.0047], [Faulkner: 1303.7221], [Hartman:1303.6955]

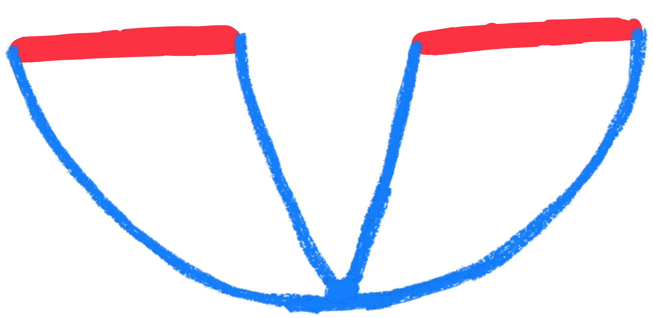
Revisitation of Dong's formula $n < 1$

[Dong, Kudler-Flam, Rath: 2312.04625]



$n < n_c < 1$

$\longrightarrow$



$n \to 0$

Density of states ρ_A at high modular energies

$$S_n(A) \sim \sigma_s \frac{g(A)}{n^{d-1}}, \qquad \rightarrow \qquad e^{\sigma_s \frac{g(A)}{n^{d-1}}} = \lim_{n \to 0} c_n \int_0^\infty dE e^{-nE} \rho_A(E)$$

Density of states ρ_A at high modular energies

$$S_n(A) \sim \sigma_s \frac{g(A)}{n^{d-1}}, \qquad \rightarrow \qquad e^{\sigma_s \frac{g(A)}{n^{d-1}}} = \lim_{n \to 0} c_n \int_0^\infty dE e^{-nE} \rho_A(E)$$

Assuming $\rho_A(E) \sim e^{k_A E^a}$ we find

$$\rho_A(E) \sim e^{k_A E^{\frac{d-1}{d}}} \quad \text{where} \quad k_A = \frac{d\sigma_s^{\frac{1}{d}}}{(d-1)^{\frac{d-1}{d}}} g^{\frac{1}{d}}(A)$$

Density of states ρ_A at high modular energies

$$S_n(A) \sim \sigma_s \frac{g(A)}{n^{d-1}}, \quad \to \quad e^{\sigma_s \frac{g(A)}{n^{d-1}}} = \lim_{n\to 0} c_n \int_0^\infty dE e^{-nE} \rho_A(E)$$

Assuming $\rho_A(E) \sim e^{k_A E^a}$ we find

$$\rho_A(E) \sim e^{k_A E^{\frac{d-1}{d}}} \quad \text{where} \quad k_A = \frac{d\sigma_s^{\frac{1}{d}}}{(d-1)^{\frac{d-1}{d}}} g^{\frac{1}{d}}(A)$$

We generalized [B,C,N,P,S: 2411.00937] away from spherical regions

$$g(A) = \int_{\sim A} \frac{u^\mu}{\beta^{d-1}(x)} n_\mu$$

Summary

- We **derived** a formula for computing $\tilde{S}_{n\to 0}$, in which the relevant state it is described by a **perfect fluid** with **vorticities** at ∂A

Summary

- We **derived** a formula for computing $\tilde{S}_{n\to 0}$, in which the relevant state it is described by a **perfect fluid** with **vorticities** at ∂A

- We **proved** the formula using **holography** and analyzed a few features for the $d = 2$ case.

Summary

- We **derived** a formula for computing $\tilde{S}_{n\to 0}$, in which the relevant state it is described by a **perfect fluid** with **vorticities** at ∂A

- We **proved** the formula using **holography** and analyzed a few features for the $d = 2$ case.

Future work

- Develop a framework to include corrections the leading $n \to 0$ a la **Fluid-Gravity correspondence**

Summary

- We **derived** a formula for computing $\tilde{S}_{n\to 0}$, in which the relevant state it is described by a **perfect fluid** with **vorticities** at ∂A

- We **proved** the formula using **holography** and analyzed a few features for the $d = 2$ case.

Future work

- Develop a framework to include corrections the leading $n \to 0$ a la **Fluid-Gravity correspondence**

- Work in progress on refined measures such as **symmetry resolved Rényi entropies**, **charged Rényi entropies,** etc in the $n \to 0$

Gracias

Additional slides

Fluid equations of a perfect/ideal fluid

$$(\partial_\mu \log \beta - d u_\mu (u_\lambda \partial_\lambda) \log \beta + u_\mu (\partial_\lambda u_\lambda) + (u_\lambda \partial_\lambda) u_\mu) = 0$$

Einstein's equations for anzats

$$R_{ab} - \frac{d}{L^2} g_{ab} = 0$$

Put metric in Fefferman Graham form

$$ds^2 = L^2 \frac{d\rho^2}{4\rho^2} + \frac{L^2}{n^2 \rho} h_{\mu\nu} dx^\mu dx^\nu$$

$$\rho \in \left[\frac{\# \epsilon^2}{n^2 \beta^2(x)}, 1 \right]$$

$$h_{\mu\nu} = \frac{16\pi^2}{d^2 \beta^2(x)} \left(\frac{1 + \rho^{\frac{d}{2}}}{2} \right)^{\frac{4}{d}} \left[\delta_{\mu\nu} - \frac{4\rho^{\frac{d}{2}}}{\left(1 + \rho^{\frac{d}{2}}\right)^2} u_\mu u_\nu \right], \qquad \rho = \left(\frac{1 - \sqrt{1 - (z/z_h(x))^d}}{1 + \sqrt{1 - (z/z_h(x))^d}} \right)^{\frac{2}{d}}$$

$$\rho \left[2h'' - 2h'h^{-1}h' + \mathrm{tr}\left(h^{-1}h'\right) h' \right] + \mathrm{Ric}(h) - (d-2)h' - \mathrm{tr}\left(h^{-1}h'\right) h = 0$$

$$\nabla_\mu \mathrm{tr}\left(h^{-1}h'\right) - \nabla^\nu h'_{\mu\nu} = 0$$

$$\mathrm{tr}\left(h^{-1}h''\right) - \frac{1}{2}\mathrm{tr}\left(h^{-1}h'h^{-1}h'\right) = 0$$