Higher-Order Newton-Cartan Gravity

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based on arXiv: 2507.05489 [hep-th] with Biel Cardona

GRASS-SYMBHOL Meeting **2025**

Toledo November 11^{th}-13^{th}

This work has been supported in part by the Ramon y Cajal fellowship RYC2023-042671-I , funded by MCIU/AEI/10.13039/501100011033 and FSE+ and in part by Fundación Séneca de la Región de Murcia FSRM/10.13039/100007801 (22581/PI/24).

1. Goal: Non-Relativistic Limit of Higher-Order Gravity Theories

2. Limit of Einstein-Hilbert(+Maxwell)

3. Non-Relativistic Limit of Higher-Order Theories
   - The Action
   - Higher-Dimensional Origin

4. Non-Relativistic Limit of the Equations of Motion
   - Non-Relativistic Ricci Scalar Squared Gravity
   - Gauss-Bonnet-Newton-Cartan Gravity

5. Conclusion & Outlooks

Higher-Order Gravity Theories

Goal

Obtain the Non-Relativistic limit of the family of higher-order theories below at the level of action and equations of motion.

Higher-Order Theories:the Action

We consider the following set of theories in D dimensions, parametrized by the couplings α, β and γ

$$\mathcal{S} = \int d^D x \sqrt{-g} \left(\mathrm{R} + \alpha\, \mathrm{R}^2 + \beta\, \mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} + \gamma\, \mathrm{R}_{\mu\nu\rho\sigma}\mathrm{R}^{\mu\nu\rho\sigma} \right),$$

We study them non-perturbatively (α, β and γ are not small).

Some Relevant Features

- The equations of motion contain terms with more than 2 derivatives.
- The pure quadratic theory defined by $\beta = -4\alpha\,, \gamma = \alpha$ is called Gauss-Bonnet gravity.
    - It has second-order equations of motion.
    - It is a topological term in 4D.

Taking the Non-Relativistic Limit

① Lorentz Breaking

The flat index $\hat{A}$ is decomposed in a longitudinal and transverse part $\hat{A} = \{0, a\}$

$$\hat{A} = \begin{cases} 0 & \textbf{Longitudinal} \\ a = 1, \dots, D-1 & \textbf{Transverse} \end{cases}$$

particle limit.

② Redefinition Ansatz

The relativistic fields (and parameters) are redefined (in an invertible way) in terms of would be non-relativistic fields (and parameters) with a dimensionless parameter c. We consider the vielbein $E_\mu{}^{\hat{A}}$ $(g_{\mu\nu} = E_\mu{}^{\hat{A}} E_\nu{}^{\hat{B}} \eta_{\hat{A}\hat{B}})$,

$$E_\mu{}^0 = c\tau_\mu \,, \qquad E_\mu{}^a = e_\mu{}^a \,.$$

and the Lorentz symmetry parameter

$$\underbrace{\Lambda_{0a} = \frac{1}{c}\lambda_a}_{\text{Boost}} \,, \qquad \underbrace{\Lambda_{ab} = \lambda_{ab}}_{\text{Transverse Rotations}} \,.$$

③ Substitution

The ansatz is substituted in the expressions we want to take the limit of. For example the Lorentz transformation rules

$$\delta E_\mu{}^0 = \Lambda^0{}_b E_\mu{}^b \,,$$

$$\delta E_\mu{}^a = \Lambda^a{}_b E_\mu{}^b + \Lambda^a{}_0 E_\mu{}^0 \,,$$

become

$$\delta\tau_\mu = -\frac{1}{c^2}\lambda_b e_\mu{}^b \,,$$

$$\delta e_\mu{}^a = \lambda^a{}_b e_\mu{}^b - \lambda^a \tau_\mu \,.$$

④ The Limit

The parameter c is sent to ∞. The limit is consistent if it does not develop **divergent** terms. In the example

$$\delta\tau_\mu = 0 \,,$$

$$\delta e_\mu{}^a = \lambda^a{}_b e_M{}^b - \lambda^a \tau_\mu \,.$$

Limit of the Action

Einstein-Hilbert Action

$$\mathcal{S}_{EH} = \int d^D x \sqrt{-g}\, \mathrm{R}$$

Index	Definition & Values
$\mu, \nu, \rho, \ldots$	D Curved ,
$\hat{A}, \hat{B}, \hat{C}, \ldots$	D Flat $(\hat{A} = \{0, a\})$,
$a, b, c \ldots$	Transverse Flat $a = 1, \ldots, D-1$.

Ansatz

$$E_\mu{}^0 = c\tau_\mu \,,$$
$$E^\mu{}_0 = \frac{1}{c}\tau^\mu$$
$$E_\mu{}^a = e_\mu{}^a \,,$$
$$E^\mu{}_a = e^\mu{}_a \,.$$

Orthogonality Relations

$$\tau^\mu \tau_\mu = 1 \,,$$
$$e^\mu{}_a e_\mu{}^b = \delta_a^b \,,$$
$$e^\mu{}_a \tau_\mu = e_\mu{}^a \tau^\mu = 0 \,,$$
$$\tau_\mu \tau^\nu + e_\mu{}^a e^\nu{}_a = \delta_\mu^\nu \,.$$

Ricci Scalar Expansion

$$\mathrm{R} = \frac{c^2}{4} t_{ab} t^{ab} + \mathcal{O}(c^0) \,,$$

with $t_{ab} = e^\mu{}_a e^\nu{}_b t_{\mu\nu}$ and $t_{\mu\nu} = 2\partial_{[\mu}\tau_{\nu]}$ **(intrinsic torsion).**

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Electric Limit

The leading order term is the result of the limit:

$$\mathcal{S}_{EH} \propto \int d^D x\, e\, t_{ab} t^{ab}$$

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Electric Limit

The leading order term is the result of the limit:

$$\mathcal{S}_{EH} \propto \int d^D x\, e\, t_{ab}t^{ab}$$

Magnetic Limit

We regard the leading order term as divergent. We need a way to cancel it!

Magnetic Limit: Cancellation Mechanism

Einstein-Hilbert+Maxwell

$$\mathcal{S} = \int d^{10}x\,\sqrt{-g}\left(\mathrm{R} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)$$

with $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$.

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Ricci Scalar Expansion

$$\mathrm{R} = \frac{c^2}{4}t_{ab}t^{ab} + \mathcal{O}(c^0)\,,$$

$$F_{\mu\nu}F^{\mu\nu} = c^2 t_{ab}t^{ab} + \mathcal{O}(c^0)\,,$$

with $t_{ab} = e^{\mu}{}_{a}e^{\nu}{}_{b}t_{\mu\nu}$ and $t_{\mu\nu} = 2\partial_{[\mu}\tau_{\nu]}$ **(intrinsic torsion).**

Ansatz

$$E_\mu{}^0 = c\tau_\mu\,,$$

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$$A_\mu = c\tau_\mu + \frac{1}{c}a_\mu\,.$$

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$$\tau^\mu\tau_\mu = 1\,,$$

$$e^\mu{}_a e_\mu{}^b = \delta_a^b\,,$$

$$e^\mu{}_a\tau_\mu = e_\mu{}^a\tau^\mu = 0\,,$$

$$\tau_\mu\tau^\nu + e_\mu{}^a e^\nu{}_a = \delta_\mu^\nu\,.$$

Magnetic Limit

The leading order term coming from the the Einstein-Hilbert lagrangian cancels against an analogous term coming from the Maxwell lagrangian. The result of the limit is the action term at sub-leading order in the expansion in c.

$$\mathcal{S} = \int d^D x\, e\left(e^{\mu a}e^{\nu}_{a}\tilde{\mathrm{R}}_{\mu\nu} + 2\tau^\mu e^{\nu a}e^{\rho}_{a}\tilde{\nabla}_\nu t_{\mu\rho} + -\frac{3}{2}t_{0a}t_0{}^a - t^{ab}f_{ab}\right) + \mathcal{O}(c^{-2})\,,$$

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The Strategy

Mimicking the 2-derivative case we can use a gauge field to cancel the divergences of the following actions

$$\mathcal{S} = \int d^D x \sqrt{-g} \left(\mathrm{R} + \alpha \, \mathrm{R}^2 + \beta \, \mathrm{R}_{\mu\nu} \mathrm{R}^{\mu\nu} + \gamma \, \mathrm{R}_{\mu\nu\rho\sigma} \mathrm{R}^{\mu\nu\rho\sigma} \right) ,$$

A Taste of the Divergences

$$\begin{aligned} \mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} = & \frac{1}{16} c^4 \left(t_{ab} t^{ab} t_{cd} t^{cd} + 4 t_a{}^b t_b{}^c t_c{}^d t_d{}^a \right) + , \\ & + \frac{1}{4} c^2 \Big[4 t^{ab} t_a{}^c (\tilde{\mathrm{R}}_{bc} + \tilde{\nabla}(t)_{b0c} - 3 t_{0b} t_{0c}) + t^{ab} t_{ab} (-2 \tilde{\nabla}(t)_{c0}{}^c + \\ & + 2 t_{0c} t_0{}^c + t^{cd} f_{cd}) + 2 (\tilde{\nabla}(t)_a{}^{ab} - 4 t^{ab} t_{0a}) \tilde{\nabla}(t)_{cb}{}^b \Big] + \mathcal{O}(c^0) , \end{aligned}$$

THE STRATEGY

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$$\mathcal{S} = \int d^D x \sqrt{-g} \left(\mathrm{R} + \alpha\, \mathrm{R}^2 + \beta\, \mathrm{R}_{\mu\nu} \mathrm{R}^{\mu\nu} + \gamma\, \mathrm{R}_{\mu\nu\rho\sigma} \mathrm{R}^{\mu\nu\rho\sigma} \right),$$

THE ISSUES

There are a few critical points:

- Adding just the Maxwell terms is not enough, we need higher-order terms ,
- Coupling between the gravity sector and Maxwell field could be necessary ,
- The higher-order terms will produce extra divergences (at orders c^4 and c^2). These should all be cancelled otherwise the contribution of the EH term is removed from the action.

THE TOOLS

We add the following terms to the action, each with an arbitrary coupling and we fix the couplings requiring full cancellation of the divergences ,

$$\mathcal{B} = \Big\{ (F_{\mu\nu}F^{\mu\nu})^2 \ , F^{\mu}{}_{\nu}F^{\nu}{}_{\rho}F^{\rho}{}_{\sigma}F^{\sigma}{}_{\mu} \, , F^{\mu\nu}F^{\rho\sigma}\mathrm{R}_{\mu\nu\rho\sigma} \, , F^{\mu\rho}F^{\nu}{}_{\rho}\mathrm{R}_{\mu\nu} \, ,$$
$$F_{\mu\nu}F^{\mu\nu}\mathrm{R} \, , \nabla_{\mu}F_{\nu\rho}\nabla^{\mu}F^{\nu\rho} \, , \nabla_{\mu}F^{\mu\rho}\nabla_{\nu}F^{\nu}{}_{\rho} \Big\} \, .$$

Voluntarily overparametrized (avoid integration by parts).

REMARKS

- In the Einstein-Hilbert-Maxwell case the coefficient of the Maxwell term can be modified by redefinition.
- In the higher-order case not all the coeffcients in $\mathcal{B}$ can be arbitrarily fine-tuned via non-perturbative redefinitions.
- Cancellation of divergences would be highly non-trivial!

The Result

- It is possible to cancel all the divergences coming from the higher order terms.
- For each of the quadratic terms there is only one! combinations in $\mathcal{B}$ cancelling the divergent part (only certains theories admit the non-relativistic limit).
- The coeffcients do not depend on the dimension.

Relativistic Theories Admitting a Non-Relativistic Limit

For each of the quandratic terms in the action the following combinations guarantee a finite limit of the action

$$\mathcal{L}_\alpha = \left(\mathrm{R} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)^2 ,$$

$$\mathcal{L}_\beta = \mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} + \frac{1}{16} F^4 + \frac{1}{4} F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu - \frac{1}{2} \nabla_\mu F^{\mu\nu} \nabla_\rho F_\nu{}^\rho - F^{\mu\rho} F^\nu{}_\rho \mathrm{R}_{\mu\nu} ,$$

$$\mathcal{L}_\gamma = \mathrm{R}_{\mu\nu\rho\sigma}\mathrm{R}^{\mu\nu\rho\sigma} + \frac{3}{8} F^4 + \frac{5}{8} F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu + \nabla_\mu F_{\nu\rho} \nabla^\mu F^{\nu\rho} - \frac{3}{2} F^{\mu\nu} F^{\rho\sigma} \mathrm{R}_{\mu\nu\rho\sigma} .$$

Limit of the Action: Selected Theories

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The Relativistic Theory (Quadratic Part)

$$\mathcal{L}_\alpha = \left(\mathrm{R} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)^2$$

$$\Downarrow$$

The Non-Relativistic Limit of the Quadratic Part

$$\overset{(0)}{\mathcal{L}}_\alpha = \left[-\operatorname{Ric}(\mathrm{J}) + 2\tilde{\nabla}\mathrm{R}(\mathrm{H})_{a0}{}^{a} - \frac{3}{2}\mathrm{R}(\mathrm{H})_{0a}\mathrm{R}(\mathrm{H})_{0}{}^{a} \right]^2$$

Limit of the Action: Selected Theories

The Relativistic Theory (Quadratic Part)

$$\mathcal{L}_\beta = \mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} + \frac{1}{16}F^4 + \frac{1}{4}F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu - \frac{1}{2}\nabla_\mu F^{\mu\nu}\nabla_\rho F_\nu{}^\rho - F^{\mu\rho}F^\nu{}_\rho \mathrm{R}_{\mu\nu}$$

The Non-Relativistic Limit of the Quadratic Part

$$\begin{aligned}
\overset{(0)}{\mathcal{L}}_\beta = {} & \mathrm{Ric(J)}_{ab}\mathrm{Ric(J)}^{ab} + 2\mathrm{Ric(J)}_{ab}\tilde{\nabla}\mathrm{R(H)}^{ab}{}_0 - 2\mathrm{Ric(J)}_{0a}\tilde{\nabla}\mathrm{R(H)}^{ba}{}_b + \\
& + 2\mathrm{Ric(J)}_{0a}\mathrm{R(H)}^{ab}\mathrm{R(H)}_{0b} + \frac{1}{2}\mathrm{R(G)}_{0c}{}^c\mathrm{R(H)}^{ab}\mathrm{R(H)}_{ab} + \mathrm{Ric(J)}_{ab}\mathrm{R(H)}_0{}^a\mathrm{R(H)}_0{}^b + \\
& - \tilde{\nabla}\mathrm{R(H)}_a{}^{ab}\tilde{\nabla}\mathrm{R(H)}_{00b} + \frac{1}{2}\tilde{\nabla}\mathrm{R(H)}_{a0}{}^a\tilde{\nabla}\mathrm{R(H)}_{b0}{}^b + \tilde{\nabla}\mathrm{R(H)}^{ab}{}_0\tilde{\nabla}\mathrm{R(H)}_{(ab)0} + \\
& - \tilde{\nabla}\mathrm{R(H)}_{00a}\mathrm{R(H)}^{ab}\mathrm{R(H)}_{0b} - \tilde{\nabla}\mathrm{R(H)}_{a0}{}^a\mathrm{R(H)}_{0b}\mathrm{R(H)}_0{}^b + \\
& + \tilde{\nabla}\mathrm{R(H)}^{ab}{}_0\mathrm{R(H)}_{0a}\mathrm{R(H)}_{0b} + \frac{3}{4}\mathrm{R(H)}_0{}^a\mathrm{R(H)}_{0a}\mathrm{R(H)}_0{}^b\mathrm{R(H)}_{0b}
\end{aligned}$$

Limit of the Action: Selected Theories

The Relativistic Theory (Quadratic Part)

$$\mathcal{L}_\gamma = \mathrm{R}_{\mu\nu\rho\sigma}\mathrm{R}^{\mu\nu\rho\sigma} + \frac{3}{8}F^4 + \frac{5}{8}F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu + \nabla_\mu F_{\nu\rho}\nabla^\mu F^{\nu\rho} - \frac{3}{2}F^{\mu\nu}F^{\rho\sigma}\mathrm{R}_{\mu\nu\rho\sigma}$$

The Non-Relativistic Limit of the Quadratic Part

$$\begin{aligned}\overset{(0)}{\mathcal{L}}_\gamma = {} & \mathrm{R(J)}^{abcd}\mathrm{R(J)}_{abcd} - \frac{10}{3}\mathrm{R(J)}_0{}^{(ab)c}\tilde{\nabla}\mathrm{R(H)}_{abc} + 2\mathrm{R(G)}^{b(ac)}\tilde{\nabla}\mathrm{R(H)}_{abc} + \\ & + 2\mathrm{R(G)}_{0ab}\mathrm{R(H)}^{ac}\mathrm{R(H)}^b{}_c + \frac{8}{3}\mathrm{R(J)}_0{}^{(ac)b}\mathrm{R(H)}_{ab}\mathrm{R(H)}_{0c} + \\ & - 2\tilde{\nabla}\mathrm{R(H)}_{00}{}^a\mathrm{R(H)}_{ab}\mathrm{R(H)}_0{}^b + 2\tilde{\nabla}\mathrm{R(H)}^{ab}{}_0\mathrm{R(H)}_{0a}\mathrm{R(H)}_{0b} + \\ & - 2\tilde{\nabla}\mathrm{R(H)}_0{}^{ab}\tilde{\nabla}\mathrm{R(H)}_{0ab} + 2\tilde{\nabla}\mathrm{R(H)}^{ab}{}_0\tilde{\nabla}\mathrm{R(H)}_{ab0} + \frac{3}{4}\mathrm{R(H)}_0{}^a\mathrm{R(H)}_{0a}\mathrm{R(H)}_0{}^b\mathrm{R(H)}_{0b}\end{aligned}$$

Beyond Quadratic Gravity

Prescription

The non-relativistic limit of any theory whose Lagrangian is a function of R, $\mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu}$ and $\mathrm{R}_{\mu\nu\rho\sigma}\mathrm{R}^{\mu\nu\rho\sigma}$,

$$\mathcal{L} = f(\mathrm{R}, \mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu}, \mathrm{R}_{\mu\nu\rho\sigma}\mathrm{R}^{\mu\nu\rho\sigma})$$

is well defined upon the following substitution:

$$\mathcal{L} = f(\mathcal{L}_{EHM}, \mathcal{L}_{\beta}, \mathcal{L}_{\gamma})\,.$$

Why only those theories?

- The limit knows about some physical features of the possible higher-order theories?
- Can the existence of the non-relativistic limit be used to gain informations on the higher-order gravity theories?
- Can we characterize more precisely the theories admitting the limit?

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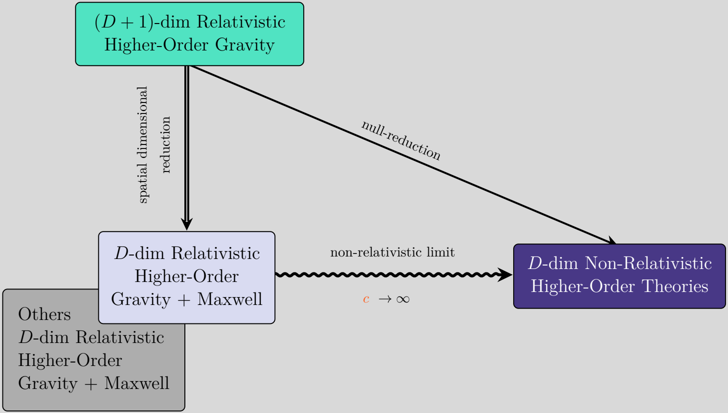
Pure Gravitational Theory in D+1 Dimensions

The Lagrangian admitting a finite non-relativistic limit $\mathcal{L}_\alpha, \mathcal{L}_\beta$ and $\mathcal{L}_\gamma$ can be obtained considering their pure gravitational part in one dimension higher and performing a compactification followed by a truncation of the scalar field.

Higher-Dimensional Origin of the Theories with Finite Limit



TAKING THE LIMIT OF THE EQUATIONS OF MOTION

- The limit $c \to \infty$ just select the leading order of the expansion of the equations of motion.
- Subtlety!

$$[X] = c^n \overset{(n)}{[X]} + c^{n-2} \overset{(n-2)}{[X]} + \mathcal{O}(c^{n-4}) = 0\,, \quad [Y] = c^n \overset{(n)}{[X]} + c^{n-2} \overset{(n-2)}{[Y]} + \mathcal{O}(c^{n-4}) = 0\,.$$

The limit gives

$$\overset{(n)}{[X]} = 0\,, \qquad \overset{(n)}{[X]} = 0\,.$$

We have lost and equation!

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To avoid loosing equations of motion in the limit we should combine them before sending c to infinity:

$$[X] + [Y] = 2c^n \overset{(n)}{[X]} + c^{n-2} \left(\overset{(n-2)}{[X]} + \overset{(n-2)}{[Y]} \right) + \mathcal{O}(c^{n-4})\,,$$

$$[X] - [Y] = c^{n-2} \left(\overset{(n-2)}{[X]} - \overset{(n-2)}{[Y]} \right) + \mathcal{O}(c^{n-4})\,.$$

The limit gives

$$\overset{(n)}{[X]} = 0\,, \qquad \overset{(n-2)}{[X]} - \overset{(n-2)}{[Y]} = 0\,.$$

Equations of Motion: Einstein-Hilbert-Maxwell Case

The equations of motion coming from the action we have considered are:

$$[G]^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} F^{\mu}{}_{\rho} F^{\nu\rho} = 0\,,$$

$$[A]^{\mu} = \nabla_{\nu} F^{\nu\mu} = 0\,,$$

and with flat indices:

$$[G]_{\hat{A}\hat{B}} = [G]^{\mu\nu} E_{\mu\hat{A}} E_{\nu\hat{B}} = 0\,, \qquad [A]^{\hat{A}} = [A]^{\mu} E_{\mu}{}^{\hat{A}} = 0\,.$$

The Poisson Equation in Newtonian Gravity

The Poisson equation is characteristic of Newtonian gravity (and Newton-Cartan gravity), $\Delta\Psi = 4\pi G\rho$, with ρ mass density distribution and Ψ Newton potential. We could expect an analogous equation from our limit.

Where is the Poisson Equation?

We can consider a general scalar combination

$$[P_+] := \mathtt{A}\, E^{\mu}{}_{a} E^{\nu a} [G]_{\mu\nu} + \mathtt{C} E^{\mu}{}_{0} E^{\nu}{}_{0}\, [G]_{\mu\nu} + \mathtt{B}\, E_{\mu}{}^{0} [A]^{\mu}$$

where $\mathtt{A}, \mathtt{B}$ and $\mathtt{C}$ are three constant parameters. The expansion of $[P_+]$

$$[P_+] = c^2 [\overset{(2)}{P}_+] + [\overset{(0)}{P}_+] + c^{-2} [\overset{(-2)}{P}_+] + \mathcal{O}(c^{-4})\,,$$

has a term that potentially could be interpreted as a Poisson equation at order c^{-2}. Indeed it contains

$$e^{\mu a} e^{\nu}{}_{a} \partial_\mu \partial_\nu a_0$$

that can be identified with the characteristic Laplacian term acting on the Newton's potential.

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The Issue!

The order c^{-2} is sub-sub-leading: it cannot be obtained from the limit of the corresponding equations unless...

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THE ISSUE!

The order c^{-2} is sub-sub-leading: it cannot be obtained from the limit of the corresponding equations unless...

THE SOLUTION

...the leading (c^2) and sub-leading (c^0) orders cancels by choosing $\mathtt{A}$, $\mathtt{B}$ and $\mathtt{C}$ properly .

Where is the Poisson Equation?

We can consider a general scalar combination

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where $\mathtt{A}, \mathtt{B}$ and $\mathtt{C}$ are three constant parameters. The expansion of $[P_+]$

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Poisson Expansion

$$[\overset{(2)}{P_+}] = \frac{1}{4}(\mathtt{C} - 2\mathtt{B}) t_{ab} t^{ab}\,,$$

$$[\overset{(0)}{P_+}] = \frac{1}{2}(3\mathtt{A} - D\mathtt{A} + \mathtt{C})\tilde{\mathrm{R}}_a{}^a + (2\mathtt{A} - D\mathtt{A} + \mathtt{B})\tilde{\nabla}(t)_{a0}{}^a + \\ + \frac{1}{4}(-5\mathtt{A} - \mathtt{C} + 3D\mathtt{A}) t_{0a} t_0{}^a + \frac{1}{2}(-3\mathtt{A} - 2\mathtt{B} + D\mathtt{A}) t^{ab} f_{ab}\,,$$

Einstein-Hilbert-Maxwell and the Poisson Equation

Where is the Poisson Equation?

We can consider a general scalar combination

$$[P_+] := \mathtt{A}\, E^{\mu}{}_{a} E^{\nu a} [G]_{\mu\nu} + \mathtt{C} E^{\mu}{}_{0} E^{\nu}{}_{0}\, [G]_{\mu\nu} + \mathtt{B}\, E_{\mu}{}^{0} [A]^{\mu}$$

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The Issue!

The order c^{-2} is sub-sub-leading: it cannot be obtained from the limit of the corresponding equations unless...

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...the leading (c^2) and sub-leading (c^0) **cannot** be canceled by choosing the proper combination...

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The Solution

We can add, by hand, another equation (a constraint)

$$F_{\mu\nu} = 0\,.$$

[**Bergshoeff:2015uaa**]

Einstein-Hilbert-Maxwell and the Poisson Equation

Proper Combinations

$$[G]_{\{ab\}}\,, \qquad [G]_{0a}\,,$$
$$[P_\pm] := \mathtt{A}\,E^{\mu}{}_{a}E^{\nu a}[G]_{\mu\nu} \pm \mathtt{A}(D-3)E^{\mu}{}_{0}E^{\nu}{}_{0}\,[G]_{\mu\nu}\,.$$

$$+$$

Constraint

$$F_{\mu\nu} = 0$$

The Effect of the Constraint

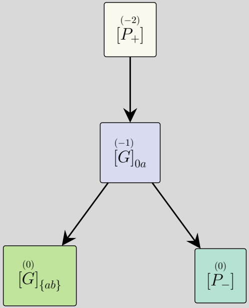
$$F_{\mu\nu} = c\,t_{\mu\nu} - \frac{1}{c} f_{\mu\nu} = 0 \longrightarrow t_{\mu\nu} = \frac{1}{c^2} f_{\mu\nu}\,.$$

The constraints allows to shift some c power to a lowest order. Its limit implies **zero torsion** $t_{\mu\nu} = 0$.

The Poisson Equation

$$\overset{(-2)}{[P]} = (D-2)\mathtt{A}\tilde{\mathrm{R}}_{00} = (D-2)\mathtt{A}\mathrm{R}(\mathrm{G})_{0a}{}^{a}\,,$$

Boost Irreb (Recucible-Indecomposable)



Towards Higher Derivative Non-Relativistic Equations of Motion

Remarks

- We should just add the higher-derivative contributions to the equations of motion.
- The combination for the Poisson equations is fully fixed by the 2-derivative terms.

What could go wrong?

The constraint could be not enough to kill the terms at order leading and subleading produced by the higher-order terms of the action.

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Does it really happen?

Yes

The Poisson of Ricci Scalar Squared

$$\overset{(2)}{[P]} = \frac{1}{4}(\mathtt{C} - 2\mathtt{B})t_{ab}t^{ab}\left[1 + 4\tilde{\nabla}(t)_{c0}{}^{c} + 2\tilde{\mathrm{R}}_{c}{}^{c} - 3t_{0c}t_{0}{}^{c} - 2f^{cd}t_{cd}\right],$$

$$\overset{(0)}{[P]} = \frac{1}{2}(3\mathtt{A} - D\mathtt{A} + \mathtt{C})\tilde{\mathrm{R}}_{a}{}^{a} + \frac{1}{2}(5\mathtt{A} - D\mathtt{A} + \mathtt{C})\alpha(\tilde{\mathrm{R}}_{a}{}^{a})^{2} + \\ + 2(D\mathtt{A} - 2\mathtt{A} - \mathtt{C})h^{\mu\nu}\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}\tilde{\mathrm{R}}_{a}{}^{a} + \ldots,$$

The Issue

The constraint $F_{\mu\nu}$ is not enough to remove all the terms at order c^{0}.

$\Rightarrow$

Constraints

$$F_{\mu\nu} = 0,$$
$$\langle C\rangle := \mathrm{R}^{2} + 2\nabla_{\mu}\nabla^{\mu}\mathrm{R} = 0.$$

Limit of the Constraints

The two constraints after the limit become:

$$t_{\mu\nu} = \mathrm{R(H)}_{\mu\nu} = 0\,,$$
$$\overset{(0)}{\langle C\rangle} := \mathrm{Ric(J)}^2 - 2h^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\mathrm{Ric(J)} = 0\,,$$

Full Set of EOMs

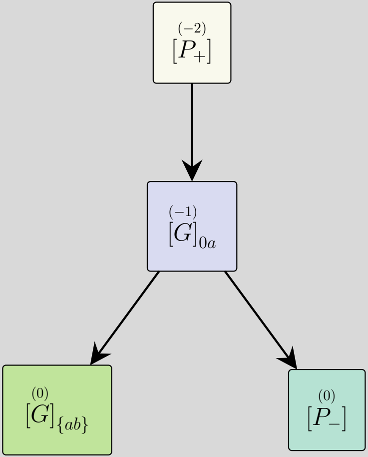
The full set of non-relativistic equations is:

$$\overset{(-2)}{[P_+]} := \mathrm{R(G)}_{0a}{}^{a}(1-2\mathrm{Ric(J)}) + 2\tau^\mu\tau^\nu\tilde{\nabla}_\mu\partial_\nu\mathrm{Ric(J)} = 0\,,$$
$$\overset{(0)}{[P_-]} := \mathrm{Ric(J)}(1-3\mathrm{Ric(J)}) = 0\,,$$
$$\overset{(-1)}{[G]}_{0a} := -\,\mathrm{Ric(J)}_{0a} + 2\mathrm{Ric(J)}_{0a}\mathrm{Ric(J)} + 2\tau^{(\mu}e^{\nu)}{}_a\tilde{\nabla}_\mu\partial_\nu\mathrm{Ric(J)} = 0\,,$$
$$\overset{(0)}{[G]}_{\{ab\}} := -\,\mathrm{Ric(J)}_{\{ab\}}(1-2\mathrm{Ric(J)}) + 2e^\mu{}_{\{a}e^\nu{}_{b\}}\tilde{\nabla}_\mu\partial_\nu\mathrm{Ric(J)} = 0\,,$$

Equations of Motion & Constraints Structure

Boost Representation (Recucible-Indecomposable)

$$
\begin{aligned}
\delta_G \mathrm{R}(\mathrm{H})_{\mu\nu} &= 0\,, \\
\delta_G \overset{(0)}{\langle C \rangle} &= 0\,, \\
\delta_G \overset{(-2)}{[P_+]} &= -\,2\lambda^a \overset{(-1)}{[G]}_{0a}\,, \\
\delta_G \overset{(0)}{[P_-]} &= 0\,, \\
\delta_G \overset{(-1)}{[G]}_{0a} &= -\,\lambda^b \overset{(0)}{[G]}_{\{ab\}} + \frac{\lambda_a}{D-1} \overset{(0)}{[P_-]}\,, \\
\delta_G \overset{(0)}{[G]}_{\{ab\}} &= 0\,.
\end{aligned}
$$



Action & EOMs

$$\mathcal{S}_{GB} = \int d^D x \sqrt{-g} \Big[\mathcal{L}_{EHM} + \alpha(\mathcal{L}_\alpha - 4\mathcal{L}_\beta + \mathcal{L}_\gamma) \Big] = 0\,,$$

$$\mathcal{L}_\alpha = \left(\mathrm{R} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)^2 ,$$

$$\mathcal{L}_\beta = \mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} + \frac{1}{16} F^4 + \frac{1}{4} F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu - \frac{1}{2} \nabla_\mu F^{\mu\nu} \nabla_\rho F_\nu{}^\rho - F^{\mu\rho} F^\nu{}_\rho \mathrm{R}_{\mu\nu}\,,$$

$$\mathcal{L}_\gamma = \mathrm{R}_{\mu\nu\rho\sigma}\mathrm{R}^{\mu\nu\rho\sigma} + \frac{3}{8} F^4 + \frac{5}{8} F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\sigma F^\sigma{}_\mu + \nabla_\mu F_{\nu\rho} \nabla^\mu F^{\nu\rho} - \frac{3}{2} F^{\mu\nu} F^{\rho\sigma} \mathrm{R}_{\mu\nu\rho\sigma}\,.$$

ACTION & EOMS

$$
\begin{aligned}
[A]^\nu :=&\nabla_\mu\left[F^{\mu\nu} - \frac{3}{2}\alpha F^2 F^{\mu\nu} + 3\alpha F^{\mu\alpha}F_{\alpha\beta}F^{\nu\beta}\right] + \\
&+ 8\alpha \mathrm{R}^{\rho[\mu}\nabla_\mu F^{\nu]}{}_\rho + \alpha \mathrm{R}^{\mu\nu\rho\sigma}\nabla_\mu F_{\rho\sigma} + 2\alpha\nabla_\mu F^{\mu\nu}\mathrm{R} = 0\,,\\
[G]_{\alpha\beta} :=& \mathrm{R}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\mathrm{R} - \frac{1}{2}F_{\alpha\mu}F_\beta{}^\mu + \frac{1}{8}g_{\alpha\beta}F^2 + \\
&+ \alpha\Big[-\frac{3}{2}F_\alpha{}^\mu F_\mu{}^\nu F_\nu{}^\rho F_{\rho\beta} + \frac{3}{4}F_\alpha{}^\mu F_{\beta\mu}F^2 + \frac{3}{16}g_{\alpha\beta}F_\sigma{}^\mu F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\sigma - \frac{3}{32}g_{\alpha\beta}F^4 + \\
&+ \frac{3}{4}g_{\alpha\beta}F^{\mu\nu}F^{\rho\sigma}\mathrm{R}_{\mu\nu\rho\sigma} + 2F_\mu{}^\nu F^{\mu\rho}\mathrm{R}_{\alpha\rho\beta\nu} - \frac{3}{2}F^{\nu\rho}(F_\beta{}^\mu \mathrm{R}_{\alpha\mu\nu\rho} + F_\alpha{}^\mu \mathrm{R}_{\beta\mu\nu\rho}) + \\
&- \frac{1}{2}\mathrm{R}_{\alpha\beta}F^2 - 2F_\beta{}^\mu F_\mu{}^\nu \mathrm{R}_{\alpha\nu} - 2F_\alpha{}^\mu F_\mu{}^\nu \mathrm{R}_{\beta\nu} + 3F_\alpha{}^\mu F_\beta{}^\nu \mathrm{R}_{\mu\nu} - 2g_{\alpha\beta}F_\mu{}^\nu F^{\mu\rho}\mathrm{R}_{\nu\rho} + \\
&- F_{\alpha\mu}F_\beta{}^\mu \mathrm{R} + \frac{1}{4}g_{\alpha\beta}F^2\mathrm{R} + \frac{1}{2}\nabla_\alpha F_{\mu\nu}\nabla_\beta F^{\mu\nu} - \nabla_\mu F_\alpha{}^\mu \nabla_\nu F_\beta{}^\nu + \nabla_\alpha F_{\beta\mu}\nabla_\nu F^{\mu\nu} + \\
&+ \nabla_\beta F_{\alpha\mu}\nabla_\nu F^{\mu\nu} + \nabla_\mu F_{\alpha\nu}\nabla^\mu F_\beta{}^\nu - \frac{1}{2}g_{\alpha\beta}\nabla_\mu F_{\nu\rho}\nabla^\mu F^{\nu\rho} - g_{\alpha\beta}\nabla_\mu F^{\mu\nu}\nabla^\rho F_{\nu\rho} + \\
&- 4\mathrm{R}_\alpha{}^\mu \mathrm{R}_{\beta\mu} + 2g_{\alpha\beta}\mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu} + 2\mathrm{R}_{\alpha\beta}\mathrm{R} - \frac{1}{2}g_{\alpha\beta}\mathrm{R}^2 + \\
&- 4\mathrm{R}^{\mu\nu}\mathrm{R}_{\alpha\mu\beta\nu} + 2\mathrm{R}_\alpha{}^{\mu\nu\rho}\mathrm{R}_{\beta\mu\nu\rho} - \frac{1}{2}g_{\alpha\beta}\mathrm{R}_{\mu\nu\rho\sigma}\mathrm{R}^{\mu\nu\rho\sigma}\Big] = 0\,,
\end{aligned}
$$

POISSON

$$[P] = c^{2}\overset{(2)}{[P]} + \overset{(0)}{[P]} + c^{-2}\overset{(-2)}{[P]} + \mathcal{O}(c^{-2}) \,,$$

with

$$\begin{aligned}\overset{(2)}{[P]} = (\mathtt{C} - 2\mathtt{B})\alpha\Bigg[&\frac{1}{4}t_{ab}t^{ab}\left(\alpha^{-1} + 2\tilde{\mathrm{R}}_{c}{}^{c} + \frac{1}{3}t_{0}{}^{c}t_{0c} - 2t^{cd}f_{cd}\right) + \\ &+ t^{ab}\left(-t_{0a}\tilde{\nabla}(t)_{cb}{}^{c} + \frac{2}{3}t_{0}{}^{c}\tilde{\nabla}(t)_{(ac)b} - t_{a}{}^{c}\tilde{\mathrm{R}}_{bc} - \frac{1}{6}t_{a}{}^{c}t_{0b}t_{0c} + t_{a}{}^{c}t_{b}{}^{d}f_{cd}\right) + \\ &+ \tilde{\nabla}(t)_{a}{}^{ab}\tilde{\nabla}(t)_{cb}{}^{c} + \frac{2}{3}\tilde{\nabla}(t)^{abc}\tilde{\nabla}(t)_{(ab)c}\Bigg] \,,\end{aligned}$$

$$\overset{(0)}{[P]} = \frac{1}{2}(3\mathtt{A} + \mathtt{C} - D\mathtt{A})\tilde{\mathrm{R}}_{a}{}^{a} + \frac{1}{2}(5\mathtt{A} + \mathtt{C} - D\mathtt{A})\alpha\left[(\tilde{\mathrm{R}}_{a}{}^{a})^{2} - 4\tilde{\mathrm{R}}^{ab}\tilde{\mathrm{R}}_{ab} + \tilde{\mathrm{R}}^{abcd}\tilde{\mathrm{R}}_{abcd}\right] + \ldots \,,$$

Scalar Field Trick

Using a Scalar Field

$$\mathcal{L}_{\text{new}} = \mathcal{L}_1 + \mathrm{e}^{\mathtt{a}\Phi}\left[\mathcal{L}_2 + \mathtt{b}\partial_\mu\Phi\partial^\mu\Phi\right]$$

with a and b two constants. Then the field equations become:

$$[\Phi]^{\text{new}} := \mathtt{a}\mathcal{L}_2 + \partial_\mu\Phi(\ldots)^\mu\,,$$

$$[G]^{\text{new}}_{\mu\nu} := [G]^{(1)}_{\mu\nu} + \mathrm{e}^{\mathtt{a}\Phi}[G]^{(2)}_{\mu\nu} + \partial_\mu\Phi(\ldots)^\mu\,,$$

$$[A]^{\text{new}}_{\mu} := [A]^{(1)}_{\mu} + \mathrm{e}^{\mathtt{a}\Phi}[A]^{(2)}_{\mu} + \partial_\mu\Phi(\ldots)^\mu\,,$$

where the superscripts $^{(1)}$ and $^{(2)}$ denote respectively the contributions to the field equation coming from $\mathcal{L}_1$ and $\mathcal{L}_2$ and dots denote irrelevant contribution for our treatment.

Scalar Field Ansatz & Constraint

Now consider the trivial ansatz for the scalar field $\Phi = \phi$ and imposing the constraint $\partial_\mu\Phi = 0$, implies that EOMs become:

$$[\phi]^{\text{new}} = \mathtt{a}\mathcal{L}_2\,,$$

$$[G]^{\text{new}}_{\mu\nu} = [G]^{(1)}_{\mu\nu} + \mathrm{e}^{\mathtt{a}\phi}[G]^{(2)}_{\mu\nu}\,,$$

$$[A]^{\text{new}}_{\mu} = [A]^{(1)}_{\mu} + \mathrm{e}^{\mathtt{a}\Phi}[A]^{(2)}_{\mu}\,.$$

Poisson

Considering the combination

$$[P] := \mathtt{A}\, E^{\mu}{}_{a} E^{\nu a} [G]_{\mu\nu} + \mathtt{C} E^{\mu}{}_{0} E^{\nu}{}_{0}\, [G]_{\mu\nu} - \mathtt{B}\, E_{\mu 0} [A]^{\mu} + \mathtt{D} [\phi]\,,$$

the expansion is unmodified at order c^2, while, at order c^0, we get

$$\overset{(0)}{[P]} = \frac{1}{2}(3\mathtt{A} + \mathtt{C} - D\mathtt{A} + 2\mathtt{D}\mathtt{a})\tilde{\mathrm{R}}_{a}{}^{a} + \\ + \frac{1}{2}(5\mathtt{A} + \mathtt{C} - D\mathtt{A})\Big[(\tilde{\mathrm{R}}_{a}{}^{a})^2 - 4\tilde{\mathrm{R}}^{ab}\tilde{\mathrm{R}}_{ab} + \tilde{\mathrm{R}}^{abcd}\tilde{\mathrm{R}}_{abcd}\Big]\,,$$

where we have used the constraints $F_{\mu\nu} = 0$ to shift the terms proportional to the intrinsic torsion to a lower order in c. We have set, for simplicity, without losing generality, $\phi = 0$ (a different constant value amounts to a relative constant between the two- and four-derivative terms that can be reabsorbed in a redefinition). This implies that the cancellation can be realized by taking $\mathtt{C}$ and $\mathtt{D}$ to be

$$C = (D - 5)\mathtt{A}\,, \qquad \mathtt{D}\mathtt{a} = \mathtt{A}\,.$$

Limit of the Constraints

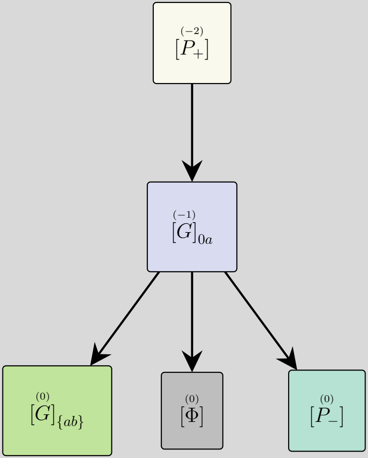
$$t_{\mu\nu} = \mathrm{R(H)}_{\mu\nu} = 0\,, \qquad \phi = 0\,,$$

Equations of Motion

$$
\begin{aligned}
[\phi] &:= \mathrm{Ric(J)} = 0\,,\\
\overset{(-2)}{[P_+]} &:= \mathrm{R(G)}_{0a}{}^{a} + 4\mathrm{R(G)}_{ab}{}^{a}\mathrm{R(G)}^{bc}{}_{c} + \frac{1}{2}\mathrm{R(G)}^{abc}\mathrm{R(G)}_{abc} - 2\mathrm{R(G)}_{0a}{}^{a}\mathrm{Ric(J)} +\\
&\quad + 4\mathrm{R(G)}_{0}{}^{ab}\mathrm{Ric(J)}_{ab} + \frac{1}{2}\mathrm{R(J)}^{0abc}(\mathrm{R(J)}_{0abc} - 2\mathrm{R(G)}_{bca}) = 0\,,\\
\overset{(0)}{[P_-]} &:= \mathrm{R(J)}^{abcd}\mathrm{R(J)}_{abcd} - 4\mathrm{Ric(J)}^{ab}\mathrm{Ric(J)}_{ab} = 0\,,\\
\overset{(-1)}{[G]}_{0a} &:= \mathrm{R(G)}_{ab}{}^{b} - 2\mathrm{R(J)}^{bc}{}_{a}{}^{d}\mathrm{R(G)}_{bcd} + 4\mathrm{Ric(J)}_{ab}\mathrm{R(G)}^{bc}{}_{c} - 2\mathrm{Ric(J)}\mathrm{R(G)}_{ac}{}^{c} +\\
&\quad + 4\mathrm{Ric(J)}_{ab}\mathrm{R(G)}^{bc}{}_{c} + 4\mathrm{Ric(J)}^{bc}\mathrm{R(G)}_{abc} = 0\,,\\
\overset{(0)}{[G]}_{\{ab\}} &:= -\mathrm{Ric(J)}_{ab} + 2\mathrm{R(J)}_{a}{}^{cde}\mathrm{R(J)}_{bcde} - 4\mathrm{Ric(J)}_{a}{}^{c}\mathrm{Ric(J)}_{bc} +\\
&\quad - 4\mathrm{Ric(J)}^{cd}\mathrm{R(J)}_{acbd} = 0\,,
\end{aligned}
$$

Boost Representation (Recucible-Indecomposable)

$$
\begin{aligned}
\delta_G \phi &= 0\,, \\
\delta_G \mathrm{R(H)}_{\mu\nu} &= 0\,, \\
\delta_G [\phi] &= 0\,, \\
\delta_G \overset{(-2)}{[P_+]} &= -\,2\lambda^a \overset{(-1)}{[G]}_{0a}\,, \\
\delta_G \overset{(0)}{[P_-]} &= 0\,, \\
\delta_G \overset{(-1)}{[G]}_{0a} &= -\,\lambda^b \overset{(0)}{[G]}_{\{ab\}} - 2\lambda_a \overset{(0)}{[P_-]} + \\
&\quad + \lambda_a [\phi](1 - 2[\phi])\,, \\
\delta_G \overset{(0)}{[G]}_{\{ab\}} &= 0\,,
\end{aligned}
$$



Conclusion & Outlooks

- Taking the non-relativistic limit of higher-deriviative quadratic theory at the level of the action requires coupiling the gravity sector to a gauge field,
- At the level of the EOMs requires introducing further constraints on the system,

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- Change the foliation.
- Study solutions.
- Include further cases (cubic gravity and ...).
- Can the non-relativistic limit be used to acquire info on higher-order gravity theories?

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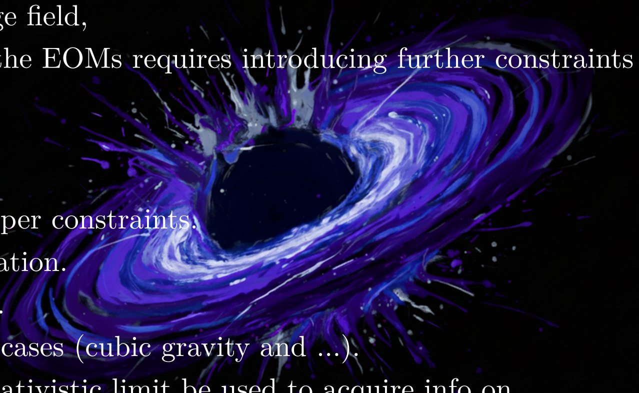

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Thank You!