UNDERSTANDING NON INVERTIBLE SYMMETRIES

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MOTIVATION

- Global symmetries are (supposed to be) broken in QG
  Banks-Dixon'88, ...
- Ganging global symmetries
  - → SUGRA → gauged SUGRA (flux compactification, ...)
  - → non-invertible symmetries
    - → gauging a
      - discrete subgroup (Cordova, Shao, ...)
      - continuous subgroup (Gª Etxebarría-Iqbal)

MOTIVATION

- Questions:
  - are (gauged) SUGRAs related to this phenomenon?
  - underlying anomaly-free criterion for gauge groups?
  - Hint: additional topological terms emerge when gaugings are turned on (even in even dimensions!) (e.g.: $\mathcal{N}=8$ $D=4$)

* A long term goal? : anomaly-free criterion based on < topological terms? / residual global symmetries?

MOTIVATION

- Global symmetry (0-form) $\longrightarrow$ $d(\underbrace{* j_A}_{(d-1)\text{-form}}) = 0$ $\Longrightarrow$ $U(\Sigma_{(d-1)}) = e^{i\alpha \int_\Sigma * j_A}$

topological

- p-form global symmetry $\longrightarrow$ $d(* j_A) = 0$ $\Longrightarrow$ $U(\Sigma_{(d-p-1)}) = e^{i\alpha \int_\Sigma * j_A}$

(d-p-1)-form

- Examples:

(a) $\phi \to e^{i\alpha}\phi$ $U(1)$

$* j = \phi^\dagger * d\phi + c.c.$

(d-1)-form

(b) Maxwell in 4D:

$a\int F \wedge * F + b \int F \wedge F$ $\quad U(1)_e \times U(1)_m$

$* j = * F \qquad * j = F \qquad$ (2-form)

(c) Maxwell in 5D:

$\frac{a}{2}\int F \wedge * F + b \int A \wedge F \wedge F$

$* j = a * F + \frac{b}{3} A \wedge F$ (3-form)

MOTIVATION

- Usual (invertible) symmetries:

$$U_{g_1}(\Sigma) \times U_{g_2}(\Sigma) = U_{g_1 \times g_2}(\Sigma)$$

- Non invertible symmetries (fusion rules)

$$U_{g_1}(\Sigma) \times U_{g_2}(\Sigma) = \sum_k N_{12}{}^k \, U_{g_k}(\Sigma) \qquad \text{(OPE - like)}$$

- How to construct these operators?

$\exp i\alpha \int \star j$ is not enough!! (gauge invariance, ...)

MOTIVATION

- One generic case: for a $U(1)$ p-form current J_p satisfying

$$\underline{d * J_p = G_{d-p+1}}; \qquad G_{d-p+1} = d\,\mathbb{K}_{d-p} \ \text{(locally)}$$

$$\hookrightarrow * \tilde{J}_p = * J_p - \mathbb{K}_{d-p} \qquad \text{(Page current)}$$

$$\hookrightarrow U(\Sigma_p) = \exp i\alpha \int_\Sigma * J_p - \mathbb{K}_{d-p}$$

Example: $\int F \wedge * F + \int F \wedge F \wedge A \longrightarrow (* \tilde{J})_3 = * F + A \wedge F$

<u>not</u> gauge invariant

MOTIVATION

- $U(\Sigma_3) = \exp i\alpha \int \star F + A \wedge F$

  $\hookrightarrow$ for $\alpha = \frac{1}{N}$: $D_{1/N}(\Sigma_3) = \int \mathcal{D}a \big|_{\Sigma_3} \exp i \int \frac{\star F}{N} + \frac{N}{2} a \wedge da - a \wedge F$

  $N \in \mathbb{Z}$

  gauge invariant

- Integrating out $\boxed{a = \frac{A}{N}}$ we recover $U(\Sigma_3)$

MOTIVATION

- Interestingly, the operator $D_{1/N}$ is not invertible:

$$D_{1/N}(\Sigma)\times D^{+}_{1/N}(\Sigma) = \int Da \int D\bar{a}\ \exp \int_\Sigma i\frac{N}{4\pi}\left(a\,da - \bar{a}\,d\bar{a}\right) + \frac{i}{2\pi}(a-\bar{a})dA$$

$$\neq \mathbb{1}$$

(condensation operator)

- For general $\alpha = \frac{p}{N}$; $p, N \in \mathbb{Z} \longrightarrow$ coupling to a TQFT $\mathcal{A}^{N,p}$

- Other realizations? Codimension-1 objects (defects $\equiv$ domain walls)

MOTIVATION

- Codimension-1 object

$$S = \int_{S_L} \mathcal{L}(\Phi_L) + \int_S \mathcal{L}(\Phi_L, \Phi_R, b) + \int_{S_R} \mathcal{L}(\Phi_R)$$

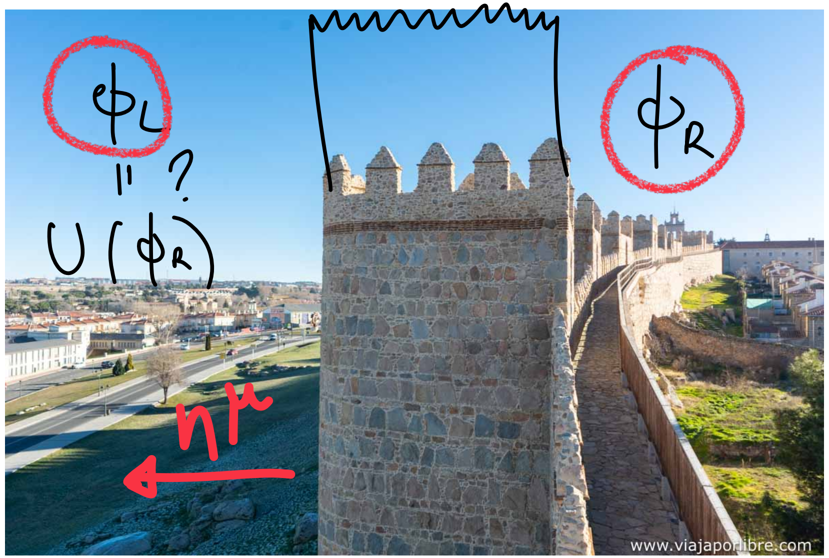
- Energy momentum conservation:

$$n^\mu \left(T_{\mu\nu}(\Phi_L) - T_{\mu\nu}(\Phi_R)\right)\Big|_S = 0$$

- For a symmetry of the action $\Phi \to U(\Phi)$

$$S = \int_{S_L \cup S_R} \mathcal{L}(\Phi) + \int_S b \wedge \left(\Phi_L - U(\Phi_R)\right)$$

$b \to$ Lagrange multiplier



OUTLINE

1. Motivation

2. The (1+1)D compactified boson (and 4D Maxwell)

3. Some attempts in superstring effective actions

4. Conclusions and prospects

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(1+1)D COMPACT BOSON

$$\mathcal{L}(R;\phi) = \frac{R^2}{4\pi} d\phi \wedge * d\phi$$

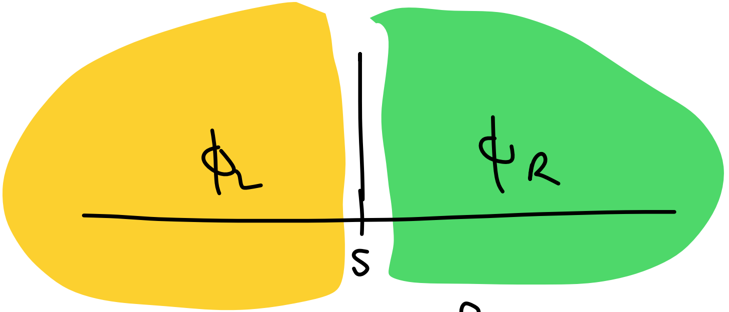
Symmetries
$$\begin{cases} \mathbb{Z}_2 : & \phi \longrightarrow -\phi \\ U(1): & \phi \longrightarrow \phi + c, \quad c \equiv \text{constant} \\ U(1): & \tilde{\phi} \longrightarrow \tilde{\phi} + c, \quad d\tilde{\phi} = -iR^2 * d\phi \end{cases}$$

$$m = \int \frac{d\tilde{\phi}}{2\pi}$$

$$w = \int \frac{d\phi}{2\pi}$$

Duality: $d\phi \longrightarrow d\tilde{\phi} = -iR^2 * d\phi$

$R \longrightarrow \frac{1}{R}$



Adding a defect: $S = \int_{S_L} \mathcal{L}(R;\phi_L) + \int_{S_R} \mathcal{L}(R;\phi_R) + \int_S \mathcal{L}_S(\phi_L, \phi_R, b)$

(1+1)D COMPACT BOSON

- How to glue $\phi_L = U(\phi_R)$?

$$d\phi_L\big|_S = \alpha\, d\phi_R\big|_S + \beta\, i{*}d\phi_R\big|_S$$

- $n^\mu \left(T_{\mu\nu}^{(L)} - T_{\mu\nu}^{(R)}\right)\Big|_S = 0 \implies \begin{matrix} \alpha^2+\beta^2 = 1 \\ \alpha\beta = 0 \end{matrix} \quad \begin{cases} \alpha = \pm 1,\ \beta = 0 \\ \alpha = 0,\ \beta = \pm 1 \end{cases}$

$$d\phi_L = d\phi_R \quad (\mathbb{1}) \leftarrow$$
$$d\phi_L = -d\phi_R \quad (\mathbb{Z}_2)$$
$$d\phi_L = i{*}d\phi_R \quad (T) \leftarrow$$
$$d\phi_L = -i{*}d\phi_R \quad (\mathbb{Z}_2 \times T)$$

- Variational principle: $S = S_{bulk} + S_S$

$$\delta S_{bulk} \underset{\text{on shell}}{=} \int_S \delta\phi_L * d\phi_L - \int_S \delta\phi_R * d\phi_R$$

(1+1)D COMPACT BOSON

- $$\boxed{d\phi_L\big|_s = d\phi_R\big|_s \quad (1)}$$

$$\delta S_{bulk} = \int \left(\delta\phi_L \star d\phi_L - \delta\phi_R \star d\phi_R \right)$$

$$S_s = \frac{iK}{2\pi}\int b\,(d\phi_L - d\phi_R) \longrightarrow \delta S_s = \frac{iK}{2\pi}\int \delta b\,(d\phi_L - d\phi_R) - \delta\phi_L\, db + \delta\phi_R\, db$$

$$\delta S_{bulk} + \delta S_s = \int \delta\phi_L \underbrace{\left(\frac{R^2}{4\pi}\star d\phi_L - \frac{iK}{2\pi}db\right)} - \delta\phi_R \underbrace{\left(\frac{R^2}{4\pi}\star d\phi_R - \frac{iK}{2\pi}db\right)} + \delta b \underbrace{(d\phi_L - d\phi_R)}$$

$$\| \atop 0$$

$$\hookrightarrow \begin{cases} d\phi_L = d\phi_R \\ db = -\frac{i}{2}\frac{R^2}{K}\star d\phi_L = -\frac{i}{2}\frac{R^2}{K}\star d\phi_R \end{cases} \Longrightarrow S_s = \frac{i}{2\pi}\int_s b\,(d\phi_L - d\phi_R)$$

$$\boxed{= \frac{1}{4\pi}\frac{R^2}{K}\int \star d\phi_L\, \phi_R}$$

(1+1)D COMPACT BOSON

$$\boxed{d\phi_L\big|_S = i * d\phi_R\big|_S \qquad (T)}$$

$$S_S = \int b\,(d\phi_L - i * d\phi_R)$$

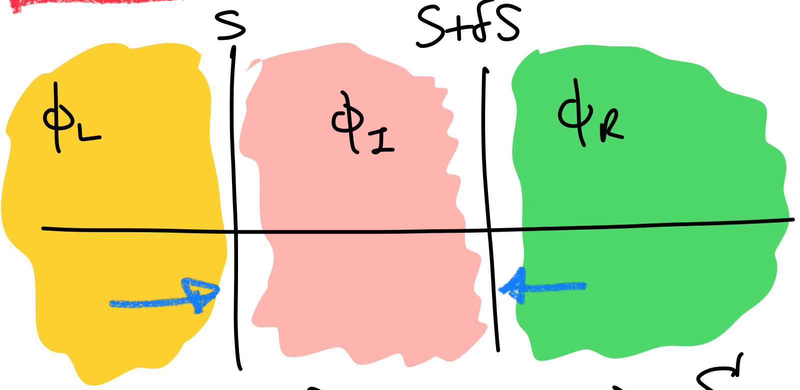
$$\delta S = \int \delta b\,\underbrace{(d\phi_L - i * d\phi_R)} + \delta\phi_L\,\underbrace{(* d\phi_L - db)} + \delta\phi_R\,\underbrace{(- * d\phi_R + i * db)}$$

$$\Rightarrow \begin{cases} d\phi_L = i * d\phi_R \\ db = * d\phi_L = i\, d\phi_R \Rightarrow b = i\phi_R \end{cases}$$

$$\Rightarrow \quad S_S = i \int \phi_R\,(d\phi_L - \cancel{i * d\phi_R}) \Longrightarrow \boxed{S_S = i \int_S \phi_R\, d\phi_L}$$

(1+1)D COMPACT BOSON

- What is $T \times T$? Two T-type defects at S and $S + \delta S$



$$S = \int_{S_L \cup S_I \cup S_R} \mathcal{L}(R;\phi) + \int_S \phi_L \, d\phi_I + \int_{S+\delta S} \phi_R \, d\phi_I$$

$\delta S \to 0 \Longrightarrow$

$$S = \int_{S_L \cup S_R} \mathcal{L}(R;\phi) + \frac{iR^2}{2\pi} \int_S d\phi_I \, (\phi_R - \phi_L) \quad ; \quad R^2 = N$$

This can be rewritten as:

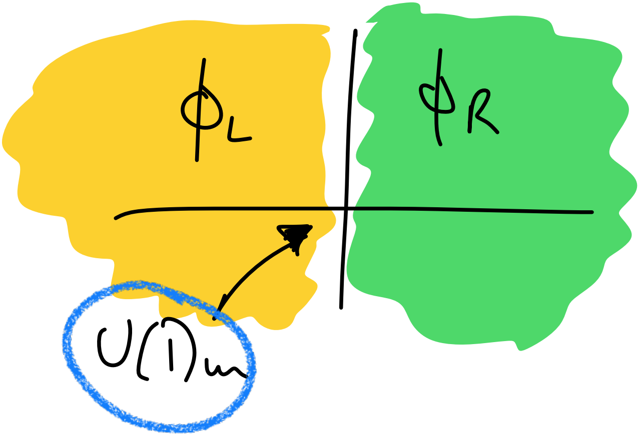
$$S = \frac{N}{4\pi} \int_{S_L} d\phi_L \wedge * d\phi_L + \frac{N}{4\pi} \int_{S_R} d\phi_R \wedge * d\phi_R + \frac{i}{2\pi} \int_S d\varphi \left(\phi_L - \phi_R + \frac{2\pi}{N} \eta \right)$$

$\varphi \sim \varphi + 2\pi$ (compactified boson)

$\eta = 0, \dots, N-1$

(1+1)D COMPACT BOSON

- On the other hand, if we consider the insertion of a $U(1)_m$ defect:



$$S = \frac{N}{4\pi}\int_{S_L \cup S_R} d\phi \wedge * d\phi + \beta \frac{N}{2\pi}\int_S * d\phi$$

$\downarrow$ (Lagrange multiplier φ)

$$S = \frac{N}{4\pi}\int_{S_L \cup S_R} d\phi \wedge * d\phi + \beta \frac{N}{2\pi}\int_S * d\phi + \frac{i}{2\pi}\int_S d\varphi(\phi_L - \phi_R)$$

$\downarrow$ $\phi_L \to \phi_L - \beta\,\Theta(S_L)$

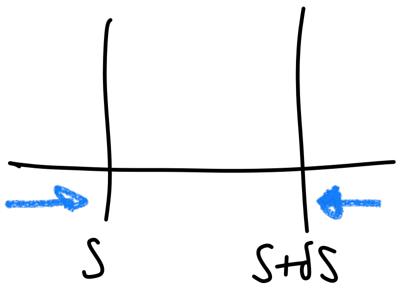
$$S = \frac{N}{4\pi}\int_{S_L \cup S_R} d\phi \wedge * d\phi + \frac{i}{2\pi}\int_S d\varphi(\phi_L - \phi_R - \beta)$$

SIMILAR TO THE 2-DEFECT ACTION $\left(\beta = -\frac{2\pi\eta}{N}\right)$

$$\boxed{T \times T = \sum_{\eta=0}^{N-1} e^{-\frac{2\pi}{N}\frac{N}{2\pi}\eta \oint * d\phi}}$$

(1+1)D COMPACT BOSON

- Summarizing



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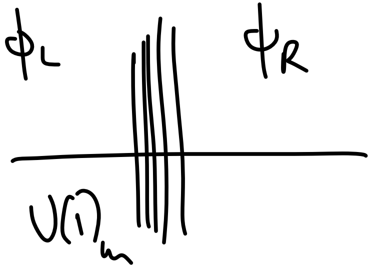

$$T \times T = \sum_{\eta=0}^{N-1} e^{\oint d\phi_L}$$

Fuchs-Gaberdiel-Runkel-Schweigert '07

$$\begin{cases} N=1 \to \text{invertible} \\ N \neq 1 \Rightarrow \text{sum of } N \ U(1)_m \text{ defects} \end{cases}$$

(non invertible)

* (1) Gauging the $\mathbb{Z}_N^m$ subgroup of $U(1)^m \Rightarrow R \longrightarrow R/N$

$\mathbb{Z}_N^w$ $\quad U(1)^w \Rightarrow R \longrightarrow RN$

* (2) $R=\sqrt{N} \xrightarrow{\text{gauge } \mathbb{Z}_2} R=\frac{1}{\sqrt{N}} \xrightarrow{\text{T-duality}} R=\sqrt{N}$ $\begin{cases} N=1 \to \text{invertible} \\ N\neq 1 \to \text{non invertible} \end{cases}$

4D MAXWELL & DEFECTS

- $\mathcal{L} = \frac{1}{4\pi e^2} F \wedge \star F + i \frac{\theta}{8\pi^2} F \wedge F$; $F = dA$

- Symmetries $\begin{cases} \mathbb{Z}_2 \text{ charge conjugation} \\ U(1)_e \times U(1)_m \end{cases}$ $\quad n = \int \frac{\tilde{F}}{2\pi}$; $m = \int \frac{F}{2\pi}$

$$\tilde{F} = d\tilde{A} = -\frac{i}{e^2} \star F + \frac{\theta}{2\pi} F$$

- S-duality: $\tau \to \frac{1}{\tau}$; $\tau = \frac{\theta}{2\pi} + \frac{i}{e^2}$

$\hookrightarrow$ $\tau = i$ (self-dual point): $F \longrightarrow \star F = \tilde{F}$

- Defect: $S = \int_{S_L} \mathcal{L}(e^2, \theta; A_L) + \int_{S_R} \mathcal{L}(e^2, \theta; A_R) + \int \mathcal{L}_S(A_L, A_R, b)$

4D MAXWELL & DEFECTS

- How to glue the fields? $A_L = U(A_R)$?

$$F_L|_S = \alpha F_R|_S + i\beta * F_R|_S$$

- $(T_{\mu\nu}^{(L)} - T_{\mu\nu}^{(R)})n^\mu = 0 \implies \begin{pmatrix} F_L \\ i*F_L \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} F_R \\ i*F_R \end{pmatrix}$

(a) $\varphi = \pi \longrightarrow \mathbb{Z}_2$ symmetry $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(b) $\varphi = \pi/2 \longrightarrow S$ - symmetry $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(c) $\cos\varphi = -\frac{\theta}{2\pi}$; $\sin\varphi = \frac{1}{e^2}$? (rational θ/π, e) $\longrightarrow$ non invertible!!

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SUPERGRAVITY

García-Valdecasas'23

- 11D SUGRA

$\boxed{U(1)_e^{(3)} \times U(1)_m^{(6)}}$

$$S = \frac{1}{2\kappa_{11}^2}\int \sqrt{-g}\,R - \frac{1}{2} F_4 \wedge * F_4 - \frac{1}{6} A_3 \wedge F_4 \wedge F_4$$

$$d\left(* F_4 - \frac{1}{2} A_3 \wedge F_4\right) = 0 \implies U_\alpha(\Sigma_7) = \exp i\alpha \int_{\Sigma_7} * F_4 - \frac{1}{2} \underline{A_3} \wedge F_4$$

not gauge invariant!!

$$\underset{(\alpha = 1/N)}{\Rightarrow} \quad D_{1/N} = \int D c_3\Big|_{\Sigma_7} \exp i \int_{\Sigma_7} \frac{* F_4}{N} + \frac{N}{2} c_3 \wedge d c_3 - c_3 \wedge F_4$$

$$\underset{(\alpha = p/N)}{\Rightarrow} \quad D_{p/N} = \exp i \frac{p}{N} \int * F_4 \times \underbrace{A_7^{N,p}\left[\frac{F_4}{N}\right]}_{7D\ \text{TQFT}} \implies \underline{\Gamma_{\mathbb{Q}}^{(3)} \times U(1)_m^{(6)}}$$

SUPERGRAVITY

García-Valdecasas'23

- Type IIA

  $U(1)^{(6)}_m \times U(1)^{(7)}_m$

  B_6 , C_7

$$\begin{cases} dH_3 = 0 \\ dF_2 = 0 \\ d\tilde{F}_4 = F_2 \wedge H_3 \end{cases}$$

Page currents:

$\mathbb{Z}_N^{(6)} \times \mathbb{Z}_N^{(7)}$

$d(\tilde{F}_4 - A_1 \wedge H_3) = 0$ ✓

$d(\tilde{F}_6 - A_3 \wedge H_3) = 0$?

$d(\tilde{F}_8 - A_5 \wedge H_3) = 0$?

- Type IIB

  $U(1)^{(6)}_m \times U(1)^{(8)}_m$

$$\begin{cases} dH_3 = 0 \\ dF_1 = 0 \\ \vdots \end{cases}$$

Page currents:

$d(\tilde{F}_3 - A_0 H_3) = 0$ ✓

$d(\tilde{F}_5 - A_2 \wedge H_3) = 0$?

- Problems: higher-order anomaly when gauging → constraints for $D^{(6)}_{1/N}$, $D^{(8)}_{1/N}$

- Solution? Non invertibility à la G$^{\underline{o}}$-Etxebarría-Iqbal

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CONCLUSIONS

- Non invertible symmetries
  - ↳ gauging a (discrete) subgroup of the global symmetry group
  - ↳ realization with p-form symmetries
- Several constructions
  - $\exp i\alpha \int \star J \Rightarrow D_{1/N} = \cdots \Rightarrow D_{p/N} = \int \star F \times \underbrace{\mathcal{A}^{N,p}}_{\text{TQFT}}$
  - codimension-1 construction
    - classification of all possible defects from symmetries $\Phi_L = U(\Phi_R)$
    - fusion rules → non invertible for dualities!!!

PROSPECTS

- Domain walls with dualities
  $\Phi_L = U(\Phi_R)$
  - T-duality (T-fects) ?
  - Hodge duality ?
  - S-duality ?

- Gaugings
  - non invertible mechanism?
    - ↳ global symmetry relates dual theories
  - emerging topological terms (anomaly inflow?)
    - ↳ higher codimension defects !!
  - domain wall solutions in 9D, 8D SUGRAs ?

ThnKs

REFERENCES:

- lectures
  - Shao
  - Schafer-Nameki
- Articles
  - García-Etxebarría-Iqbal '22
  - Niro-Roumpedakis-Sela '23
  - Cordova-Shao+ '21 '22

4D MAXWELL & DEFECTS

- For $\boxed{\sin\varphi = 0}$

$$\mathcal{L}_S = i\frac{K}{2\pi}\, a \wedge (dA_R - dA_L)$$

$K \in \mathbb{Z}$

a → auxiliary gauge field

- Variational principle:

$$EOM(a): \quad F_L|_S - F_R|_S = 0$$

$$EOM(A_L|_S): \quad \frac{K}{2\pi} da = -\frac{i}{2\pi e^2} \star F_R|_S + \frac{\theta}{4\pi^2} F_R|_S$$

$$EOM(A_R|_S): \quad \frac{K}{2\pi} da = -\frac{i}{2\pi e^2} \star F_L|_S + \frac{\theta}{4\pi^2} F_L|_S$$

$F_L|_S = F_R|_S$

Integration over Σ_2

$Km = n_R = n_L$

$K=1 \rightarrow m = n$

$K \neq 1 \rightarrow$ constraints on $n_{R,L}$

4D MAXWELL & DEFECTS

- For $\boxed{\sin\varphi \neq 0}$

$$\underbrace{\delta S_{bulk}}_{\substack{\text{on-shell} \\ \text{in } S_L \text{ and } S_R}} = \int_{S^1} -i\frac{N_L}{2\pi}\delta A_L \wedge dA_L - i\frac{N_R}{2\pi}\delta A_R \wedge dA_R + \frac{N}{2\pi}\left(\delta A_L \wedge dA_R + \delta A_R \wedge \delta A_L\right)$$

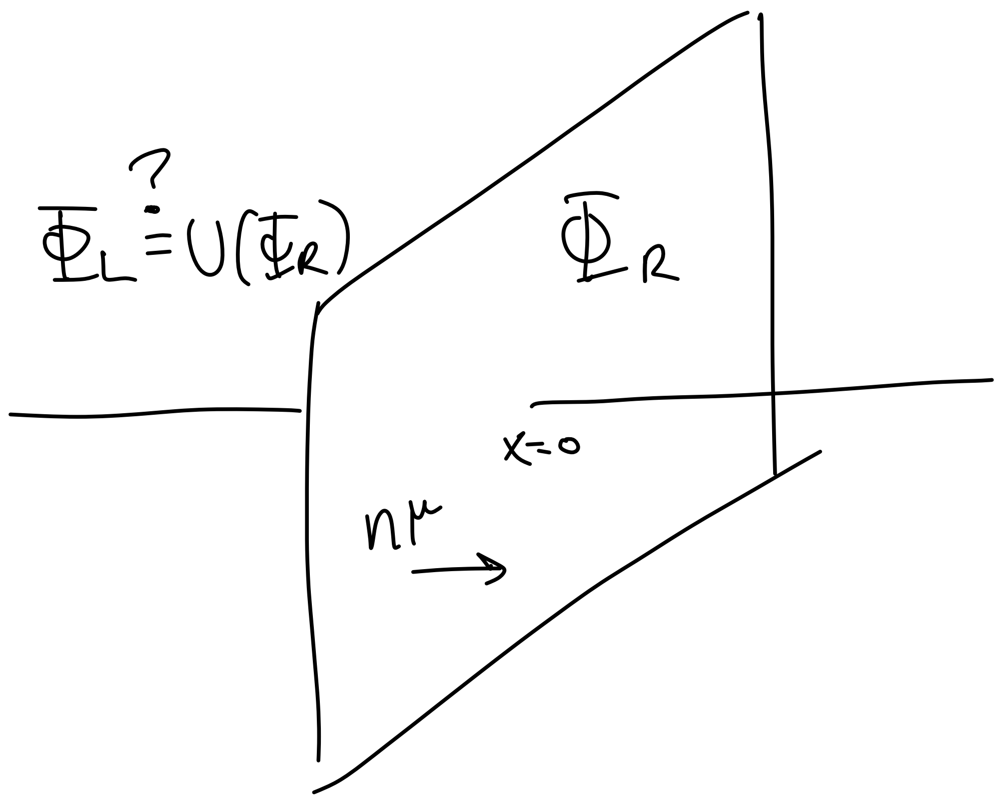
(total derivative)

$$N_{L,R} = \frac{1}{e^2\tan\varphi} \mp \frac{\theta}{2\pi} \;;\quad N = \frac{1}{e^2\sin\varphi}$$

- Variationally: $\mathcal{L}_S = \frac{i}{4\pi}\left(N_L\, A_L \wedge dA_L + N_R\, A_R \wedge dA_R - 2N\, A_L \wedge dA_R\right)$

then $\delta S = \delta\left(S_L + S_R + S_S\right) = 0$

- Gauge invariance: $N_L, N_R, N \in \mathbb{Z}$


$\Phi_L \overset{?}{=} U(\Phi_R)$
Φ_R
$x=0$
n^μ