Entropic charcaterization of Non-Invertible symmetries in 2D-CFT

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Work In Progress

in collaboration w/

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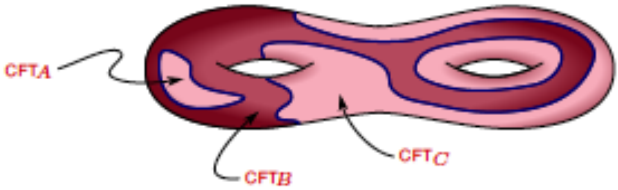

Outline

1. **Introduction**
2. Topological lines in 2D-CFT. Verlinde Lines
3. Entropic portray of fusion symmetries in 2D-CFT
4. Results

The Power of Symmetry

[Kapustin-Seiberg, Gaiotto-Kapustin-Seiberg-Willet. . .]

- Global symmetries constrain RG flows, correlation functions, and many physical observables in QFT.

- Global symmetries $\sim$ Topological (defect) operators,

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The virtue of defects in 4D gauge theories and 2D CFTs

Nadav Drukker,[a] Davide Gaiotto[b] and Jaume Gomis[c]

Generalizations of symmetry

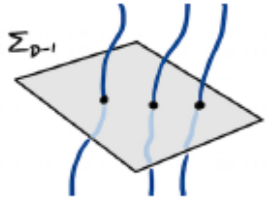
To understand this, let's review ordinary 1-index currents first.

$$\nabla_\mu j^\mu = 0 \qquad d \star j = 0$$

An ordinary current counts particles; "catch them all" by integrating on a co-dimension 1 subspace. In Euclidean:

$$U(\mathcal{M}_{d-1}) = \exp\left(i\alpha Q\right) \qquad Q = \int_{\mathcal{M}_{d-1}} \star j$$

defines a U(1)-valued topological codimension-1 surface operator (fancy way to talk about "conserved charge"; call this a 0-form symmetry)



Higher form symmetries

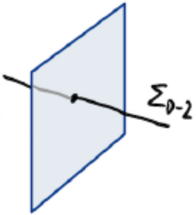
Now a 2-index current [Kapustin-Seiberg, Gaiotto-Kapustin-Seiberg-Willet...]

$$\nabla_\mu J^{\mu\nu} = 0 \qquad d \star J = 0$$

A 2-index current counts strings; as they don't end in space or time, "catch them all" by integrating on a co-dimension 2 subspace:

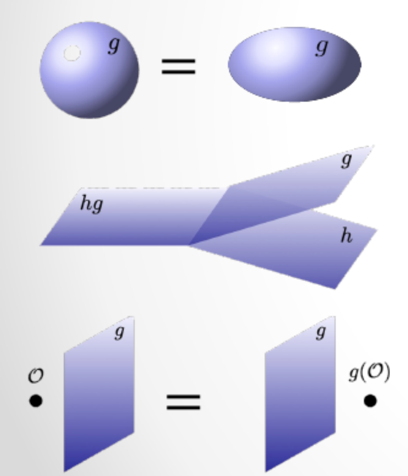
$$U(\mathcal{M}_{d-2}) = \exp\left(i\alpha Q\right) \qquad Q = \int_{\mathcal{M}_{d-2}} \star J$$

Local current defines a U(1)-valued topological codimension-2 surface operator. This is called a 1-form symmetry.



Symmetry and Topological defects

- Topological: deformations of Σ do not modify correlators of $U_g(\Sigma)$
- Fusion: Group multiplication law $U_g(\Sigma) \times U_h(\Sigma) = U_{hg}(\Sigma)$.
- Linking: Local operator crossing the defect $\rightarrow$ a symmetry operation.



The Power of Non-Inv-Symmetry

"What's Done Cannot Be Undone"

- There are codimension-1 topological defects that are NOT the charge operators of any 0-form global symmetries.
- These topological defects are equally powerful in constraining the dynamics of QFT or implementing dualities and are called Non-Invertible.
- Why are these duality defects non-invertible? Their action on operators in the theory does not preserve the dimension of the operator
- They are ubiquitous in 2D CFT (e.g. 2D Ising model $\rightarrow$ Krammers-Wannier duality, free compact boson $\rightarrow$ T-duality c.f José Fernández-Melgarejo talk).

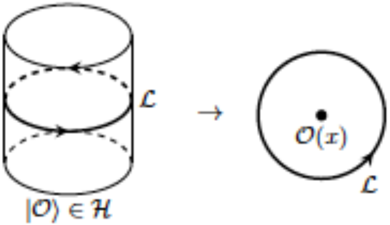
Motivation

- At the moment, CMT applications of the idea of fusion category symmetries remain in the realm of relatively formal developments.
- One application is to understand topological order as SSB.
- The idea is that by suitably refining and generalizing our notions of symmetry, we can incorporate all ‘beyond-Landau’ examples into a *Generalized Landau Paradigm*.
- Can SSB of non-invertible symmetries lead to physically distinguishable ground states? $\Longrightarrow$
- Today: Entropic characterization of ground states with these symmetries.

Outline

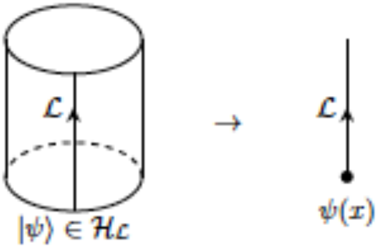
Verlinde Lines. Action on Hilbert Space

Via the operator-state map, the action of a topological line $\mathcal{L}$ on a state $|\mathcal{O}\rangle$ in the Hilbert space $\mathcal{H}$ is mapped to the Euclidean configuration of the line encircling the corresponding local operator $\mathcal{O}(x)$.



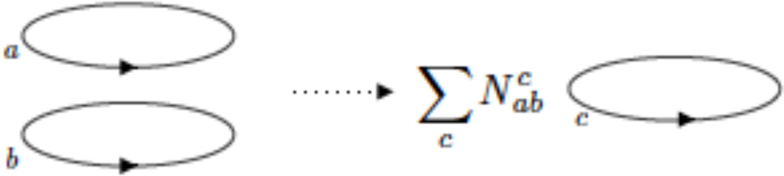
Verlinde Lines. Defect Hilbert space

Under the operator-state map, a state $|\psi\rangle$ in the Hilbert space $\mathcal{H}_{\mathcal{L}}$ twisted by the topological line $\mathcal{L}$ is mapped to a point operator $\psi(x)$ attached to the topological line $\mathcal{L}$. Here $\psi(x)$ need not have integer spin $h - \bar{h}$



Verlinde Lines. Fusion

$$\mathcal{L}_a \times \mathcal{L}_b = \sum_c N_{ab}^c \, \mathcal{L}_c$$

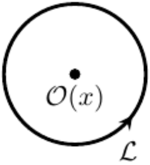


Ising CFT

The critical Ising model has three Virasoro primaries: $\mathbb{I}_{0,0}$, $\varepsilon_{\frac{1}{2},\frac{1}{2}}$, $\sigma_{\frac{1}{16},\frac{1}{16}}$

Topological lines: Trivial $\mathbb{I}$, invertible line η ($\mathbb{Z}_2$ global symmetry), non-invertible $\mathcal{N}$ line (KW-duality).

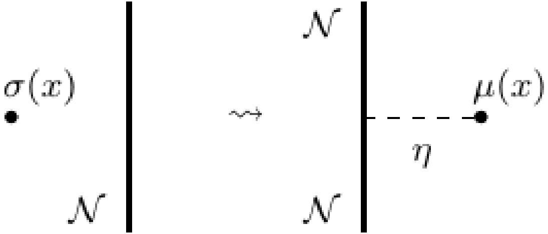
$$\begin{array}{cccc} & \mathbb{I} & \varepsilon & \sigma \\ \eta: & 1 & 1 & -1 \\ \mathcal{N}: & \sqrt{2} & -\sqrt{2} & 0 \end{array}$$



Ising CFT

Definition of $\mathcal{N}$: passing through the KW-wall, the spin σ and the disorder operator μ are interchanged.

The latter is not a local operator, but rather must be attached to a branch cut across which the $\mathbb{Z}_2$ symmetry acts.



Fusion Rules

$$\eta \times \eta = \mathbb{I}$$
$$\eta \times \mathcal{N} = \mathcal{N}$$
$$\mathcal{D} \times \eta = \mathcal{N}$$
$$\mathcal{N} \times \mathcal{N} = \mathbb{I} + \eta$$

The last, arises as the Kramers-Wannier duality only keeps track of the locations of domain walls, and erases the information about the overall spin flip.

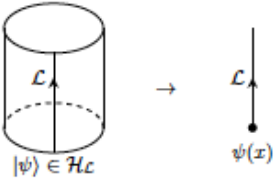
The RG flows generated by a perturbation by a local operator can only generate operators that pass freely through the wall.

Ising CFT. Defect Hilbert Space

The Hilbert spaces of defect operators at the endpoints of η and $\mathcal{N}$ are determined by the fusion coefficients and are spanned by

$$\mathcal{H}_\eta \; : \; \psi_{\frac{1}{2},0}, \quad \tilde{\psi}_{0,\frac{1}{2}}, \quad \mu_{\frac{1}{16},\frac{1}{16}},$$
$$\mathcal{H}_\mathcal{N} \; : \; s_{\frac{1}{16},0}, \quad \tilde{s}_{0,\frac{1}{16}}, \quad \Lambda_{\frac{1}{16},\frac{1}{2}}, \quad \tilde{\Lambda}_{\frac{1}{2},\frac{1}{16}},$$

where we only listed the primaries. Note that the defect operator μ is the disorder operator in the critical Ising model.



Outline

Entropic portray of fusion symmetries in 2D-CFT

- Non invertible symmetries manifest themselves with definite "order" and "disorder" operators which close specific fusion algebras.
- Both order and disorder operators, generate self-consistent von Neumann algebras $\Longrightarrow$ obvious information theoretic quantities to address is the von Neumann entropy or related.

Approaches

- Symmetry Resolved Entanglement Entropy. Pablo Saura's talk
- Relative Entropy

Relative Entropy in CFT

Relative entropy measures the distinguishability between states ρ and σ.

Relative Entropy

$$S(\rho||\sigma) = \mathrm{Tr}\rho \log \rho - \mathrm{Tr}\rho \log \sigma \geq 0$$

It can be written in terms of the modular operator $K_\sigma = -\log \sigma$ as

$$S(\rho||\sigma) = \Delta\langle K_\sigma \rangle - \Delta S$$

w/

$$\Delta\langle K_\sigma \rangle = \mathrm{Tr}\rho K_\sigma - \mathrm{Tr}\sigma K_\sigma \qquad \Delta S = S(\rho) - S(\sigma) .$$

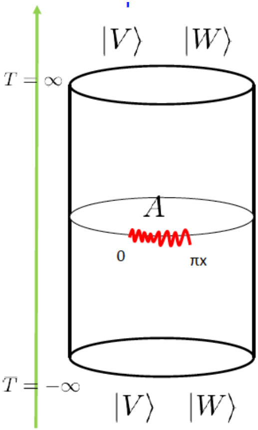
Positivity of relative entropy $\implies$

- Rigorous formulation of Bekenstein bound in QFT [Casini]
- Proof of: Generalized second law [Wall]... // Quantum Bousso bound [Bousso, Casini, Fisher Maldacena] // Averaged null energy condition [Faulkner et al]

Relative Entropy between two "excited" states

[Lashkari, Ugajin et al]

Reduced density matrices $\rho_W = \mathrm{Tr}_{\bar{A}}|W\rangle\langle W|$ and $\rho_V = \mathrm{Tr}_{\bar{A}}|V\rangle\langle V|$.



Sketch of the derivation

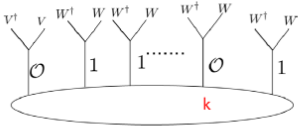
1. Replica Trick

$$S(\rho_V||\rho_W) = \lim_{n\to 1} \frac{1}{n-1} \left(\log \mathrm{Tr}\rho_V^n - \log \mathrm{Tr}\rho_V \rho_W^{n-1}\right) .$$

2. Write each term by a correlation function

$$\mathrm{Tr}\rho_V \rho_W^{n-1} = \frac{\langle V^\dagger(w_0)V(w_0') \prod_{k=1}^{n-1} W^\dagger(w_k)W(w_k)\rangle_{\Sigma_n}}{\langle V^\dagger(w)V(w')\rangle_{\Sigma_1} \langle W^\dagger(w)W(w')\rangle_{\Sigma_1}^{n-1}}$$

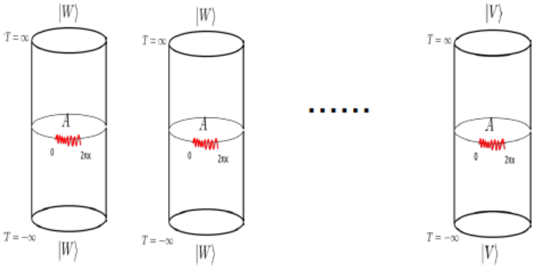
3. Expand these correlation functions in the $x \to 0$ limit, using OPEs [Alcaraz, Berganza, Sierra]



Replica Trick I

Each term can be computed by a Path integral on the *n* sheet cover of the cylinder with the bdy condition specifying the excited states

$$\mathrm{tr}\rho_V\,\rho_W^{n-1} =$$



Replica Trick II

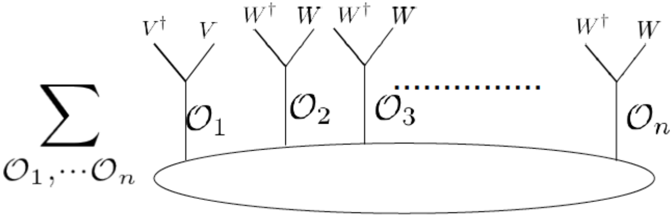
We can write $S(\rho_V||\rho_W) = \lim_{n\to 1} S_n(\rho_V||\rho_W)$ in terms of the correlation functions of V and W in the following way (compact notation)

$$\mathrm{Tr}\rho_V^n = \frac{\langle \underbrace{V^\dagger V \cdots V^\dagger V}_{n} \rangle_{\Sigma_n}}{\langle V^\dagger V \rangle_{\Sigma_1}^n}$$

$$\mathrm{Tr}\rho_V \rho_W^{n-1} = \frac{\langle V^\dagger V \underbrace{W^\dagger W \cdots W^\dagger W}_{n-1} \rangle_{\Sigma_n}}{\langle V^\dagger V \rangle_{\Sigma_1} \langle W^\dagger W \rangle_{\Sigma_1}^{n-1}}$$

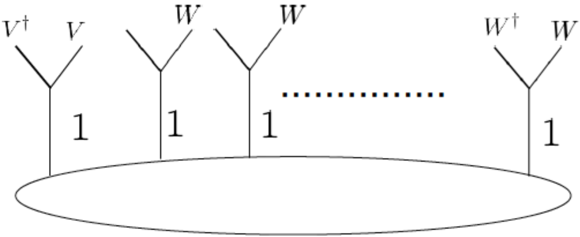
OPE expansion

- By using the OPEs n times, the $2n$ point functions can be expressed as a sum over all exchanges of internal operators $\mathcal{O}_\alpha$.
- Each term with the fixed internal operators scales like $x^{\sum_\alpha \Delta_\alpha}$ leading to an expansion with respect to the subsystem size x
- In the small subsystem size limit $x \ll 1$, the effects of light operators dominate in the correlation function.



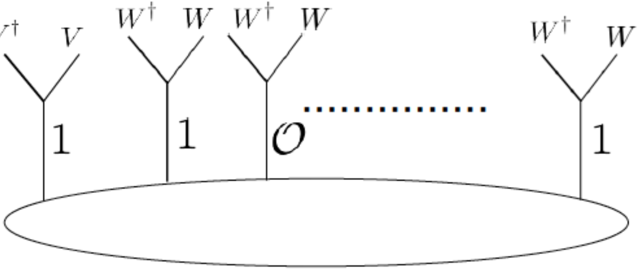
OPE expansion

The first term in the expansion comes from the exchange of the n identity operators. If we keep only this term, the correlation function gets factorized $\langle V^\dagger V \rangle_{\Sigma_n} \langle W^\dagger W \rangle_{\Sigma_n}^{n-1}$ and he contribution of this part in the relative entropy is vaninshing



OPE expansion

The next contribution comes from the single lightest operator exchange. However, this term is zero because one point functions in a CFT is vanishing $\implies$ First non trivial contribution comes from double operator exchange.



Outline

Defect Operators related to the NI line $\mathcal{N}$

We apply previous result to the case $V = s_{\frac{1}{16},0} \in \mathcal{H}_{\mathcal{N}}$ and $W = \mathbb{I}$.

$$S(\rho_s||\rho_{\mathbb{I}}) = \lim_{n\to 1} \frac{1}{n-1} \log \left(1 + (C_{ss}^{\psi})^2 \sin\left(\frac{\pi x}{n}\right) \sum_{i<j}^{n-1} \langle \psi_i \psi_j \rangle_n \right)$$

$$= \lim_{n\to 1} \frac{\sum_{i<j}^{n-1} \langle \psi_i \psi_j \rangle_n}{n(n-1)} (C_{ss}^{\psi})^2 \, \pi x + \mathcal{O}(x^2)$$

where $\psi_{\frac{1}{2},0} \in \mathcal{H}_\eta$ and we have considered the OPE $s \times s = 1 + \psi$

$$s(w)s(0) = w^{-2(h+\bar{h})} + C_{ss}^{\psi} w^{(\delta+\bar{\delta})-2(h+\bar{h})} \psi(0) + \cdots$$

where δ and $\bar{\delta}$ are the conformal dimensions of the operator ψ and $C_{ss}^{\psi} = 1/\sqrt{2}$

Applying

$$\sum_{i<j}^{n-1}\langle\psi_i\psi_j\rangle_n = (n-1)\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\delta+1\right)}{4\Gamma\left(\delta+\frac{3}{2}\right)} + \mathcal{O}((n-1)^2)$$

we obtain

$$S(\rho_s||\rho_{\mathbb{I}}) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{4\Gamma\left(2\right)}\frac{1}{2}\pi x = \frac{\pi^2 x}{16}\,.$$

and

$$S(\rho_s||\rho_{\mathbb{I}}) = -\Delta S_s + \mathcal{O}(x^2) \implies$$

$\Delta S_s < 0$ in contrast to what happens in general for a bulk operator insertion $\Delta S_\Phi \sim \mathcal{O}(x^2) > 0$ [Alcaraz, Berganza, Sierra]

Defect Operators related to the I-line η

Here $V = \psi_{\frac{1}{2},0} \in \mathcal{H}_\eta$ and $W = \mathbb{I}$. Using the OPE of this operator one finds

$$S(\rho_\psi || \rho_{\mathbb{I}}) = \lim_{n\to 1} \frac{1}{n-1}\left(\log \frac{\langle\psi\psi\rangle_n^n}{\langle\psi\psi\rangle_1^n} - \log \frac{\langle\psi\psi\rangle_n}{\langle\psi\psi\rangle_1}\right)$$
$$= \lim_{n\to 1} \frac{n-1}{n-1} \log \frac{\langle\psi\psi\rangle_n}{\langle\psi\psi\rangle_1} = 0\,,$$

a significant sign of invertibility?

Ongoing/Future work

1. Treat non RCFT as the free compactified boson
2. Holographic Computations?

regarding (2)

- Holographic computation of Charged Renyi Entropies $\sim$ SREE (Pablo Saura's talk)
- Holographic computations of relative entropies in our setting $\Longrightarrow$ NI symmetries in higher dimensions : |

Thanks for attention GRASS-SYMBHOL!