Superrotations at spatial infinity

Javier Matulich

IFT, Madrid

Work in progress in collaboration with A. Fiorucci and R. Ruzziconi

GRASS-SYMBHOL Meeting, Ávila, November 2023

Outline

Table of Contents

Introduction

• The infinite dimensional BMS (Bondi-van der Burg-Metzner-Sachs) group was shown long ago to be the group of asymptotic symmetries of gravity in asymptotically flat spacetime and it was originally discovered in the asymptotic analysis at null infinity in the context of gravitational radiation.

• The BMS symmetries do not only leave invariant the boundary conditions at null infinity, but are also exact symmetries of the theory leaving the action invariant up to a surface term. Therefore, they should appear in any description, in particular, in slices adapted to spatial infinity.

• Remarkably, since there is no outgoing flux at spatial infinity, any relevant symmetry can be generated by a conserved charge that can be determined by standard canonical techniques.

• However, Hamiltonian analyses at spatial infinity did not exhibit any sign of BMS_4.

[REGGE - Teitelboim, 1974]

Introduction

- It was not until recently that this discrepancy was resolved by considering appropriate "parity-twisted boundary conditions" at spatial infinity on spacelike hypersurfaces leading to a full agreement with the result at null infinity.

  [Henneax-Troessaert, 2018-2019]

- Canonical realization of the super-BMS algebra where the fermionic generators can be considered as being the "square roots" of the BMS_4 supertranslations.

  [Fuentealba, Henneaux, Majumdar, J.M., Neogi, 2020-2021]

- The asymptotic structure of gravity at spatial infinity in five spacetime dimensions, which revealed interesting new features not uncovered before.

  [Fuentealba, Henneaux, J.M., Troessaert, 2021-2022]

- Extension of the BMS_4 group by adding logarithmic supertranslations.

  [Fuentealba, Henneaux, Troessaert, 2022]

- The BMS group can be enlarged by extending the Lorentz algebra, the so-called “superrotations”.

→ Barnich - Troessaert : $(\mathrm{Vir} \times \mathrm{Vir}) \ltimes S$

[Barnich, Troessaert, 2010]

→ Campiglia - Laddha : $\mathrm{Diff}(S^2) \ltimes S$

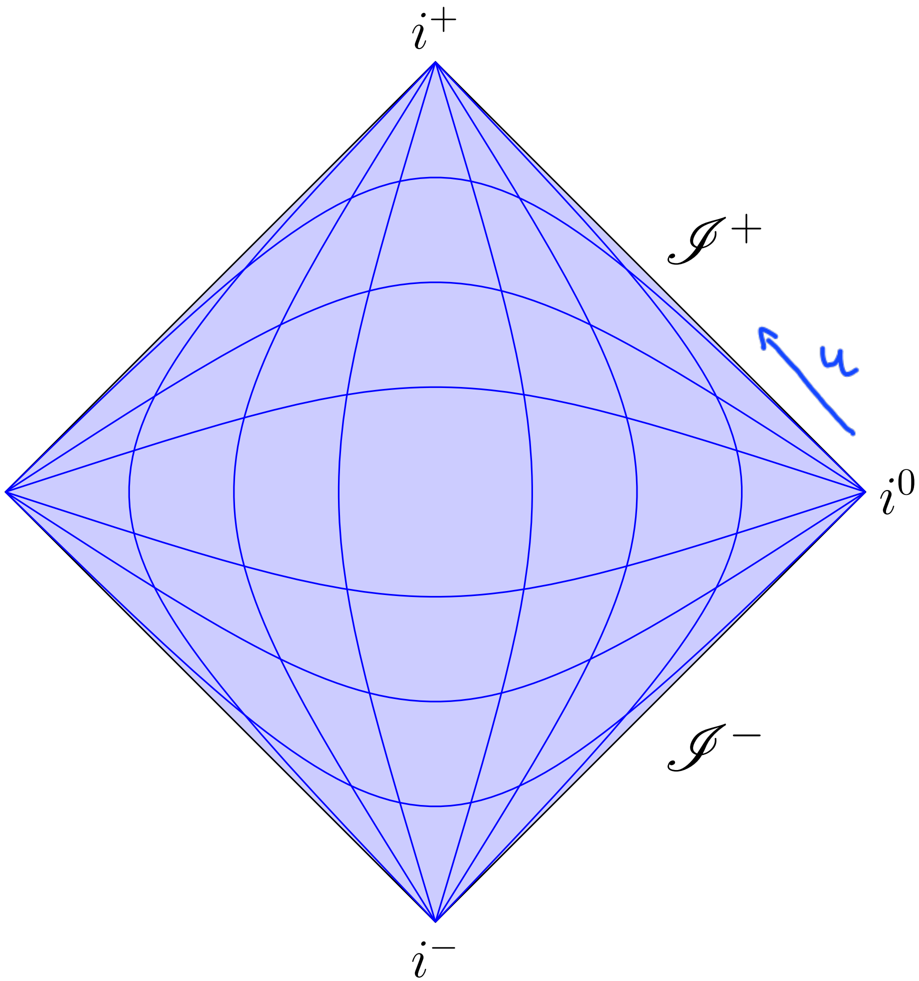
[Campiglia - Laddha, 2014]

[Compère, Fiorucci, Ruzziconi, 2020]

- Unfortunately, superrotations have been elusive in spatial infinity settings.

Table of Contents

Minkowski spacetime



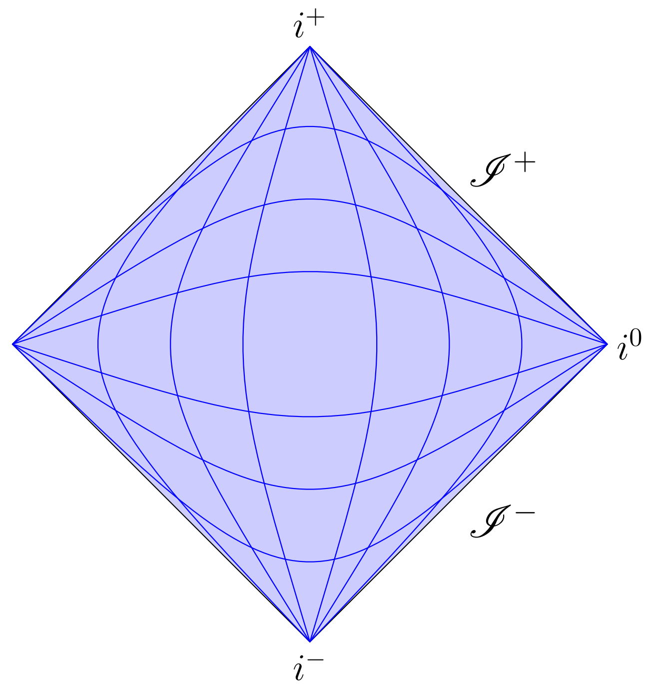
Retarded Bondi coordinates $(u, r, x^A = (z, \bar{z}))$:

$$ds^2 = -du^2 - 2du dr + 2r^2 \mathring{q}_{AB} dx^A dx^B$$

Asymptotically flat metrics

- The solution space of four-dimensional asymptotically flat metrics reads as

$$
\begin{aligned}
ds^2 = & \left(\frac{2m_B}{r} + \mathcal{O}(r^{-2})\right) du^2 \\
& - 2\left(1 + \mathcal{O}(r^{-2})\right) du dr \\
& + \left(r^2 \mathring{q}_{AB} + r C_{AB} + \mathcal{O}(r^{-1})\right) dx^A dx^B \\
& + \left(\frac{1}{2}\partial_B C^B_A + \frac{2}{3r}\left(N_A + \frac{1}{4} C^B_A \partial_C C^C_B\right) + \mathcal{O}(r^{-2})\right) du dx^A
\end{aligned}
$$



$m_B(u, x^A)$: Bondi mass.

$N_A(u, x^A)$: Angular momentum aspect.

$N_{AB} = \partial_u C_{AB}$: Bondi news (encoding GW)

Symmetry group of asymptotically flat spacetimes

What do we expect?

- The isometry group of Minkowski space, i.e., the Poincaré group: 4 spacetime traslations + 6 Lorentz transformations (3 boosts, 3 rotations).

What do we actually have?

- An infinite dimensional extension of Poincaré: the Bondi-Metzner-van der Burg; Sachs (BMS) group.

There exist a vector field preserving the asymptotic form of the metric given by $\xi = \xi^u \partial_u + \xi^z \partial_z + \xi^{\bar{z}} \partial_{\bar{z}} + \xi^r \partial_r$ such that

$$\xi^u = \mathcal{T}(z, \bar{z}); \quad \text{supertranslations} \quad u \to u + \mathcal{T}(z, \bar{z})$$

- Supertranslations shift the retarded time by a different amount at each point on the sphere.
- There exist an even bigger extension by considering arbitrary functions that parametrize rotations $\mathcal{Y}$ (superrotations)

$$\xi^u = \mathcal{T}(z, \bar{z}) + u\alpha\,(\mathcal{Y}, \bar{\mathcal{Y}})\ ;\ \xi^z = \mathcal{Y} + \mathcal{O}(r^{-1})\ ;\ \xi^r = -r\alpha + \mathcal{O}(r^0)$$

Table of Contents

Electromagnetism

[HENNEAUX-TROESSAERT, 1803.10194]

- Hamiltonian action for Electromagnetism

LAGRANGE multiplier

$$S_H = \int dt \left\{ \int d^3x \, \pi^i \dot{A}_i - \left(\frac{1}{2} \pi^i \pi_i + \frac{1}{4} F^{ij} F_{ij} + A_t \mathcal{G} \right) + F_\infty \right\}$$

vector potential

CONJUGATE MOMENTA

$\mathcal{G} = -\partial_i \pi^i \approx 0$

Electromagnetism

[Henneaux-Troessaert, 1803.10194]

- Hamiltonian action for Electromagnetism

Lagrange multiplier

$$S_H = \int dt \left\{ \int d^3x \, \pi^i \dot{A}_i - \left(\frac{1}{2} \pi^i \pi_i + \frac{1}{4} F^{ij} F_{ij} + A_t \mathcal{G} \right) + F_\infty \right\}$$

vector potential

conjugate momenta

$\mathcal{G} = -\partial_i \pi^i \approx 0$

- The EM field and its conjugate are usually taken to decay as

$$A_i = \frac{1}{r} \bar{A}_i + O\left(r^{-2}\right) \quad , \quad \pi^i = \frac{1}{r^2} \bar{\pi}^i + O\left(r^{-3}\right)$$

arbitrary functions on the 2-sphere

These conditions will be constrainted in order to guarantee the finiteness of the symplectic structure.

Electromagnetism

[Henneaux-Troessaert, 1803.10194]

- Hamiltonian action for Electromagnetism

Lagrange multiplier

$$S_H = \int dt \left\{ \int d^3x\, \pi^i \dot{A}_i - \left(\frac{1}{2}\pi^i \pi_i + \frac{1}{4} F^{ij} F_{ij} + A_t \mathcal{G} \right) + F_\infty \right\}$$

vector potential

conjugate momenta

$\mathcal{G} = -\partial_i \pi^i \approx 0$

- The EM field and its conjugate are usually taken to decay as

$$A_i = \frac{1}{r} \bar{A}_i + O\left(r^{-2}\right) \quad , \quad \pi^i = \frac{1}{r^2} \bar{\pi}^i + O\left(r^{-3}\right)$$

arbitrary functions on the 2-sphere

These conditions will be constrainted in order to guarantee the finiteness of the symplectic structure.

- BC are invariant under gauge transformations generated by the first-class constraint $\mathcal{G}$

$$\delta_\epsilon A_i = \partial_i \epsilon \quad , \quad \delta_\epsilon \pi^i = 0$$

$\epsilon = \bar{\epsilon}(x^A) + O(r^{-1})$

- The generator of the gauge symmetries is given by

$$G[\epsilon] = \int d^3x \epsilon \mathcal{G} + \oint d^2 S_i \epsilon \bar{\pi}^i \approx \oint d^2 S_i \epsilon \bar{\pi}^i$$

Electric charge $\epsilon = 1$
(imp. gauge trans.)

Poincaré Transformations

- A general deformation of constant time hypersurface can be parametrized by normal ξ and tangential ξ^i components. In particular, a general Poincaré tranformation corresponds to

$$\xi = \underset{\text{Lorentz boosts}}{\underset{\downarrow}{b_i x^i}} + \underset{\text{standard translations}}{\underset{\downarrow}{a^\perp}} \qquad\qquad \xi^i = \underset{\text{Spatial Rotations}}{\underset{\downarrow}{b^i{}_j x^j}} + \underset{\text{standard translations}}{\underset{\downarrow}{a^i}}$$

- The fields transform as

$$\begin{aligned} \delta A_i &= \xi \frac{\pi_i}{\sqrt{g}} + \xi^j F_{ji} + \partial_i \zeta \longrightarrow \text{Gauge trans.} \\ \delta \pi^i &= \partial_m \left(\sqrt{g} F^{mi} \xi \right) + \mathcal{L}_\xi \pi^i \end{aligned}$$

Finiteness of the symplectic form

- The symplectic form (kinetic term) possesses a logaritmic divergence

$$\int \frac{dr}{r} \boxed{\int d^2x \left(\bar{\pi}^r \dot{A}_r + \bar{\pi}^A \dot{A}_A\right)}$$

Impose extra conditions so that $\int d^2x \,(\quad) = 0$ $\rightarrow$ PARITY CONDITIONS

$\theta \rightarrow \pi - \theta$
$\varphi \rightarrow \varphi + \pi$
$r \rightarrow r$

Finiteness of the symplectic form

- The symplectic form (kinetic term) possesses a logaritmic divergence

$$\int \frac{dr}{r} \boxed{\int d^2x \left(\bar{\pi}^r \dot{\bar{A}}_r + \bar{\pi}^A \dot{\bar{A}}_A\right)}$$

Impose extra conditions so that $\int d^2x \,(\quad) = 0$ $\rightarrow$ PARITY CONDITIONS

- Radial components

$$\bar{A}_r(-x^A) = -\bar{A}_r(x^A)$$

$$\bar{\pi}^r(-x^A) = \bar{\pi}^r(x^A)$$

- Coulomb field (moving frame)
- Gauge invariant
- Charge: $\boxed{G[\epsilon] \simeq \oint d^2x \, \bar{\epsilon}_{\text{even}} \, \bar{\pi}^r}$

Finiteness of the symplectic form

- The symplectic form (kinetic term) possesses a logaritmic divergence

$$\int \frac{dr}{r} \boxed{\int d^2x \left(\bar{\pi}^r \dot{\bar{A}}_r + \bar{\pi}^A \dot{\bar{A}}_A\right)}$$

Impose extra conditions so that $\int d^2x\,(\quad) = 0$ $\longrightarrow$ PARITY CONDITIONS

- Radial components

$$\bar{A}_r(-x^A) = -\bar{A}_r(x^A)$$

$$\bar{\pi}^r(-x^A) = \bar{\pi}^r(x^A)$$

- Coulomb field (moving frame)
- Gauge invariant
- Charge: $\boxed{G[\epsilon] \approx \oint d^2x\, \bar{\epsilon}_{\text{even}}\, \bar{\pi}^r}$

- Angular components

$$\bar{A}_A(-x^B) = -\bar{A}_A(x^B) + \partial_A \Phi(x^B)$$

$$\bar{\pi}^A(-x^B) = \bar{\pi}^A(x^B)$$

- "Twisted" parity cond. of the form of a gauge trans.
- $\Phi(-x^B) = -\Phi(x^B)$ / $\partial_A \bar{\pi}^A = 0$
- Accommodate solutions.

Finiteness of the symplectic form

- The symplectic form (kinetic term) possesses a logaritmic divergence

$$\int \frac{dr}{r} \boxed{\int d^2x \left(\bar{\pi}^r \dot{\bar{A}}_r + \bar{\pi}^A \dot{\bar{A}}_A\right)} = 0 \;!$$

Impose extra conditions so that $\int d^2x (\quad) = 0$ → PARITY CONDITIONS

- Radial components

$$\bar{A}_r(-x^A) = -\bar{A}_r(x^A)$$

$$\bar{\pi}^r(-x^A) = \bar{\pi}^r(x^A)$$

- Coulomb field (moving frame)
- Gauge invariant
- Charge: $\boxed{G[\epsilon] \approx \oint d^2x \, \bar{\epsilon}_{even} \, \bar{\pi}^r}$

- Angular components

- "Twisted" parity cond. of the form of a gauge trans.
- $\Phi(-x^B) = -\Phi(x^B)$ / $\partial_A \bar{\pi}^A = 0$
- Accommodate solutions.

$$\bar{A}_A(-x^B) = -\bar{A}_A(x^B) + \partial_A \Phi(x^B)$$

$$\bar{\pi}^A(-x^B) = \bar{\pi}^A(x^B)$$

Invariance of the symplectic structure

- We want Poincaré to be a symmetry of the theory, in particular, it should leave the symplectic structure invariant

$$\Omega = \int d^3x d_V \pi^i d_V A_i$$

EXTERIOR DERIVATIVE
IN PHASE SPACE

Invariance of the symplectic structure

- We want Poincaré to be a symmetry of the theory, in particular, it should leave the symplectic structure invariant

$$\Omega = \int d^3x d_V \pi^i d_V A_i$$

EXTERIOR DERIVATIVE
IN PHASE SPACE

- A transformation defined by the vector field X es canonical if

$$d_V(\iota_X \Omega) = 0$$

Invariance of the symplectic structure

- We want Poincaré to be a symmetry of the theory, in particular, it should leave the symplectic structure invariant

$$\Omega = \int d^3x d_V \pi^i d_V A_i$$

exterior derivative
in phase space

- A transformation defined by the vector field X es canonical if

$$d_V(\iota_X \Omega) = 0$$

- Evaluating for Lorentz boosts $\xi = br$

$$d_V(\iota_b \Omega) = \oint d^2x \sqrt{\bar{\gamma}}\, d_V \bar{A}_r \bar{D}^B (b d_V \bar{A}_B) \neq 0$$

EVEN + odd

Something must be done in order to accomodate Lorentz boosts.

• The problem can be solved by considering a surface degree of freedom at infinity such that we add to the symplectic form

$$-\oint d^2x\sqrt{\bar\gamma}d_V\bar A_r d_V\bar\Psi$$

odd

where $\bar\Psi$ transforms under boosts as

$$\delta_b\bar\Psi=\bar D^B(b\bar A_B)+b\bar A_r$$

• The problem can be solved by considering a surface degree of freedom at infinity such that we add to the symplectic form

$$-\oint d^2x\sqrt{\bar{\gamma}}d_V\bar{A}_r d_V\bar{\Psi}$$

odd

where $\bar{\Psi}$ transforms under boosts as

$$\delta_b\bar{\Psi} = \bar{D}^B(b\bar{A}_B) + b\bar{A}_r$$

• The transformation of A_i under boosts is modified by adding a gauge transformation

$$\delta_\xi A_i = \frac{1}{\sqrt{g}}\xi\pi^i + \partial_i(\xi\Psi)$$

such that Ψ is any function that matches $\bar{\Psi}$ at infinity as

$$\Psi = \frac{1}{r}\bar{\Psi} + O(r^{-1})$$

• Finally, since the symplectic form is invariant, boosts define canonical transformations!

Complete action

- We can extend the surface dof $\bar{\Psi}$ into a dynamical bulk field Ψ by introducting the constraint $\pi_\Psi \approx 0$ and its Lagrange multiplier λ such that the full action is now

$$
\begin{aligned}
S_H \quad = \quad & \int dt \left\{ \int d^3x \left(\pi^i \dot{A}_i + \pi_\Psi \dot{\Psi} \right) - \oint d^2x \sqrt{\bar{\gamma}} \bar{A}_r \dot{\bar{\Psi}} \right. \\
& \left. - \int d^3x \left(\frac{1}{2\sqrt{g}} \pi^i \pi_i + \frac{\sqrt{g}}{4} F^{ij} F_{ij} \right) - \int d^3x \left(\lambda \pi_\Psi + A_t \mathcal{G} \right) \right\}
\end{aligned}
$$

$$\pi_\Psi = \frac{1}{r} \pi_\Psi^{(1)} + O(r^{-2}) \qquad \lambda = \frac{\bar{\lambda}}{r} + O(r^{-2})$$

Complete action

- We can extend the surface dof $\bar{\Psi}$ into a dynamical bulk field Ψ by introducting the constraint $\pi_\Psi \approx 0$ and its Lagrange multiplier λ such that the full action is now

$$\begin{aligned} S_H \quad = \quad & \int dt \left\{ \int d^3x \left(\pi^i \dot{A}_i + \pi_\Psi \dot{\Psi} \right) - \oint d^2x \sqrt{\bar{\gamma}} \bar{A}_r \dot{\bar{\Psi}} \right. \\ & \left. - \int d^3x \left(\frac{1}{2\sqrt{g}} \pi^i \pi_i + \frac{\sqrt{g}}{4} F^{ij} F_{ij} \right) - \int d^3x \left(\lambda \pi_\Psi + A_t \mathcal{G} \right) \right\} \end{aligned}$$

- The action is invariant under arbitrary shifts of $\bar{\Psi}$ parametrized by μ, such that

$$\delta_{\mu,\epsilon} \Psi = \mu \quad , \quad \delta_{\mu,\epsilon} A_i = \partial_i \epsilon \quad , \quad \delta_{\mu,\epsilon} \pi^i = 0 \quad , \quad \delta_{\mu,\epsilon} \pi_\Psi = 0$$

Complete action

- We can extend the surface dof $\bar{\Psi}$ into a dynamical bulk field Ψ by introducting the constraint $\pi_\Psi \approx 0$ and its Lagrange multiplier λ such that the full action is now

$$
\begin{aligned}
S_H \;=\; & \int dt \Big\{ \int d^3x \left(\pi^i \dot{A}_i + \pi_\Psi \dot{\Psi} \right) - \oint d^2x \sqrt{\bar{\gamma}} \bar{A}_r \dot{\bar{\Psi}} \\
& - \int d^3x \left(\frac{1}{2\sqrt{g}} \pi^i \pi_i + \frac{\sqrt{g}}{4} F^{ij} F_{ij} \right) - \int d^3x \left(\lambda \pi_\Psi + A_t \mathcal{G} \right) \Big\}
\end{aligned}
$$

- The action is invariant under arbitrary shifts of $\bar{\Psi}$ parametrized by μ, such that

$$
\delta_{\mu,\epsilon}\Psi = \mu \quad , \quad \delta_{\mu,\epsilon} A_i = \partial_i \epsilon \quad , \quad \delta_{\mu,\epsilon}\pi^i = 0 \quad , \quad \delta_{\mu,\epsilon}\pi_\Psi = 0
$$

- The generator is then given by

$$
G_{\mu,\epsilon} = \int d^3x \left(\mu \pi_\Psi + \epsilon \mathcal{G} \right) + \oint d^2x \left(\underset{\text{EVEN}}{\bar{\epsilon}\bar{\pi}^r} - \sqrt{\bar{\gamma}} \underset{\text{odd}}{\bar{\mu}} \bar{A}_r \right)
$$

IMPROPER GAUGE TRANS. !

- This combination form the full angle-dependent $U(1)$ transformation at null infinity.

Asymptotic symmetry algebra

- Poincaré transformations are now canonical transformations generated by

$$P_{\xi,\xi^i} = \int d^3x \left(\xi\mathcal{H}^{EM} + \xi^i\mathcal{H}_i^{EM}\right) + \mathcal{B}^{EM}_{(\xi,\xi^i)}$$

$$\mathcal{H}^{EM} = \Psi\partial_i\pi^i + A_i\partial^i\pi_\Psi + \frac{1}{2\sqrt{g}}\pi_i\pi^i + \frac{\sqrt{g}}{4}F_{ij}F^{ij}$$

$$\mathcal{H}_i^{EM} = F_{ij}\pi^j - A_i\partial_j\pi^j + \pi_\Psi\partial_i\Psi$$

$$\mathcal{B}^{EM}_{(\xi,\xi^i)} = \oint d^2x \left[b\left(\bar{\Psi}\bar{\pi}^r + \sqrt{\bar{\gamma}}\bar{A}_B\bar{D}^B A_r\right) + Y^B\left(\bar{A}_B\bar{\pi}^r + \sqrt{\bar{\gamma}}\bar{\Psi}\partial_B\bar{A}_r\right)\right]$$

- The algebra of the generators is given by

$$\left\{P_{\xi_1,\xi_1^i}, P_{\xi_2,\xi_2^i}\right\} = P_{\hat{\xi},\hat{\xi}^i} \quad , \quad \left\{G_{\mu,\epsilon}, P_{\xi,\xi^i}\right\} = G_{\hat{\mu},\hat{\epsilon}}$$

$$\left\{G_{\mu_1,\epsilon_1}, G_{\mu_2,\epsilon_2}\right\} = 0$$

where

$$\hat{\xi} = \xi_1^i\partial_i\xi_2 - \xi_2^i\partial_i\xi_1 \quad , \quad \hat{\xi}^i = \xi_1^j\partial_j\xi_2^j - \xi_2^j\partial_j\xi_1^j + g^{ij}\left(\xi_1\partial_j\xi_2 - \xi_2\partial_j\xi_1\right)$$

$$\hat{\mu} = \nabla^i\left(\Psi\partial_i\epsilon\right) - \xi^i\partial_i\mu \quad , \quad \hat{\epsilon} = \xi\mu - \xi^i\partial_i\epsilon$$

- The algebra is the ***semi-direct sum of the Poincaré algebra and the abelian algebra*** parametrized by $\bar{\mu}$ and $\bar{\epsilon}$. (angle dependent $U(1)$ transformation)

Table of Contents

Gravity

[Henneaux-Troessaert, 1904.04495]

- The hamiltonian action of pure gravity in four spacetime dimensions can be written as

$$S[g_{ij},\pi^{ij},N,N^i] = \int dt \left\{ \int d^3x \, (\underbrace{\pi^{ij}}_{\text{conjugate momentum}} \underbrace{\dot{g}_{ij}}_{\text{spatial metric}} - \underbrace{N}_{\text{lapse}} \mathcal{H}^{grav} - \underbrace{N^i}_{\text{shift}} \mathcal{H}_i^{grav}) - \underbrace{B_\infty^{grav}}_{\text{well-defined action principle}} \right\}$$

$$\underbrace{\mathcal{H}^{grav}}_{\text{Hamiltonian constraint}} = -\sqrt{g}R + \frac{1}{\sqrt{g}}(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2), \quad \underbrace{\mathcal{H}_i^{grav}}_{\text{momentum constraint}} = -2\nabla_j \pi_i^j$$

Gravity

[Henneaux-Troessaert, 1904.04495]

- The hamiltonian action of pure gravity in four spacetime dimensions can be written as

$$S[g_{ij}, \pi^{ij}, N, N^i] = \int dt \left\{ \int d^3x \, (\pi^{ij} \dot{g}_{ij} - N\mathcal{H}^{grav} - N^i \mathcal{H}_i^{grav}) - B_\infty^{grav} \right\}$$

conjugate momentum — spatial metric — Lapse — Shift — well-defined action principle

$$\mathcal{H}^{grav} = -\sqrt{g}R + \frac{1}{\sqrt{g}}(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2), \quad \mathcal{H}_i^{grav} = -2\nabla_j \pi_i^j$$

Hamiltonian constraint — momentum constraint

- We can consider the fall-off of the metric and its conjugate momentum

[Regge-Teitelboim, '74]

$$g_{ij} = \eta_{ij} + \frac{\bar{h}_{ij}}{r} + O(r^{-1}) \quad , \quad \pi^{ij} = \frac{\bar{\pi}^{ij}}{r^2} + O(r^{-3})$$

$\bar{h}_{ij}$, $\bar{\pi}^{ij}$: depend on the angles

Gravity

[Henneaux-Troessaert, 1904.04495]

- The hamiltonian action of pure gravity in four spacetime dimensions can be written as

$$S[g_{ij},\pi^{ij},N,N^i] = \int dt \left\{ \int d^3x \, (\pi^{ij}\dot{g}_{ij} - N\mathcal{H}^{grav} - N^i\mathcal{H}_i^{grav}) - B_\infty^{grav} \right\}$$

conjugate momentum — spatial metric — Lapse — Shift — well-defined action principle

$$\mathcal{H}^{grav} = -\sqrt{g}R + \frac{1}{\sqrt{g}}(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2), \quad \mathcal{H}_i^{grav} = -2\nabla_j \pi_i^j$$

Hamiltonian constraint — momentum constraint

- We can consider the fall-off of the metric and its conjugate momentum

[Regge-Teitelboim, '74]

$$g_{ij} = \eta_{ij} + \frac{\bar{h}_{ij}}{r} + O(r^{-1}) \quad , \quad \pi^{ij} = \frac{\bar{\pi}^{ij}}{r^2} + O(r^{-3})$$

$\bar{h}_{ij}$, $\bar{\pi}^{ij}$: depend on the angles

and go directly to the kinetic term and check finiteness

$$\int d^3x\pi^{ij}\dot{h}_{ij} \sim \int \frac{dr}{r} \int \underbrace{d\theta\, d\varphi\, \sin\theta\, \bar{\pi}^{ij}\dot{\bar{h}}_{ij}}_{\text{odd}} = 0$$

unit normal to the sphere

$$\bar{h}_{ij}(-\mathbf{n}^k) = \bar{h}_{ij}(\mathbf{n}^k)$$

$$\bar{\pi}^{ij}(\mathbf{n}^k) = -\bar{\pi}^{ij}(\mathbf{n}^k)$$

• One possible solution is to consider parity conditions involving a "twist" given by an improper gauge transformations of the form of a diffeomorphism

$$g_{ij} = \frac{(\bar{h}_{ij})^{even}}{r} + (\partial_i \zeta_j + \partial_j \zeta_i) + O(r^{-2})$$

$$\pi^{ij} = \frac{(\bar{\pi}^{ij})^{odd}}{r^2} + (\partial_i \partial_j V - \triangle V) + O(r^{-3})$$

- ζ_i, V : EVEN, $O(1)$
- Accommodate Taub-nut
- It complicates charges, etc...

• One possible solution is to consider parity conditions involving a "twist" given by an improper gauge transformations of the form of a diffeomorphism

$$g_{ij} = \frac{(\bar{h}_{ij})^{even}}{r} + (\partial_i \zeta_j + \partial_j \zeta_i) + O(r^{-2})$$

$$\pi^{ij} = \frac{(\bar{\pi}^{ij})^{odd}}{r^2} + (\partial_i \partial_j V - \triangle V) + O(r^{-3})$$

- ζ_i, V : EVEN, $O(1)$
- Accommodate Taub-nut
- It complicates charges, etc...

• In order to eliminate divergences of the symplectic term and the Hamiltonian, as well as to guarantee that the bulk pieces of the boost and rotation generators are convergent integrals we have to impose

$$\mathcal{H}^{grav} = O(r^{-5}) \quad , \quad \mathcal{H}_i^{grav} = O(r^{-5})$$

• The absence of canonical generator for the boosts can be solved by demanding that the leading order of the mixed radial-angular components of the metric is zero so that

$$g_{rA} \equiv h_{rA} = \bar{\lambda}_A + O(r^{-1}) = O(r^{-1})$$

Invariance of the boundary conditions

- Under deformations of the constant time hypersurface parametrized by ξ and ξ^i, the canonical variables transform as

$$
\begin{aligned}
\delta g_{ij} &= 2\xi g^{-\frac{1}{2}}\left(\pi_{ij} - \frac{1}{2}g_{ij}\pi\right) + \mathcal{L}_{\xi} g_{ij} \\
\delta \pi^{ij} &= -\xi g^{\frac{1}{2}}\left(R^{ij} - \frac{1}{2}g^{ij}R\right) + \frac{1}{2}\xi g^{-\frac{1}{2}}\left(\pi_{mn}\pi^{mn} - \frac{1}{2}\pi^2\right) \\
&\quad -2\xi g^{-\frac{1}{2}}\left(\pi^{im}\pi_m{}^j - \frac{1}{2}\pi^{ij}\pi\right) + g^{\frac{1}{2}}\left(\xi^{|ij} - g^{ij}\xi^{|m}{}_{|m}\right) \\
&\quad +\mathcal{L}_{\xi}\pi^{ij}
\end{aligned}
$$

Invariance of the boundary conditions

- Under deformations of the constant time hypersurface parametrized by ξ and ξ^i, the canonical variables transform as

$$
\begin{aligned}
\delta g_{ij} &= 2\xi g^{-\frac{1}{2}}\left(\pi_{ij} - \frac{1}{2} g_{ij}\pi\right) + \mathcal{L}_\xi g_{ij} \\
\delta \pi^{ij} &= -\xi g^{\frac{1}{2}}\left(R^{ij} - \frac{1}{2} g^{ij} R\right) + \frac{1}{2}\xi g^{-\frac{1}{2}}\left(\pi_{mn}\pi^{mn} - \frac{1}{2}\pi^2\right) \\
&\quad -2\xi g^{-\frac{1}{2}}\left(\pi^{im}\pi_m{}^j - \frac{1}{2}\pi^{ij}\pi\right) + g^{\frac{1}{2}}\left(\xi^{|ij} - g^{ij}\xi^{|m}{}_{|m}\right) \\
&\quad +\mathcal{L}_\xi \pi^{ij}
\end{aligned}
$$

- The boundary conditions are then invariant under hypersurface deformations that behave asymptotically as (in spherical coordinates)

$$
\xi = br + T + C_{(b)} + O(r^{-1}) \quad , \quad \xi^A = Y^A + \frac{1}{r}\left(\bar{D}^A W + C^A_{(b)}\right) + O(r^{-2}),
$$

$$
\xi^r = W + O(r^{-1}), \quad \bar{D}_A \bar{D}_B b + \bar{\gamma}_{AB} b = 0, \quad \mathcal{L}_Y \bar{\gamma}_{AB} = 0,
$$

b : boosts
Y^A : spatial Rot.
T, W : functions on the sphere.

$C_{(b)}$: to have a well-def. generator
$C^A_{(b)}$: to maintain $\bar{h}_{rA} = 0$

Surface terms and charge generators

- The generator of the transformations is given by

$$P_\xi^{grav}[g_{ij},\pi^{ij}] = \int d^3x\ (\xi\mathcal{H} + \xi^i\mathcal{H}_i) + \mathcal{B}_\xi^{grav}[g_{ij},\pi^{ij}]$$

where the boundary term can be determined by integrating

$$\begin{aligned} d_V\mathcal{B}_\xi &= -\oint G^{ijkl}(\xi(d_V g_{ij})_{|k} - \xi_{,k} d_V g_{ij})d^2S_l \\ &\quad -\oint \left(2\xi_k d_V \pi^{kl} + (2\xi^k\pi^{jl} - \xi^l\pi^{jk})d_V g_{jk}\right) d^2S_l \end{aligned}$$

and G^{ijkl} is De Witt supermetric,

$$G^{ijkl} = \sqrt{g}\Big(\frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - g^{ij}g^{kl}\Big).$$

- The generator is ***finite***, ***integrable*** and ***canonical***.

BMS_4 algebra

• The asymptotic symmetry algebra are canonical transformations generated by P^{grav}_{ξ} such that

$$\{P^{grav}_{\xi_1}, P^{grav}_{\xi_2}\} = P^{grav}_{\hat{\xi}}$$

where

$$\begin{aligned}
\hat{Y}^A &= Y_1^B \partial_B Y_2^A + \bar{\gamma}^{AB} b_1 \partial_B b_2 - (1 \leftrightarrow 2) \\
\hat{b} &= Y_1^B \partial_B b_2 - (1 \leftrightarrow 2) \\
\hat{T} &= Y_1^A \partial_A T_2 - 3 b_1 W_2 - \partial_A b_1 \bar{D}^A W_2 - b_1 \bar{D}_A \bar{D}^A W_2 - (1 \leftrightarrow 2) \\
\hat{W} &= Y_1^A \partial_A W_2 - b_1 T_2 - (1 \leftrightarrow 2)
\end{aligned}$$

• BMS_4 algebra in an unfamiliar parametrization.

• Similarly to EM odd W and even T combine to yield the arbitrary function that paramatrize supertranslations in the null infinity parametrization.

Table of Contents

Superrotations (super-Lorentz) at spatial infinity

- We will propose some boundary conditions invariant under $Diff(S^2) \ltimes \mathcal{S}$.

- In the null infinity analysis, the asymptotic conditions are asymptotically locally Minkowski, this implies that the metric at the boundary is not fixed.

- We have to relax the boundary conditions in a way that they are invariant under hypersurface deformations that behave asymptotically as

$$\xi = br + T + \frac{\epsilon}{r} + O\left(r^{-2}\right) ,$$

$$\xi^r = W + \frac{\epsilon^r}{r} + O\left(r^{-2}\right) ,$$

$$\xi^A = Y^A + \frac{\epsilon^A}{r} + \frac{\epsilon^A_{(2)}}{r^2} + O\left(r^{-3}\right) ,$$

where b, Y^A, T, W, ϵ, ϵ^r, ϵ^A and $\epsilon^A_{(2)}$ are arbitrary functions of the angles.

Asymptotic conditions

- This can be done by considering asymptotically locally flat spacetimes, where we now allow the angular components of the background metric to vary at infinity

$$
g_{rr} = 1 + \frac{\bar{h}_{rr}}{r} + O\left(r^{-2}\right), \quad g_{rA} = \bar{\lambda}_A + \frac{\bar{h}_{rA}}{r} + O\left(r^{-2}\right),
$$
$$
g_{AB} = r^2 \bar{G}_{AB} + r\bar{h}_{AB} + O\left(r^0\right)
$$
$$
\pi^{rr} = r\bar{P}^{rr} + \bar{\pi}^{rr} + O\left(r^{-1}\right), \quad \pi^{rA} = \frac{\bar{\pi}^{rA}}{r} + O\left(r^{-1}\right)
$$
$$
\pi^{AB} = \frac{\bar{P}^{AB}}{r} + \frac{\bar{\pi}^{AB}}{r^2} + O\left(r^{-3}\right)
$$

- This asymptotic conditions are invariant under super-Lorentz.

$\mathrm{Diff}(S^2) \ltimes S$

Symplectic structure

- The kinetic term in the Hamiltonian action yields the symplectic form

$$\Omega = \int d^3x d_V \pi^{ij} d_V h_{ij} \, .$$

- It is easy to check that the symplectic structure is divergent for our boundary conditions

$$\begin{aligned}\Omega = \int d^3x \Big[& r \, d_V \bar{P}^{AB} d_V \bar{G}_{AB} \\ & + \left(d_V \bar{P}^{AB} d_V \bar{h}_{AB} + d_V \bar{\pi}^{AB} d_V \bar{G}_{AB} + d_V \bar{P}^{rr} d_V \bar{h}_{rr} \right) \\ & + \frac{1}{r} \Big(d_V \bar{P}^{AB} d_V \bar{h}^{(2)}_{AB} + d_V \bar{\pi}^{AB} d_V \bar{h}_{AB} + \bar{\pi}^{AB}_{(2)} d_V \bar{G}_{AB} \\ & \quad + d_V \bar{P}^{rr} d_V \bar{h}^{(2)}_{rr} + d_V \bar{\pi}^{rr} d_V \bar{h}_{rr} \Big) + O\left(r^{-2}\right) \Big]\end{aligned}$$

Symplectic structure

- We can use the ambiguities present in the formalism to renormalize the symplectic structure, by adding the boundary terms to the action

$$
\begin{aligned}
\int_{\Sigma_r \to \infty} d^3x \Big[& r\, \bar{P}^{AB} \partial_t \bar{G}_{AB} \longrightarrow i(\Omega_r) \sim \text{constraints.} \\
& + \left(\bar{P}^{AB} \partial_t \bar{h}_{AB} + \bar{\pi}^{AB} \partial_t \bar{G}_{AB} + \bar{P}^{rr} \partial_t \bar{h}_{rr} \right) \longrightarrow i(\Omega) \sim r\, \delta B_r^{\text{GRAV}} \\
& + \frac{1}{r} \left(\bar{P}^{AB} \partial_t \bar{h}^{(2)}_{AB} + \bar{\pi}^{AB} \partial_t \bar{h}_{AB} + \bar{\pi}^{AB}_{(2)} \partial_t \bar{G}_{AB} + \bar{P}^{rr} \partial_t \bar{h}^{(2)}_{rr} + \bar{\pi}^{rr} \partial_t \bar{h}_{rr} \right) \Big]
\end{aligned}
$$

$\hookrightarrow i(\Omega_{\log}) \sim \text{constraints}$

- The renormalization procedure makes the charge finite!

- Unfortunately, the charge is non-integrable.

Surface terms and charge generators

• The generator of the transformations can be obtained from $i_\xi \Omega = -d_V P_\xi^{grav}$ which in this setup is equivalent to the Regge-Teitelboim approach

$$P_\xi^{grav}[g_{ij}, \pi^{ij}] = \int d^3x \, \left(\xi \mathcal{H} + \xi^i \mathcal{H}_i\right) + \mathcal{B}_\xi^{grav}[g_{ij}, \pi^{ij}]$$

where the boundary term can be determined by integrating

$$\begin{aligned} d_V \mathcal{B}_\xi &= -\oint G^{ijkl}(\xi (d_V g_{ij})_{|k} - \xi_{,k} d_V g_{ij}) d^2 S_l \\ &\quad -\oint \left(2\xi_k d_V \pi^{kl} + (2\xi^k \pi^{jl} - \xi^l \pi^{jk}) d_V g_{jk}\right) d^2 S_l \end{aligned}$$

and G^{ijkl} is De Witt supermetric,

$$G^{ijkl} = \sqrt{g}\Big(\frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - g^{ij}g^{kl}\Big).$$

Charge

$$
\begin{aligned}
\mathcal{B}_\xi = \int d^2x d_V \Big\{ & 2Y^A \left[\bar{G}_{AB}\bar{\pi}^{rB}_{(2)} + \bar{h}_{AB}\bar{\pi}^{rB}\right] + 2W\left[\bar{\pi}^{rr} - \bar{\pi}^A_A + \bar{h}_{rr}\bar{P}^{rr} - \bar{P}^{AB}\bar{h}_{AB}\right] \\
& + \tilde{T}\sqrt{\bar{G}}\bar{h}_{rr} + \epsilon^r\bar{P} + \tilde{\epsilon}\sqrt{\bar{G}} + b\left[\sqrt{\bar{G}}\left(2\bar{k}^{(2)} + \bar{k}^2 + \bar{k}^A_B\bar{k}^B_A - 3\bar{h}_{rr}\bar{k}\right) + \frac{2}{\sqrt{\bar{G}}}\bar{\pi}^{rA}\bar{\pi}^r_A\right]\Big\} \\
+ \int d^2x \Big\{ & d_V\bar{P}^{AB}\left[2W\bar{k}_{AB} + \epsilon^r\bar{G}_{AB}\right] \\
& + d_V\bar{G}_{AB}\left[b\sqrt{\bar{G}}\left(\bar{k}^{(2)AB} + \frac{1}{4}(\bar{h} - 3\bar{h}_{rr})\bar{k}^{AB}\right)\right. \\
& \left. + \frac{2b}{\sqrt{\bar{G}}}\bar{\pi}^{rA}\bar{\pi}^{rB} + W(-\bar{\pi}^{AB} + \bar{h}_{rr}\bar{P}^{AB}) + \frac{1}{4}\tilde{T}\sqrt{\bar{G}}\bar{k}^{AB}\right]\Big\}
\end{aligned}
$$

where

$$\tilde{T} = 2T + b(\bar{h} + 3\bar{h}_{rr})$$

$$\tilde{\epsilon} = -4\epsilon + b\left(-\frac{13}{5}\bar{k}^{(2)} - \bar{k}^2 - \frac{11}{5}\bar{k}^A_B\bar{k}^B_A + \frac{9}{10}\bar{h}_{rr}\bar{k} - \frac{2}{5}\bar{h}^{(2)}_{rr} + \frac{3}{5}\bar{h}^{(2)} + \frac{2}{\bar{G}}\bar{\pi}^{rA}\bar{\pi}^r_A\right) - 2T\bar{k}$$

- The generator is ***finite***, but ***non-integrable***.

Integrability of the charge

- There are in principle two ways of integrating the charge.

a. We can introduce a couple of boundary fields C_{AB} and F^{AB} along with a modification of the symplectic structure

$$\tilde{\Omega} = \int d^2x \left(d_V F^{AB} \wedge d_V G_{AB} + d_V \bar{P}^{AB} \wedge d_V C_{AB} \right)$$

such that $i_\xi \tilde{\Omega}$ makes the charge integrable.

b. We can follow the same procedure with fields that are already present in the A.C.

↳ Candidates: $\bar{h}^{(2)}_{AB}$ and $\bar{\Pi}^{AB}_{(2)}$

• We can integrate the charge by imposing the following transformation law for the fields C_{AB} and F^{AB}

$$
\begin{aligned}
\mathcal{B}_\xi = \int d^2x d_V \Big\{ & 2Y^A \left[\bar{G}_{AB}\bar{\pi}^{rB}_{(2)} + \bar{h}_{AB}\bar{\pi}^{rB}\right] + 2W\left[\bar{\pi}^{rr} - \bar{\pi}^A_A + \bar{h}_{rr}\bar{P}^{rr} - \bar{P}^{AB}\bar{h}_{AB}\right] \\
& + \tilde{T}\sqrt{\bar{G}}\bar{h}_{rr} + \epsilon^r \bar{P} + \tilde{\epsilon}\sqrt{\bar{G}} + b\left[\sqrt{\bar{G}}\left(2\bar{k}^{(2)} + \bar{k}^2 + \bar{k}^A_B\bar{k}^B_A - 3\bar{h}_{rr}\bar{k}\right) + \frac{2}{\sqrt{\bar{G}}}\bar{\pi}^{rA}\bar{\pi}^r_A\right]\Big\} \\
& + \int d^2x \Big\{ d_V \bar{P}^{AB} \underbrace{\left[2W\bar{k}_{AB} + \epsilon^r \bar{G}_{AB}\right]}_{\delta_\xi \bar{C}_{AB}} \\
& \qquad + d_V \bar{G}_{AB} \underbrace{\left[b\sqrt{\bar{G}}\left(\bar{k}^{(2)AB} + \frac{1}{4}(\bar{h} - 3\bar{h}_{rr})\bar{k}^{AB}\right) + \frac{2b}{\sqrt{\bar{G}}}\bar{\pi}^{rA}\bar{\pi}^{rB} + W(-\bar{\pi}^{AB} + \bar{h}_{rr}\bar{P}^{AB}) + \frac{1}{4}\tilde{T}\sqrt{\bar{G}}\bar{k}^{AB}\right]}_{\delta_\xi F^{AB}} \Big\}
\end{aligned}
$$

• This allows us to to get rid of the non-integrable pieces in the charge by means of the term in the contraction of the symplectic form proportional to

$$
i_\xi \tilde{\Omega} \sim \int \left(\underline{\delta_\xi F^{AB}} d_V \bar{G}_{AB} - d_V P^{AB} \underline{\delta_\xi C_{AB}}\right).
$$

- We still have to deal with the contributions coming from

$$i_\xi \tilde{\Omega} \sim \int \left(\delta_\xi P^{AB} d_V C_{AB} - d_V F^{AB} \delta_\xi \bar{G}_{AB} \right).$$

- The computation leads to the following contribution to the charge

$$\tilde{\mathcal{B}}_\xi = -\int d^2x d_V \left\{ b\mathcal{H}^{(C,F)} + Y^A \mathcal{H}_A^{(C,F)} \right\},$$

where $\mathcal{H}^{(C,F)}$ and $\mathcal{H}_A^{(C,F)}$ correspond to the Hamiltonian and Momentum constraints for linearized gravity in a curved background with a positive cosmological constant

$$\begin{aligned}
\mathcal{H}^{(C,F)} =& -\sqrt{\bar{G}} \left(\bar{D}^A \bar{D}^B C_{AB} - \triangle \bar{C} - \bar{R}^{AB} C_{AB} + \frac{1}{2} C^{AB} \bar{R} \right) - \sqrt{\bar{G}}\bar{C} \quad \Lambda = \frac{1}{2} \\
&+ \frac{2}{\sqrt{\bar{G}}} \left(F^{AB} P_{AB} - \bar{P}\bar{F} + P^{AC} P^B{}_C C_{AB} - \bar{P} C_{AB} P^{AB} \right) \\
&- \frac{1}{2\sqrt{\bar{G}}} \left(P^{AB} P_{AB} - \bar{P}^2 \right) \bar{C}, \\
\mathcal{H}_A^{(C,F)} =& -2\bar{D}_B F^B{}_A + P^{BC} \left(\bar{D}_A C_{BC} - 2\bar{D}_B C_{CA} \right).
\end{aligned}$$

- The transformation laws of the boundary fields have to be supplemented by precisely the (Hamiltonian) transformation laws of a spin-2 field in a curved background.

- Thus, the final expression for the charge is integrable and found to be

odd odd

$$\mathcal{B}_\xi = \oint d^2x \Big[2\underline{Y}^A (\bar{G}_{AB}\bar{\pi}^{rB}_{(2)} + \bar{h}_{AB}\bar{\pi}^{rB}) + 2\underline{W}(\bar{\pi}^{rr} - \bar{\pi} + \bar{h}_{rr}\bar{P} - \bar{P}^{AB}\bar{h}_{AB})$$
$$+ 2\underline{T}\sqrt{\bar{G}}\bar{h}_{rr} + \underline{b}\sqrt{\bar{G}}(2\bar{k}^{(2)} + \bar{k}^2 + \bar{k}^A_B\bar{k}^B_A - 3\bar{h}_{rr}\bar{k}) + \frac{2b}{\sqrt{\bar{G}}}\bar{\pi}^{rA}\bar{\pi}^r_A \Big] + \bar{\mathcal{B}}_\xi$$

EVEN odd

$$\left[\delta\sqrt{\bar{G}} = \bar{D}_A Y^A - \frac{b}{\sqrt{\bar{G}}}\bar{P} = 0\right]$$

- Finally, the generator of the asymptotic symmetries is given by

$$\mathcal{G}_\xi[g_{ij}, \pi^{ij}] = \int d^3x \left(\xi\mathcal{H} + \xi^i\mathcal{H}_i\right) + \mathcal{B}_\xi$$

• The asymptotic symmetry algebra is

$$\{\mathcal{G}_{\xi_1}, \mathcal{G}_{\xi_2}\}[g_{ij}, \pi^{ij}] \approx \mathcal{G}_{\hat{\xi}}[g_{ij}, \pi^{ij}] + \Lambda_{\xi_1,\xi_2}[g_{ij}, \pi^{ij}]$$

where $\hat{\xi}$ is parametrized by

$$\begin{aligned}
\hat{Y}^A &= Y_1^B \partial_B Y_2^A + \bar{G}^{AB} b_1 \partial_B b_2 - (1 \leftrightarrow 2), \\
\hat{b} &= Y_1^A \partial_A b_2 - (1 \leftrightarrow 2), \\
\hat{T} &= Y_1^A \partial_A T_2 - 3 b_1 W_2 - \bar{G}^{AB} \partial_A b_1 \bar{D}_B W_2 \\
&\quad - b_1 \bar{G}^{AB} \bar{D}_A \bar{D}_B W_2 - (1 \leftrightarrow 2), \\
\hat{W} &= Y_1^A \partial_A W_2 - b_1 T_2 - (1 \leftrightarrow 2).
\end{aligned}$$

and the nonlinear term

$$\Lambda_{\xi_1,\xi_2} = 2 \oint d^2x \big(b_1 T_2 - b_2 T_1\big) \bar{P} \bar{h}_{rr}.$$

• Despite the resemblance with the previous results, the structure constants now depend explicitly on $\bar{G}_{AB}$ and $\bar{D}_A Y^A = \frac{b}{\sqrt{\bar{G}}} \bar{P}$ thus forming a Lie algebroid.

Table of Contents

Comments and Remarks

- There is another potential way of solving the integrability problem by means of the sub-sub-leading terms in the asymptotic expansion, i.e. $\bar{h}^{(2)}_{AB}$ and $\bar{\pi}^{AB}_{(2)}$. (Still in progress but promising)

- It will be important to understand the connection with the results at null infinity.

- Canonical realization of the superrotation symmetry of Barnich and Troessaert ($(Vir \times Vir) \ltimes \mathcal{S}$) at spatial infinity.

- Supergravity, Higher spacetime dimensions, etc...