Noether-Wald charge in supergravity, fermions, and Killing supervector in superspace.

Igor Bandos

Department of Physics and EHU Quantum Center, University of the Basque Country, UPV/EHU, Bilbao, Spain,
& IKERBASQUE, the Basque Foundation for Science, Bilbao, Spain

based on 2305.10617 [hep-th] with Tomás Ortín

GRASS-SYMBHOL Meeting, Ávila, Spain, November 16, 2023

Outline

Outline

Killing vectors and Killing equation in GR

- In GR a Killing vector $k_\mu(x)$ parametrizes diffeomorphism which leaves the solution of the Einstein equation invariant,

$$-\delta_k g_{\mu\nu} = \nabla_\mu k_\nu + \nabla_\mu k_\mu =: 2\nabla_{(\mu} k_{\nu)} = 0 \ .$$

  Hence the standard form of the Killing equation is

$$\nabla_{(\mu} k_{\nu)} = 0 \ .$$

- The tetrade (vielbein) formalism with $g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu$ this corresponds to the statement that the vielbein 1-form $e^a = dx^\mu e^a_\mu$ is left invariant up to gauge transformations, SO(1,3) local Lorentz transformations,

$$-\delta_k e^a = \mathcal{L}_k e^a = \iota_k de^a + dk^a = e^b L_{(k)b}{}^a \ .$$

  In a manifestly covariant form, using the spin connection $\omega^{ab} = dx^\mu \omega^{ab}_\mu = e^c \omega_c{}^{ab}$, covariant derivative $\mathcal{D}k^a = e^b \mathcal{D}_b$ and torsion $\mathcal{D}e^a = de^a - e^b \wedge \omega_b{}^a = T^a = \frac{1}{2} e^c \wedge e^b T_{bc}{}^a$, eq. $-\delta_k e^a = 0$ reads

$$\boxed{\mathcal{D}k^a + \iota_k T^a = e^b P_{(k)b}{}^a} \ , \qquad P_{(k)}{}^{ab} = -\iota_k \omega^{ab} + L_{(k)b}{}^a \ .$$

Killing equation and momentum map

- In this equivalent form of the Killing equation,

$$\boxed{\mathcal{D}k^a + \iota_k T^a = e^b P_{(k)b}{}^a} \,,$$

- $P_{(k)}{}^{ab} = -\iota_k\omega^{ab} + L_{(k)b}{}^a = -P_{(k)}{}^{ba}$ is called *momentum map*,

$$\iota_k\omega^{ab} = k^\mu \iota_k\omega_\mu{}^{ab} = -\iota_k\omega_\mu{}^{ba} \,, \qquad \iota_k T^a = e^c k^b T_{bc}{}^a \,,$$

$$\text{and} \qquad \mathcal{D}k^a = e^b \mathcal{D}_b k^a \,,$$

- so that, when $T^a = 0$, all the contributions in the expression for $\mathcal{D}_b k^a$ become antisymmetric and thus $\mathcal{D}_{(b}k_{a)} = 0$ follows,

$$\mathcal{D}_{(b}k_{a)} = 0 \qquad \Leftarrow \qquad T^a = 0 \,.$$

- This is equivalent to $\nabla_{(\mu}k_{\nu)} = 0$ as far as the spin connection and affine connection obey 'metricity conditions',

$$\begin{aligned} \nabla_\mu e^a_\nu \quad &= \quad \partial_\mu e^a_\nu + \Gamma_{\mu\nu}{}^\rho e^a_\rho - e^b_\nu \omega_{\mu b}{}^a = 0 \\ &\Rightarrow \qquad \nabla_\mu g_{\nu\rho} = 0 \,. \end{aligned}$$

- In the case of supergravity (SUGRA), the set of gauge symmetries includes also supersymmetry (SUSY) transformations which must be taken into account when defining the Killing vector. Also the presence of fermionic fields (gravitini) results in a modification of the Killing eqs.
- Such a modification of Killing equations in supergravity can be obtained within the spacetime component approach, but SUSY transformation properties of both Killing vector and Killing eqs remain obscure in it.
- In our 2305.10617 [hep-th] with Tomás Ortín we used the superfield approach to $\mathcal{N} = 1$ $D = 4$ supergravity to determine these properties and used the results of this stage to construct a new Noether-Wald charge of SUGRA with fermionic contributions.
- For $\mathcal{N} = 2$ supergravity, where the black hole solutions with fermionic hairs exist, such a charge should contribute in the generalization of the black hole thermodynamics.
- Our results in 2305.10617 [hep-th], obtained in the frame of $\mathcal{N} = 1$ SUGRA in $D = 4$, which will be the subject of this talk, can be considered as a proof of concept basis for a future study of such generalized thermodynamics.

The previous study of the related issues

- Our formalism to approach Noether-Wald charges [Lee, Wald 1990, Wald 1993, Iyer, Wald 1994] is close to [Barnich, Brandt 2002, 2003].
- To the best of our knowledge, the concept of Killing supervector was introduced in the book by Buchbinder and Kuzenko [1995uq]
- and developed
  - by Kuzenko with collaborators [2012vd,2015lca,2019tys]
  - by Howe and Lindström [2015bdd,2018lwu]
  - and, in pure spinor superstring perspective, by Chandia with collaborator [2022uyy].
- The problem of BH thermodynamics and charges in supergravity was adressed in 2020 by Aneesh, Chakraborty, Hoque and Virmani [2020fcr]. However, they have not taken into account supersymmetry when defining the Killing vector, which resulted in that their proposal for Noether-Wald charge is not supersymmetric invariant.

Outline

Supergravity action and gauge symmetries

- The first order action for SUGRA (which can be used also in 1.5 order formalism) is

$$S = \int_{M^4} \mathcal{L}_4(e^a, \psi^\alpha, \bar{\psi}^{\dot\alpha}, \omega^{ab}) \, ,$$
$$\mathcal{L}_4 = \frac{1}{2}\epsilon_{abcd}\mathcal{R}^{ab} \wedge e^c \wedge e^d + 4\mathcal{D}\psi \wedge \sigma^{(1)} \wedge \bar{\psi} - 4\psi \wedge \sigma^{(1)} \wedge \mathcal{D}\bar{\psi},$$

where

$$\mathcal{R}^{ab} = (d\omega - \omega \wedge \omega)^{ab} = \tfrac{1}{2} e^d \wedge e^c \mathcal{R}_{cd}{}^{ab}, \quad \psi^\alpha = dx^\mu \psi^\alpha_\mu = (\bar{\psi}^{\dot\alpha})^*,$$

$\sigma^{(1)} := (e^a \sigma_{a\alpha\dot\alpha})$, and, for simplicity, we have set $16\pi G_N^{(4)} = 1$.

- This action is invariant under three gauge symmetries: Lorentz SO(1,3), diffeomorphisms, and supersymemtry.
- The Noether currents for these gauge symmetries, $\mathcal{J}^\mu$, are trivially conserved which is to say their dual 3-forms are exact,

$$\mathcal{J} = \tfrac{1}{3!} e^c \wedge e^b \wedge e^a \epsilon_{abcd} \mathcal{J}^d = d\mathcal{Q} \qquad \Rightarrow \qquad d\mathcal{J} \equiv 0 \, .$$

- (Remember: the external differivative $d = dx^\mu \partial_\mu$ is nilpotent, $dd = 0$).

Lorentz symmetry

- More explicitly, under a gauge symmetry $\delta_{g.s.}\mathcal{L}|_{\text{on-shell}} = d\mathcal{J}(\delta_{g.s.})$ and $\mathcal{J}(\delta_{g.s.}) = d\mathcal{Q}(\delta_{g.s.})$.
- In particular, for local Lorentz transformations

$$\delta_L e^a = e^b L_b{}^a \,, \qquad \delta_L \psi^\alpha = \tfrac{1}{4}\psi^\beta \sigma_{ab\beta}{}^\alpha L^{ab}$$

we find, after using $\boxed{T^a = \mathcal{D}e^a = -2i\psi \wedge \sigma^a \bar{\psi}}$ $(= -\frac{i}{2}\bar{\Psi} \wedge \gamma^a \Psi)$ (following from eqs for spin connection in 1st order formalism),

$$\mathcal{Q}(L) = -\tfrac{1}{2} L^{ab}\, \epsilon_{abcd}\, e^c \wedge e^d \,.$$

- If we consider $L^{ab} = \mathfrak{k}^{ab}$ such that, for a solution under consideration $\delta_{\mathfrak{k}} e^a = 0$ and $\delta_{\mathfrak{k}}\psi^\alpha = 0$, then

$$\mathfrak{Q}(\mathfrak{k}) = \int_{\Sigma^2} \mathcal{Q}(\mathfrak{k}) = -\frac{1}{32\pi G_N^{(4)}} \int_{\Sigma^2} \mathfrak{k}^{ab}\, \epsilon_{abcd}\, e^c \wedge e^d$$

is a conserved charge, $\boxed{d\mathfrak{Q}(\mathfrak{k}) \doteq 0}$ (where $\doteq$ denotes on-shell equality).

- Actually one can check that this charge can be conserved, $d\mathfrak{Q}(\mathfrak{k}) \doteq 0$, when $\delta_{\mathfrak{k}} e^a \neq 0$ but $\delta_{\mathfrak{k}}\psi^\alpha = 0$ and $\delta_{\mathfrak{k}}\omega^{ab} = 0$.

Supersymmetry

- For local supersymmetry

$$\delta_\epsilon e^a = -2i\psi\sigma^a\bar{\epsilon} + 2i\epsilon\sigma^a\bar{\psi} \;\left(= -i\bar{\epsilon}\gamma^a\psi\right) , \qquad \delta_\epsilon\psi = \mathcal{D}\epsilon , \quad \delta_\epsilon\bar{\psi} = \mathcal{D}\bar{\epsilon} ,$$

we find

$$\mathcal{Q}(\epsilon) = -4\epsilon\sigma^{(1)} \wedge \bar{\psi} + 4\psi \wedge \sigma^{(1)}\bar{\epsilon} \quad \left(\mathcal{Q}(\epsilon) = 2\bar{\psi} \wedge \gamma\gamma_5\epsilon\right) .$$

- If, we chose $\epsilon^\alpha = \kappa_s^\alpha$ such that, for a solution under consideration $\delta_{\kappa_s} e^a = 0$ and $\delta_{\kappa_s}\psi^\alpha = 0$, then

$$\mathfrak{Q}(\kappa_s) = \int_{\Sigma^2} \mathcal{Q}(\kappa_s) = \frac{1}{4\pi G_N^{(4)}} \int_{\Sigma^2}(-\epsilon\sigma^{(1)} \wedge \bar{\psi} + \psi \wedge \sigma^{(1)}\bar{\epsilon})$$

$$\left(\mathfrak{Q}(\kappa_s) = \frac{1}{8\pi G_N^{(4)}} \int_{\Sigma^2} \bar{\psi} \wedge \gamma\gamma_5\epsilon\right)$$

is a conserved charge, $\boxed{d\mathfrak{Q}(\kappa_s) \doteq 0}$.

Diffeomorphisms. Killing vectors in supergravity

- No apparent nonvanishing charge is associated with diffeomorphisms as own, but it is if we consider also some Lorentz and supersymmetry transformations induced by diffeomorphisms.
- But why we should introduce these?
- Because our aim is to consider the conserved charges associated with Killing vectors k^a and in SUGRA the definition of k^a should take into account the presence of both Lorenz symmetry and supersymmetry.
- The form of supersymmetry transformation of Killing vector is not manifest in spacetime component approach, but it is in superspace approach as we are going to show in a moment.
- This superspace (SSP) study shows the existence of superpartner κ^α of Killing vector k^a, which can be called *fermionic Killing spinor*,

$$\delta_\epsilon k^a = 2i\epsilon\sigma^a\bar{\kappa} - 2i\kappa\sigma^a\bar{\epsilon} \qquad \left(\delta_\epsilon k^a = -i\bar{\epsilon}\gamma^a\kappa\right) \ ,$$

- and this enters the Killing equation

$$\mathcal{D}k^a - 2i\psi\sigma^a\bar{\kappa} + 2i\kappa\sigma^a\bar{\psi} = e^b P_{(K)b}{}^a \ . \qquad \left(\mathcal{D}k^a + i\bar{\psi}\gamma^a\kappa = e^b P_{(K)b}{}^a\right) \ .$$

- The SSP approach also allows to find eqs. for κ (generalized Killing spinor equation), and find supersymmetry properties of Killing equation.

Outline

Curved supergravity superspace. Generalities.

- In curved superspace with coordinates

$$Z^M = (x^\mu, \theta^{\check{\underline{\alpha}}}) \, , \qquad \mu = 0,1,2,3 \, , \qquad \check{\underline{\alpha}} = 1,2,3,4$$

simple, $\mathcal{N} = 1$ D=4, supergravity is described by supervielbein 1-forms

$$E^A = dZ^M E_M^A(Z) = (E^a, E^{\underline{\alpha}}) = (E^a, E^\alpha, E^{\dot{\alpha}})$$

($\alpha = 1,2$ and $\dot{\alpha} = 1,2$ are Weyl spinor indices, $\underline{\alpha} = 1,2,3,4$ is Majorana spinor index) and spin connection $\omega^{ab} = dZ^M \omega_M^{ab}$.

- obeying the set of constraints imposed on torsion and curvature

$$T^A = DE^A = dE^A - E^B \wedge \omega_B{}^A = \tfrac{1}{2} E^C \wedge E^B T_{BC}{}^A \, ,$$
$$R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b = \tfrac{1}{2} E^D \wedge E^C R_{CD}{}^{ab}$$
$$\omega_B{}^C = \begin{pmatrix} \omega_b{}^a & 0 \\ 0 & \frac{1}{4}\omega^{cd}\Gamma_{cd} \end{pmatrix} = \begin{pmatrix} \omega_b{}^a & 0 \\ 0 & \frac{1}{4}\not{\omega} \end{pmatrix} .$$

- We will present these constraints later, but now stress that local spacetime supersymmetry of SUGRA comes from superdiffeomorphism invariance of the SSP formalism.

Killing supervectors

- Killing supervector in curved superspace of any bosonic and fermionic dimensions

$$K^A = (K^a, K^{\underline{\alpha}}) \ , \qquad a = 0, 1, ..., (D-1) \ , \qquad \underline{\alpha} = 1, ..., \mathcal{N} n_D$$

- is defined by the conditions

$$-\delta_K E^A = \boxed{DK^A + \iota_K T^A + E^B \iota_K \omega_B{}^A = E^B L_{(K)B}{}^A} \ ,$$

$$-\delta_K \omega^{ab} = \boxed{D\iota_K \omega^{ab} + \iota_K R^{ab} = DL_K^{ab}} \ ,$$

where

$$\omega_B{}^C = \begin{pmatrix} \omega_b{}^a & 0 \\ 0 & \frac{1}{4}\omega^{cd}\Gamma_{cd} \end{pmatrix} = \begin{pmatrix} \omega_b{}^a & 0 \\ 0 & \frac{1}{4}\not{\omega} \end{pmatrix}$$

- $\omega^{ab} = dZ^M \omega_M^{ab}(Z) = E^C \omega_C^{ab}(Z)$ is the spin connection 1 form in superspace, $\iota_K \omega^{ab} = K^C \omega_C^{ab}(Z)$, $\quad \iota_K T^A = E^C K^B T_{BC}^A(Z)$ and

$$L_B{}^C = \begin{pmatrix} L_b{}^a(Z) & 0 \\ 0 & \frac{1}{4}L^{cd}(Z)\Gamma_{cd} \end{pmatrix} .$$

Momentum map in superspace

- We find convenient to define the following momentum map

$$P_{(K)B}{}^{A} = -\iota_K \omega_B{}^{A} + L_B{}^{A}(K) = \begin{pmatrix} P_{(K)b}{}^{a} & 0 \\ 0 & \frac{1}{4} P^{cd}_{(K)} \Gamma_{cd} \end{pmatrix} = \begin{pmatrix} P_{(K)b}{}^{a} & 0 \\ 0 & \frac{1}{4} \not{P}_{(K)} \end{pmatrix}$$

  and write the above superKilling equations as

$$DK^{A} + \iota_K T^{A} = E^{B} P_{(K)B}{}^{A} \, ,$$

$$DP_{(K)}{}^{ab} = \iota_K R^{ab} \equiv E^{D} K^{C} R_{CD}{}^{ab}$$

- where

$$D = E^{A} D_{A} = E^{a} D_{a} + E^{\underline{\alpha}} D_{\underline{\alpha}}$$

Constraints of simple supergravity, its superspace torsion and curvature

- The curved superspace of simple, i.e. $\mathcal{N}=1$, $D=4$ supergravity,

$$D = E^A D_A = E^a D_a + E^\alpha D_\alpha + E^{\dot\alpha} D_{\dot\alpha} \; , \qquad \begin{cases} a = 0,1,2,3 \\ \alpha = 1,2, \qquad \dot\alpha = 1,2 \; , \end{cases}$$

is defined by the torsion constraints which result in

$$\begin{aligned} T^a &= DE^a = -2iE^\alpha \wedge \bar{E}^{\dot\alpha} \sigma^a_{\alpha\dot\alpha} \; , \\ T^\alpha &= DE^\alpha = \frac{1}{2} E^c \wedge E^b T_{bc}{}^\alpha \; , \\ R^{ab} &= -2i\sigma^{(1)}_{\alpha\dot\beta} \wedge E^{\dot\beta} T^{ab\alpha} + 2iE^\beta \wedge \sigma^{(1)}_{\beta\dot\alpha} T^{ab\dot\alpha} + \frac{1}{2} E^d \wedge E^c R_{cd}{}^{ab} \; , \end{aligned}$$

- where the superfield generalization of the gravitino field strength, $T_{bc}{}^\alpha$ obeys the superfield generalization of the Rarita -Schwinger eq

$$\epsilon^{abcd} T_{ab}{}^\alpha \sigma_{c\alpha\dot\beta} = 0 \qquad \Rightarrow \qquad \begin{cases} T_{ab}{}^\alpha = \frac{i}{2} \epsilon_{abcd} T^{cd\alpha} \; , \\ T_{ab}{}^{\dot\alpha} = -\frac{i}{2} \epsilon_{abcd} T^{cd\dot\alpha} \; , \end{cases} \qquad etc.,$$

and $R_{cd}{}^{ab}(Z)$ obeys the superfield generalization of the Einstein eq.

$$R_{ab}{}^{cb}(Z) - \frac{1}{2} \delta_a{}^c R_{ef}{}^{ef}(Z) = 0.$$

Generalized action of the Rheonomic approach to supergravity

- Where the above constraints comes from? One of the ways:
- they could be obtained from the generalized action [T. Regge, Neeman, D'Auria, Fré, Castellani, 80th]

$$S_{gen.act.} = \int_{\mathcal{M}^4:\theta=\theta(x)} \mathcal{L}_4(E^a(Z), E^\alpha(Z), \bar{E}^{\dot\alpha}(Z), \omega^{ab}(Z)) ,$$

$$\mathcal{L}_4 = \frac{1}{2}\epsilon_{abcd} R^{ab}(Z) \wedge E^c \wedge E^d + 4DE^\alpha \wedge \sigma^{(1)}_{\alpha\dot\alpha} \wedge \bar{E}^{\dot\alpha} + c.c.,$$

- which can be obtained from the 1st-order action by substituting

$$e^a(x) \mapsto E^a(x,\theta), \quad \psi^\alpha(x) \mapsto E^\alpha(x,\theta), \quad \omega^{ab}(x) \mapsto dZ^M \omega_M^{ab}(Z), \quad \mathcal{D} \mapsto D$$

in the Lagrangian form and replacing the integration over spacetime

$$M^4 \in \Sigma^{(4|4)} \; : \qquad \theta = 0, \qquad x = \text{arbitrary}$$

by an arbitrary surface of maximal bosonic dimensions in superspace determined by fermionic coordinate function $\theta(x)$,

$$\mathcal{M}^4 \in \Sigma^{(4|4)} \; : \qquad \theta = \theta(x), \qquad x = \text{arbitrary} .$$

Wess-Zumino gauge

- To pass to the spacetime component formulation of SUGRA, besides imposing the constraints, we should use the superdiffeomorphism symmetry and superspace local Lorentz symmetry to fix the Wess–Zumino gauge

$$\theta^{\underline{\check{\alpha}}} E_{\underline{\check{\alpha}}}{}^{A} = \theta^{\underline{\check{\alpha}}} \delta_{\underline{\check{\alpha}}}{}^{A} , \qquad \theta^{\underline{\check{\alpha}}} w_{\underline{\check{\alpha}}}{}^{ab} = 0 .$$

- or, equivalently

$$\iota_{\underline{\theta}} E^{a} = 0 , \qquad \iota_{\underline{\theta}} E^{\underline{\alpha}} = \theta^{\underline{\alpha}} , \qquad \iota_{\underline{\theta}} w^{ab} = 0 .$$

- This gauge is invariant under spacetime local Lorentz symmetry and spacetime supersymmetry only. In it

$$\theta^{\underline{\beta}} \mathcal{D}_{\underline{\beta}} =: \theta\mathcal{D} = \theta\partial := \theta^{\underline{\check{\alpha}}} \partial_{\underline{\check{\alpha}}} ,$$

$$E_N{}^{A}|_{\theta=0} = \begin{pmatrix} e^{a}_{\nu}(x) & \psi^{\underline{\alpha}}_{\nu}(x) \\ 0 & \delta_{\underline{\check{\beta}}}{}^{\underline{\alpha}} \end{pmatrix} , \qquad E_A{}^{N}|_{\theta=0} = \begin{pmatrix} e^{\nu}_{a}(x) & -\psi^{\underline{\check{\beta}}}_{a}(x) \\ 0 & \delta_{\underline{\alpha}}{}^{\underline{\check{\beta}}} \end{pmatrix} ,$$

- Notice that the curved superspace index $\underline{\check{\alpha}}$ can be identified as spinor index only after this gauge fixing.

More on WZ gauge

- Furthermore, in WZ gauge

$$(D_{\underline{\alpha}}(...))|_{\theta=0} = (E_{\underline{\alpha}}^{M}\partial_M(...))|_{\theta=0} = \partial_{\underline{\alpha}}(...)|_{\theta=0} \ ,$$

but

$$(D_a(...))|_{\theta=0} = (E_a^{M}\partial_M(...))|_{\theta=0} = e_a^{\mu}\partial_\mu((...)|) - \psi_a{}^{\underline{\alpha}}(D_{\underline{\alpha}}(...))| \ ,$$

and

$$\begin{aligned} T_{ab}{}^{\underline{\alpha}}|_{\theta=0} &= e_a^{\mu}e_b^{\nu}T_{\mu\nu}{}^{\underline{\alpha}}(x) - 2\psi_{[a|}{}^{\underline{\beta}}T_{\underline{\beta}|b]}{}^{\underline{\alpha}}|_0 - \psi_b{}^{\underline{\beta}}\psi_a{}^{\underline{\gamma}}T_{\underline{\gamma\beta}}{}^{\underline{\alpha}}|_{\theta=0} \ , \\ T_{ab}{}^{c}|_{\theta=0} &= e_a^{\mu}e_b^{\nu}T_{\mu\nu}{}^{c}(x) - 2\psi_{[a|}{}^{\underline{\beta}}T_{\underline{\beta}|b]}{}^{c}|_{\theta=0} - \psi_b{}^{\underline{\beta}}\psi_a{}^{\underline{\gamma}}T_{\underline{\gamma\beta}}{}^{c}|_{\theta=0} \ , \\ R_{cd}{}^{ab}|_{\theta=0} &= e_c^{\mu}e_d^{\nu}R_{\mu\nu}{}^{ab}(x) - 2\psi_{[c|}{}^{\underline{\alpha}}R_{\underline{\alpha}|d]}{}^{ab}|_{\theta=0} - \psi_d{}^{\underline{\beta}}\psi_c{}^{\underline{\alpha}}R_{\underline{\alpha\beta}}{}^{ab}|_{\theta=0} \ . \end{aligned}$$

- This results, in particular, in that the leading ($\theta=0$) component of the superfield generalization of 'free' Einstein equation gives the spacetime Einstein equation with energy-momentum tensor of gravitino,

$$\mathcal{R}_{ab}{}^{cb}(x) - \frac{1}{2}\delta_a{}^{c}\mathcal{R}_{ef}{}^{ef}(x) = 2\epsilon^{cdef}\left(-\mathcal{D}_d\psi_e\sigma_a\bar{\psi}_f + \psi_f\sigma_a\mathcal{D}_d\bar{\psi}_e\right) \ ,$$

- where $\mathcal{R}_{ab}{}^{cb}(x) = e_c^{\mu}e_d^{\nu}R_{\mu\nu}{}^{ab}(x)$.

SuperKilling equations of simple supergravity

- The superspace superKilling equations

$$DK^A + \iota_K T^A = E^B P_{(K)B}{}^A$$

- in the case of simple supergravity splits into

$$DK^a = 2iE^\alpha \sigma^a_{\alpha\dot\alpha} K^{\dot\alpha} + 2i\bar{E}_{\dot\alpha}\tilde\sigma^{a\dot\alpha\alpha} K_\alpha + E^b P_{(K)b}{}^a \, ,$$

$$DK^\alpha = -E^b K^a T_{ab}{}^\alpha + E^\beta P_{(K)\beta}{}^\alpha \, , \qquad P_{(K)\beta}{}^\alpha = -\frac{1}{4} P^{ab}_{(K)} \sigma_{ab\beta}{}^\alpha \, ,$$

  and c.c. of this latter.

- Using $D = E^A D_A = E^a D_a + E^\alpha D_\alpha + E^{\dot\alpha} D_{\dot\alpha}$ we find that the first of these eqs. in its turn splits into

$$\boxed{D_b K^a = P_{(K)b}{}^a \qquad \Rightarrow \qquad D^{(a}K^{b)} = 0} \, , \qquad P^{ab}_{(K)} = D^{[a}K^{b]}$$

  and

$$D_\alpha K^a = 2i\sigma^a_{\alpha\dot\alpha} K^{\dot\alpha} \, . \qquad \boxed{\bar{D}_{\dot\alpha} K^a = 2iK^\alpha \sigma^a_{\alpha\dot\alpha}} \, .$$

Outline

Complete Killing equation for supergravity

Denoting $K^a|_{\theta=0} = k^a(x)$, $K^\alpha|_{\theta=0} = \kappa^\alpha(x)$, $\mathcal{D} = dx^\mu D_\mu$,

- we find that the leading component of the first of these equations,

$$D_b K^a = P_{(K)b}{}^a \quad \Rightarrow \quad \begin{cases} D^{(a}K^{b)} = 0 \, , \\ P_{(K)}{}^{ab} = D^{[a}K^{b]} \, , \end{cases}$$

in the WZ gauge (where $(D_a(...))|_{\theta=0} = e_a^\mu \partial_\mu((...)|) - \psi_a{}^{\underline{\alpha}}(D_{\underline{\alpha}}(...))|)$ can be written in the form

$$\mathcal{D}k^a - 2i\psi\sigma^a\bar{\kappa} + 2i\kappa\sigma^a\bar{\psi} = e^b P_{(K)b}{}^a \quad \Rightarrow \begin{cases} \mathcal{D}^{(a}k^{b)} - 2i\psi^{(a}\sigma^{b)}\bar{\kappa} + c.c. = 0 \, , \\ P^{ab}_{(K)}(x) = \mathcal{D}^{[a}k^{b]} - 2i\psi^{[a}\sigma^{b]}\bar{\kappa} + c.c., \end{cases}$$

which gives the complete Killing equation of supergravity.

- The leading component of the second of the above equations, $D_\alpha K^a = 2i\sigma^a_{\alpha\dot{\alpha}} K^{\dot{\alpha}}$ define (through Lie derivative representation of superdiffeomorphism) the SUSY transformation of Killing vector

$$\delta_\epsilon k^a(x) = 2i\epsilon\sigma^a\bar{\kappa} - 2i\kappa\sigma^a\bar{\epsilon} \, .$$

Generalized Killing spinor equations for fermionic spinor

- The second superKilling equation

$$DK^{\alpha} = -E^bK^aT_{ab}{}^{\alpha} + E^{\beta}P_{(K)\beta}{}^{\alpha}\,, \qquad P_{(K)\beta}{}^{\alpha} = \tfrac{1}{4}P^{ab}_{(K)}\sigma_{ab\beta}{}^{\alpha}$$

splits into

$$\begin{aligned} D_bK^{\alpha} &= -K^aT_{ab}{}^{\alpha}\,, \\ D_{\beta}K^{\alpha} &= P_{(K)\beta}{}^{\alpha} := \frac{1}{4}P^{ab}_{(K)}\sigma_{ab\beta}{}^{\alpha}\,, \qquad \bar{D}_{\dot\beta}K^{\alpha} = 0\,. \end{aligned}$$

- The leading component of the first of these eqs in the WZ gauge leads to the generalized Killing spinor equation

$$\boxed{\mathcal{D}\kappa^{\alpha} = -\iota_k(\mathcal{D}\psi^{\alpha}) + \psi^{\beta}P_{(K)\beta}{}^{\alpha}}\,,$$

where $\iota_k(\mathcal{D}\psi^{\alpha}) = 2e^ck^b\mathcal{D}_{[b}\psi^{\alpha}_{c]}\,, \qquad P_{(K)\beta}{}^{\alpha} = \tfrac{1}{4}P^{ab}_{(K)}\sigma_{ab\beta}{}^{\alpha}\,,$

- while the leading component of the second and third equations encodes the SUSY transformation of generalized fermionic Killing spinor

$$\delta_{\epsilon}\kappa^{\alpha} = \epsilon^{\beta}P_{(K)\beta}{}^{\alpha} = \tfrac{1}{4}(\epsilon\sigma_{ab})^{\alpha}\left(\mathcal{D}^{[a}k^{b]} + 2i\kappa^{\gamma}(\sigma^{[a}\bar{\psi}^{b]})_{\gamma} + c.c.\right).$$

Superfield eqs for momentum map in simple SUGRA SSP and its consequences

- Similarly, the superspace equation for the momentum map

$$DP_{(K)}{}^{ab} = \iota_K R^{ab} \equiv E^D K^C R_{CD}{}^{ab}$$

- in the case of simple supergravity reads

$$DP^{ab}_{(K)} = K^c \left(E^d R_{cd}{}^{ab} + 2iE_{\dot\beta}\tilde\sigma_c^{\dot\beta\alpha} T^{ab}{}_\alpha + 2iE^\beta \sigma_{c\beta\dot\alpha} T^{ab\dot\alpha} \right) - -2iK_{\dot\beta}\tilde\sigma^{(1)\dot\beta\alpha} T^{ab}{}_\alpha - 2iK^\beta \sigma^{(1)}_{\beta\dot\alpha} T^{ab\dot\alpha}$$

- and encodes the spacetime equation for momentum map

$$\mathcal{D}P^{ab}_{(k,\kappa)} = \iota_k \mathcal{R}^{ab} - 2i((\kappa - \iota_k\psi)\sigma^{(1)})_{\dot\alpha} T^{ab\dot\alpha} + 2iT^{ab\alpha}(\sigma^{(1)}(\bar\kappa - \iota_k\bar\psi))_\alpha$$

- and its supersymmetry transformations

$$\delta_\epsilon P^{ab}_{(k,\kappa)} = +2ik^c \left((\epsilon\sigma_c)_{\dot\alpha} T^{ab\dot\alpha} + (\bar\epsilon\tilde\sigma_c)^\alpha T^{ab}{}_\alpha \right).$$

which can be also written as $\delta_\epsilon P^{ab}_{(k,\kappa)} = -\iota_k \delta_\epsilon \omega^{ab}$.

Eqs. and SUSY transformations and of superKilling muiltiplet

- Resuming, the supersymmetry transformations of Killing vector and its friends in simple supergravity read (in Majorana spinor notation)

$$\begin{aligned}
\delta_\epsilon k^a &= -i\bar{\epsilon}\gamma^a\kappa \, , \\
\delta_\epsilon \kappa &= -\not{P}_{(k,\kappa)}\epsilon = -\frac{1}{4}\left(\mathcal{D}^{[a}k^{b]} - i\bar{\psi}^{[a}\gamma^{b]}\kappa\right)\gamma_{ab}\epsilon \, , \\
\delta_\epsilon P^{ab}_{(k,\kappa)} &= -2i\bar{\epsilon}\,\not{k}\,\mathcal{D}^{[a}\psi^{b]} \, , \qquad \not{k} = k^a\gamma_a \, .
\end{aligned}$$

- They satisfy the following Killing, generalized Killing spinor and momentum map equations

$$\begin{aligned}
&\mathcal{D}k^a + i\bar{\psi}\gamma^a\kappa = e^b P_{(K)b}{}^a \, , \\
&\mathcal{D}\kappa = -\iota_k(\mathcal{D}\psi) - \not{P}_{(k,\kappa)}\,\psi \, , \qquad \not{P}_{(k,\kappa)} = \frac{1}{4}P^{ab}_{(k,\kappa)}\gamma_{ab} \, , \\
&\mathcal{D}P^{ab}_{(k,\kappa)} = \iota_k\mathcal{R}^{ab} + 2i(\bar{\kappa} - \iota_k\bar{\psi}))\gamma\mathcal{D}^{[a}\psi^{b]} \, , \qquad \gamma = e^a\gamma_a \, .
\end{aligned}$$

- The Killing and generalized Killing spinor equations form supermultiplet, i.e. their set is closed under the supersymmetry.

SUSY invariance of the supermultiplet of superKilling equations

- The supersymmetry transformations of the complete Killing equation of supergravity mixes it with generalized Killing spinor equation

$$\delta_\epsilon \left(\mathcal{D}k^a + i\bar{\psi}\gamma^a\kappa - e^b P_{(K)b}{}^a\right) = +i\bar{\epsilon}\gamma^a \left(\mathcal{D}\kappa + i_k(\mathcal{D}\psi) + \tfrac{1}{4}\not{P}_{(K)}\psi\right) \; .$$

- The SUSY transformations of generalized Killing spinor equation mixes this with the momentum map eq.

$$\delta_\epsilon \left(\mathcal{D}\kappa^\alpha + i_k(\mathcal{D}\psi) - \psi^\beta P_{(K)\beta}{}^\alpha\right) = \epsilon^\beta \left(\mathcal{D}P_{(K)\beta}{}^\alpha - i_k\mathcal{R}_\beta{}^\alpha - \delta_{\epsilon_k}\omega_\beta{}^\alpha\right) .$$

- Although it is tempting to state that this latter also enters the supermultiplet, this is not the case:
- $\mathcal{D}P^{ab}_{(K)} = i_k\mathcal{R}^{ab} + 2i(\bar{\kappa} - i_k\bar{\psi})\gamma\,\mathcal{D}^{[a}\psi^{b]} \equiv \mathcal{D}P_{(K)\beta}{}^\alpha - i_k\mathcal{R}_\beta{}^\alpha - \delta_{\epsilon_k}\omega_\beta{}^\alpha = 0$ can be obtained as selfconsistency conditions of Killing equation,
- so that we have actually the standard situation of supersymmetry transforming bosonic object (equation) in terms of fermionic one and fermionic object (equation) in terms of the derivative of the bosonic one.

Outline

Noether-Wald charge

- The Noether-Wald (NW) charge is the 2-form associated to the invariance under diffeomorphisms.
- The conserved charge, i.e. 2-form closed on-mass shell, should be associated with diffeomorphisms parametrized by Killing vectors.
- But, as we have seen, in supergravity the Killing vector appears accompanied by a supersymmetry and a Lorentz group transformations associated with it (induced by it).
- Thus to construct the correct Noether-Wald charge, we have to consider diffeomorphisms δ_ξ accompanied by (induced or associated) local Lorentz and local SUSY transformations, δ_{L_ξ} and δ_{ϵ_ξ}.
- Actually, only these two give a contribution to the total derivative term in the expression for on-shell variation of the Lagrangian, i.e. in the conserved current which is thus expressed in terms of momentum map $P_\xi^{ab} = -\iota_\xi \omega^{ab} + L_\xi^{ab}$ and fermionic ϵ_ξ,

$$\mathbf{J}[\xi, \epsilon_\xi] = \star(e^a \wedge e^b) \wedge \mathcal{D} P_{\xi\, ab} - \tfrac{1}{2} \bar{\psi} \wedge \gamma_5 \gamma \wedge \not{P}_\xi \psi - 2 \bar{\epsilon}_\xi \gamma_5 \gamma \wedge \mathcal{D}\psi + 2\mathcal{D}\bar{\epsilon}_\xi \wedge \gamma_5 \gamma \wedge \psi \,,$$

where $\star(e^a \wedge e^b) = \epsilon^{abcd} e_c \wedge e_d$.

- Furthermore, this form is exact,

$$\mathbf{J}[\xi,\epsilon_\xi] = \star(e^a \wedge e^b)\wedge \mathcal{D}P_{\xi\, ab} - \frac{1}{2}\bar{\psi}\wedge\gamma_5\gamma\wedge \not{P}_\xi \psi - 2\bar{\epsilon}_\xi\gamma_5\gamma\wedge\mathcal{D}\psi + 2\mathcal{D}\bar{\epsilon}_\xi\wedge\gamma_5\gamma\wedge\psi =$$

$$= d\mathbf{Q}[\xi,\epsilon_\xi]\,,$$

- where

$$\mathbf{Q}[\xi,\epsilon_\xi] = \star(e^a\wedge e^b)P_{\xi\, ab} + 2\bar{\epsilon}_\xi\gamma_5\gamma\wedge\psi$$

  is the Noether-Wald charge 2-form we were after.
- Indeed, it is manifestly invariant under diffs and local $SO(1,3)$.
- Furthermore, for Killing parameters (k,κ) it can be shown that it is on-shell closed and supersymmetry-invariant up to a total derivative

$$d\mathbf{Q}[k,\kappa] \doteq 0\,,$$

$$\delta_\epsilon \mathbf{Q}[k,\kappa] \doteq d\left(-2\bar{\kappa}\gamma_5\gamma\epsilon\right)\,.$$

- $\mathbf{Q}[k,\kappa]$ is, thus, the SUSY generalization of the Komar charge.
- Observe that it is the sum of the terms corresponding to the standard Komar charge ($\propto$ the momentum map $P_{\xi\, ab}$) and corresponding to the supercharge ($\propto \bar{\kappa}$) neither of which is separately invariant under SUSY.

Outline

Discussion

- Our definition of invariance under (super-) diffeomorphisms and the supersymmetric Noether-Wald charge provide a solid basis to study the supersymmetric thermodynamics of the black holes.
- Unavoidably, when they are present, fermions (gravitini) must play a role in it if local supersymmetry remains unbroken.
- As a general rule, gauge fields with gauge freedoms are expected to contribute to the Noether-Wald charge and, probably, other kinds of fermions will not contribute, although this needs to be proven.
- In order to develop such a supersymmetric thermodynamics
  - (themodynamics in superspace, actually, because, when fermions are present, bosonic fields such as the metric have body and soul)

  many notions of Lorentzian geometry need to be extended to this realm:
  - how are event horizons defined and characterized in this setting?
  - (How do particles move in this space?)
  - Do they coincide with (super-) Killing horizons?
  - If so, can one generalize the definition of surface gravity to them?
  - etc ...

Outlook

- Another important ingredient to make this possible generalization relevant is the existence of black-hole solutions with gravitino hair.
- It was proven that the only ones in $\mathcal{N}=1, d=4$ (Poincaré) supergravity are those whose gravitini can be removed by a supersymmetry transformation [Cordero+Teitelboim 78, Gueven 1980].
- Thus our present reasults can be considered as a proof of concept in this perspective.
- In $\mathcal{N}=2, d=4$ (Poincaré) supergravity things are different [Aichelburg+Gueven 81, Gueven 82] and a solution with non-pure-gauge gravitini built over the body of the extremal Reissner-Nordström black hole exists [Aichelburg+Gueven 83].
- Thus the natural next steps towards generalized Black hole (and Black brane) thermodynamics is the extension of our approach to $\mathcal{N}=1,2, d=4$ AdS supergravity and then to higher-dimensional supergravities.
- Work in this direction is currently going on [IB + Patrick Meessen + Tomás Otrín+: work in progress].

-
- THANK YOU FOR YOUR ATTENTION!
-