Gauge fixing in Einstein-Yang-Mills theory

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Objectives

- My goal with this talk is to give a pedagogical introduction to a canonical gauge fixing procedure in euclidean Einstein-Yang-Mills theory.
- The procedure can be applied to any gravity-coupled theory following exactly the same steps that I will follow for Einstein-Yang-Mills theory.
- In particular, it applies to Einstein and Yang-Mills theories, where it gives the expected results in terms of divergence-free tensors.
- Ultimately, our goal is to apply the procedure to the study of supersymmetric initial data sets in supergravity on compact Cauchy hypersurfaces.

Context and motivation

A semi-classical theory on a category of geometric objects Φ is defined via:

- Its configuration space $\mathrm{Conf}(\Phi)$, an infinite-dimensional manifold.
- A smooth map $\mathcal{E}\colon \mathrm{Conf}(\Phi) \to \mathcal{W}$ into an auxiliary target space $\mathcal{W}$.
- An automorphism group $\mathcal{G}(\Phi)$ of Φ equipped with a natural action both on $\mathrm{Conf}(P)$ and $\mathcal{W}$ respect to which the smooth map $\mathcal{E}$ is equivariant.

$\mathcal{E}^{-1}(0) \subset \mathrm{Conf}(P)$ is the solution space of the theory whereas:

$$\mathscr{B}(\Phi) = \mathrm{Conf}(\Phi)/\mathcal{G}(\Phi)$$

is the *physical* configuration space, where *gauge-related* configurations are identified. Similarly, $\mathcal{E}^{-1}(0)/\mathcal{G}(\Phi) \subset \mathscr{B}(\Phi)$ is the moduli space of solutions. No global gauge fixing $\xi\colon \mathscr{B}(\Phi) \to \mathrm{Conf}(\Phi)$ possible in general: Gribov ambiguity problem!

- Understanding $\mathscr{B}(\Phi)$ is of fundamental physical importance, since path integrals and partition functions are computed via integration on $\mathscr{B}(\Phi)$.
- Understanding $\mathscr{B}(\Phi)$ is of fundamental mathematical: it has been used to define diverse smooth invariants via the study of $\mathcal{E}^{-1}(0)/\mathcal{G}(\Phi) \subset \mathscr{B}(\Phi)$.

Example: euclidean Maxwell theory

- Underlying object: $\Phi = \pi\colon P \to M$ principal $\mathrm{U}(1)$ bundle on a manifold M.
- $\mathrm{Conf}(P) = \mathcal{A}_P$, the affine space of connection on P.
- Symmetry group $\mathcal{G}(P)$: *gauge transformations* of P, which consist of the group of all automorphisms of P covering the identity.

Since $\mathrm{U}(1)$ is abelian we have $\mathcal{G}(P) = \mathcal{C}^\infty(M, \mathrm{U}(1))$, which acts naturally on $\mathcal{A}_P$:

$$\mathcal{A}_P \times \mathcal{G}(P) \to \mathcal{A}_P \,, (A, u) \mapsto u^*(A) = A + u^{-1}\mathrm{d}u$$

The theory is defined via the smooth map $\mathcal{E}\colon \mathcal{A}_P \to \Omega^1(\mathfrak{g}_P)\,,\ A \mapsto \mathrm{d}_A^{\mathfrak{g}*} F_A$.

- YM connections: $\mathcal{E}^{-1}(0) \subset \mathcal{A}_P$. Moduli space: $\mathcal{E}^{-1}(0)/\mathcal{G}(P) \simeq \frac{H^1(M,\mathbb{R})}{H^1(M,\mathbb{Z})}$
- If M is simply connected, then $\mathscr{B}(P) \simeq \Omega^1(M)/\Omega^1_{ex}(M)$.

Note that $\mathcal{E}^{-1}(0)/\mathcal{G}(P)$ is a topological invariant of the underlying M!

The euclidean EYM system is determined by the following data:

- A principal bundle P over a connected and compact manifold M.
- An inner product c on the adjoint bundle $\mathfrak{g}_P$ of P.

The EYM functional determined by (P, c) is:

$$\mathcal{S}_{P,\mathfrak{c}}\colon \mathrm{Conf}(P) \to \mathbb{R}\,, \quad (g, A) \mapsto \mathcal{S}_{P,\mathfrak{c}}[g, A] = \int_M \left\{ s^g + \kappa |F_A|^2_{g,\mathfrak{c}} \right\} \nu_g\,,$$

where $\mathrm{Conf}(P) = \mathrm{Met}(M) \times \mathcal{C}(P)$ and $\kappa \in \mathbb{Z}_2$.

The variational problem of S defines the Einstein-Yang-Mills system:

$$\mathbb{G}^g = \mathcal{T}(g, A) = \frac{\kappa}{2} |F_A|^2_{g,\mathfrak{c}}\, g - \kappa F_A \circ_{g,c} F_A\,, \quad \mathrm{d}^{g*}_A F_A = 0$$

where $\mathbb{G}^g = \mathrm{Ric}^g - \frac{1}{2} s^g g$ and d^{g*}_A is the formal adjoint of d_A.

The symmetry group I

Automorphism group $\mathrm{Aut}(P)$: group of equivariant diffeomorphisms of P, that is:

$$\mathrm{Aut}(P) := \{u \in \mathrm{Diff}(P) \mid u(po) = u(p)o \;\; \forall\, o \in \mathrm{G}\} \; .$$

Every equivariant diffeomorphism $u \in \mathrm{Aut}(P)$ covers a unique diffeomorphism $f_u \colon M \to M$ of M. If the identity, we obtain the gauge group $\mathcal{G}(P) \subset \mathrm{Aut}(P)$:

$$\mathcal{G}(P) := \{u \in \mathrm{Aut}(P) \mid \pi \circ u = \pi\} \; , \quad 1 \to \mathcal{G}(P) \to \mathrm{Aut}(P) \to \mathrm{Diff}'(M) \to 1$$

The Lie algebra of $\mathrm{Aut}(P)$ is given by the G-invariant vector fields $\mathfrak{X}(P)^{\mathrm{G}}$ on P.

$$0 \to \Gamma(\mathfrak{g}_P) \to \mathfrak{X}(P)^{\mathrm{G}} = \Gamma(\mathcal{A}_P) \to \mathfrak{X}(M) \to 0 \, , \quad \mathcal{A}_P = TP/\mathrm{G}$$

A choice of connection gives $\Gamma(\mathcal{A}_P) = \mathfrak{X}(M) \oplus \Gamma(\mathfrak{g}_P)$ with bracket:

$$[v_1 + \tau_1, v_2 + \tau_2]_P = [v_1, v_2] + \nabla^A_{v_1}\tau_2 - \nabla^A_{v_2}\tau_1 + \mathcal{R}^A(v_1, v_2) + [\tau_1, \tau_2]_{\mathfrak{g}_P}$$

This gives an explicit realization of the Lie algebra $\mathfrak{aut}(P)$ of $\mathrm{Aut}(P)$.

The symmetry group II

$\mathrm{Diff}(P)$ is a tame Fréchet Lie group by Leslie and Hamilton. Charts are constructed using the exponential map associated to an auxiliary Riemannian metric on P.

Proposition

The automorphism group $\mathrm{Aut}(P) \subset \mathrm{Diff}(P)$ *is a closed tame Fréchet Lie subgroup of* $\mathrm{Diff}(P)$ *locally modeled on the tame Fréchet Lie algebra* $\mathfrak{X}(P)^{\mathrm{G}}$ *with the standard bracket of vector fields or, equivalently,* $\Gamma(\mathcal{A}_P)$ *with Lie bracket* $[\cdot,\cdot]_{\mathcal{A}_P}$.

Proof.

Prove that $\mathfrak{X}(P)^{\mathrm{G}}$ is closed in $\mathfrak{X}(P)$ and admits a closed complement. Appropriately restrict the exponential map of a G-invariant metric on P. □

Similarly, the subgroup $\mathcal{G}(P) \subset \mathrm{Aut}(P)$ is a closed tame Fréchet Lie subgroup of $\mathrm{Aut}(P)$ locally modeled on the tame Fréchet space $\Gamma(\mathfrak{g}_P)$ with Lie bracket $[\cdot,\cdot]_{\mathfrak{g}_P}$.

The moduli space of Einstein-Yang-Mills pairs

Consider the following smooth map of Fréchet manifolds:

$$\begin{aligned} \mathcal{E} := (\mathcal{E}_1, \mathcal{E}_2)\colon \mathrm{Conf}(P) &\to \Gamma(T^*M \odot T^*M) \times \Omega^1(M, \mathfrak{g}_P) \\ (g, A) &\mapsto (\mathcal{E}_1(g,A) := \mathbb{G}^g - \mathcal{T}(g,A)\,, \mathcal{E}_2(g,A) := \mathrm{d}_A^{g*} F_A) \end{aligned}$$

Space of solutions on (P, c): $\mathcal{E}^{-1}(0) \subset \mathrm{Conf}(P)$ with the subspace topology.

$\mathrm{Aut}(P)$ has a natural smooth action on $\mathrm{Conf}(P)$ via pullback:

$$\Phi\colon \mathrm{Conf}(P) \times \mathrm{Aut}(P) \to \mathrm{Conf}(P)\,, \qquad (g, A, u) \mapsto (f_u^* g, u^* A)$$

$\mathcal{E}$ is equivariant with respect to this action, that is, $\mathcal{E}(f_u^* g, u^* A) = f_u^* \mathcal{E}(g, A)$. Hence, $\mathrm{Aut}(P)$ preserves the solution space $\mathcal{E}^{-1}(0) \subset \mathrm{Conf}(P)$. The quotient:

$$\mathfrak{M}(P, c) = \mathcal{E}^{-1}(0)/\mathrm{Aut}(P)\,,$$

equipped with quotient topology is the moduli space of Einstein-Yang-Mills pairs.

Orbit map: $\Phi_{(g,A)}\colon \mathrm{Aut}(P) \to \mathrm{Conf}(P)\,, u \mapsto (f_u^* g, u^*(A))$.

Orbit through (g, A): $\mathcal{O}_{(g,A)} := \mathrm{Im}(\Phi_{(g,A)}) \subset \mathrm{Conf}(P)$.

Stabilizer: $\mathcal{I}_{(g,A)} := \{u \in \mathrm{Aut}(P) \mid (f_u^* g, u^*(A)) = (g, A)\}$

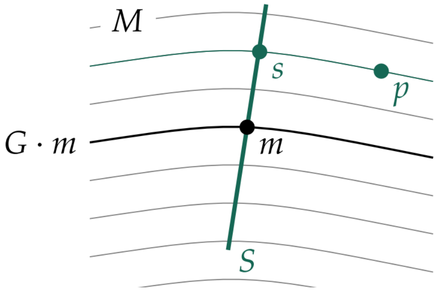
The isotropy group $\mathcal{I}_{(g,A)}$ fits non-canonically into the following short exact sequence:

$$1 \to C[\mathrm{Hol}_m(A), \mathrm{G}] \to \mathcal{I}_{(g,A)} \to \mathrm{Iso}(M, g)' \to 1\,,$$

where $C[\mathrm{Hol}_m(A), \mathrm{Aut}(P_m)]$ denotes the centralizer of the holonomy of A at $m \in M$ inside the automorphism group $\mathrm{Aut}(P_m) \cong \mathrm{G}$ of the fiber P_m and $\mathrm{Iso}(M, g)'$ denotes the Lie subgroup of the isometry group of (M, g) that can be covered by elements in $\mathcal{I}_{(g,A)}$. Recall that, since both $C[\mathrm{Hol}_m(A), \mathrm{G}]$ and $\mathrm{Iso}(M, g)'$ are finite-dimensional Lie groups it follows that $\mathcal{I}_{(g,A)}$ is a finite-dimensional Lie group: even more, it is a compact Lie group!

We would like to understand in more detail the local structure of $\mathrm{Conf}(P)/\mathrm{Aut}(P)$. How can we proceed further? Let us start with the local structure of this space!

For this, it is very useful to construct a *slice*, that is, an embedded smooth space as *transverse as possible* to the orbit $\mathcal{O}_{(g,A)}$ and nearby orbits, which produces a tubular neighbourhood of $\mathcal{O}_{(g,A)}$ via the action of $\mathrm{Aut}(P)$ as well as a local chat for $\mathrm{Conf}(P)/\mathrm{Aut}(P)$ around (g, A) modulo the natural action of the stabilizer $\mathcal{I}_{(g,A)}$.



Let $\mathcal{M}$ be a $\mathcal{G}$ manifold. A *slice* at $m \in \mathcal{M}$ is a submanifold $\mathcal{S} \subset \mathcal{M}$ containing m with the following properties:

- The submanifold $\mathcal{S}$ is invariant under $\mathcal{G}_m \subset \mathcal{G}$.
- Any $u \in \mathcal{G}$ with $(u \cdot \mathcal{S}) \cap \mathcal{S} \neq \emptyset$ satisfies $u \in \mathcal{G}_m$.
- The projection $\mathcal{G} \to \mathcal{G}/\mathcal{G}_m$ is a principal bundle admitting a local section $\chi \colon \mathcal{U} \to \mathcal{G}$ around the identity coset such that:

  $$\bar{\chi} \colon \mathcal{U} \times \mathcal{S} \to \mathcal{M}\,, \quad ([u], s) \mapsto \chi([u]) \cdot s$$

  is a diffeomorphism onto an open neighborhood of m.

If $\mathcal{G}_m$ is trivial ((g, A) has no symmetries) then a slice around m is an embedded submanifold that intersects each orbit only once. Therefore is in one to one correspondence with the orbits of $\mathcal{G}$ acting on $\mathcal{M}$ around $\mathcal{O}_m$. In particular:

$$\mathcal{M}/\mathcal{G} \simeq \mathcal{S} \quad \text{locally around } m$$

Natural candidate for a chart of the slice: the orthogonal complement to $T_{(g,A)}\mathcal{O}_{(g,A)}$!

Lemma

The adjoint differential operator $\mathrm{d}_e\Phi^*_{(g,A)}\colon T_{(g,A)}\mathrm{Conf}(P) \to \mathfrak{X}(P)^{\mathrm{G}}$ *is given by:*

$$\mathrm{d}_e\Phi^*_{(g,A)}(h,a) = (2(\nabla^{g*}h)^{\sharp_g} - (a\lrcorner^c_g F_A)^{\sharp_g}, \mathrm{d}^{g*}_A a)$$

In particular, the orthogonal complement of $T_{(g,A)}\mathcal{O}_{(g,A)}$ *in* $T_{(g,A)}\mathrm{Conf}(P)$ *is given by:*

$$T_{(g,A)}\mathcal{O}^{\perp_g}_{(g,A)} = \left\{(g,a) \in T_{(g,A)}\mathrm{Conf}(P) \mid 2\nabla^{g*}h = a\lrcorner^c_g F_A\,,\ \mathrm{d}^{g*}_A a = 0\right\}\,.$$

Here $a\lrcorner^c_g F_A \in \Omega^1(M)$ *defined by* $(a\lrcorner^c_g F_A)(v) = -\langle a, \iota_v F_A\rangle_{g,c}$.

$T_{(g,A)}\mathcal{O}^{\perp_g}_{(g,A)} \subset T_{(g,A)}\mathrm{Conf}(P)$ is the natural candidate of *infinitesimal slice* for the action of $\mathrm{Aut}(P)$ on $\mathrm{Conf}(P)$. This is verified in the following slice theorem.

The slice theorem IV

Theorem

Let $(g, A) \in \mathrm{Conf}(P)$*. Then, there exists a slice* $\mathbb{S} \subset \mathrm{Conf}(P)$ *around* $(g, A) \in \mathrm{Conf}(P)$ *which is the image of an equivariant diffeomorphism* $\mathrm{E}\colon \mathcal{U} \to \mathbb{S}$*, where* $\mathcal{U} \subset T_{(g,A)}\mathcal{O}^{\perp_g}_{(g,A)}$ *is an open neighbourhood of* $0 \in \mathcal{U}$ *in* $T_{(g,A)}\mathcal{O}^{\perp_g}_{(g,A)}$.

Proof.

Use the general slice theorem of Diez and Rudolph together with:

- $\mathrm{Diff}(P)$ is a tame Lie group acting tamely and properly on the tame Fréchet manifold $\mathrm{Met}(P) \Rightarrow \mathrm{Aut}(P)$ acts on $\mathrm{Im}(\Theta^c) \subset \mathrm{Met}(P)$ tamely and properly.
- $\mathcal{O}_{\bar{g}}$ is a closed submanifold of $\mathrm{Aut}(P)$ and the L^2 orthogonal complements of $T\mathcal{O}_{\bar{g}}$ inside $T\mathrm{Im}(\Theta^c)|_{\mathcal{O}_{\bar{g}}}$ assemble into a smooth normal subbundle $\mathcal{N}\mathcal{O}_{\bar{g}}$ of the latter.
- $\mathrm{Im}(\Theta^c)$ inherits a smooth exponential map from that of $\mathrm{Met}(P)$ whose restriction to a neighbourhood of the zero section of the normal bundle $\mathcal{N}\mathcal{O}_{\bar{g}}$ is an equivariant local diffeomorphism onto its image.

□

This is only the very beginning of the study of moduli spaces! Still, it improves our understanding of the configuration space of Einstein-Yang-Mills theory.

Corollary

There exists an open neighborhood $\mathcal{U}$ *of the identity in* $\mathrm{Aut}(P)$ *and an open neighborhood* $\mathcal{V}$ *of* (g, A) *in* $\mathrm{Conf}(P)$ *such that for any* $(g', A') \in \mathcal{V}$ *the symmetry group* $\mathcal{I}_{(g',A')}$ *of* (g', A') *is conjugate to a subgroup of* $\mathcal{I}_{(g,A)}$ *via an element in* $\mathcal{U}$*.*

Corollary

The subset of elements of $\mathrm{Conf}(P)$ *with trivial symmetry group is open in* $\mathrm{Conf}(P)$*.*

The fact that the set of metrics with trivial isometry group and connections with trivial symmetry group is open and dense in $\mathrm{Met}(M)$ and $\mathcal{C}(P)$, respectively, together with the fact that $\mathrm{Conf}(P)$ is the direct product of $\mathrm{Met}(M)$ and $\mathcal{C}(P)$ equipped with the corresponding product Fréchet structures implies, in addition, the following result.

Corollary

The set of elements with trivial symmetry group is open and dense in $\mathrm{Conf}(P)$*.*

Conjecture

General infinitesimal deformations of susy initial data are again susy.

Gracias!

Proposition

For every $(g, A) \in \mathrm{Conf}(P)$ *we have* $\mathrm{d}_e\Phi^*_{(g,A)}(\mathcal{E}(g,A)) = 0$. *Furthermore:*

$$0 \to \Gamma(\mathcal{A}_P) \xrightarrow{\mathrm{d}_e\Phi_{(g,A)}} T_{(g,A)}\mathrm{Conf}(P) \xrightarrow{\mathrm{d}_{(g,A)}\mathcal{E}} T_{(g,A)}\mathrm{Conf}(P) \xrightarrow{\mathrm{d}_e\Phi^*_{(g,A)}} \Gamma(\mathcal{A}_P) \to 0$$

is a complex if (g, A) *is an EYM pair.*

Elements $(h, a) \in T_{(g,A)}\mathrm{Conf}(P)$ in the kernel of $\mathrm{d}_{(g,A)}\mathcal{E}$ are the so-called *infinitesimal deformations* of the Einstein-Yang-Mills pair (g, A). Associated cohomology groups:

$$\mathbb{H}^0_{(g,A)} = \mathrm{Ker}(\mathrm{d}_e\Phi_{(g,A)})\,, \quad \mathbb{H}^1_{(g,A)} = \frac{\mathrm{Ker}(\mathrm{d}_{(g,A)}\mathcal{E})}{\mathrm{Im}(\mathrm{d}_e\Phi_{(g,A)})}\,, \quad \mathbb{H}^2_{(g,A)} = \frac{\mathrm{Ker}(\mathrm{d}_e\Phi^*_{(g,A)})}{\mathrm{Im}(\mathrm{d}_{(g,A)}\mathcal{E})}$$

Elements in the vector space $\mathbb{H}^0_{g,A}$ correspond to the *infinitesimal symmetries* of (g, A), whereas elements in the vector space $\mathbb{H}^1_{g,A}$ correspond to the *infinitesimal deformations* of (g, A) modulo the infinitesimal action of $\mathrm{Aut}(P)$. ($\mathbb{H}^3_{(g,A)}$ defined similarly).

Theorem

For every Einstein-Yang-Mills pair (g, A) *the vector space of essential deformations* $\mathbb{H}^1_{(g,A)}$ *is finite-dimensional and isomorphic to the vector space of obstructions* $\mathbb{H}^2_{(g,A)}$.

Proof.

The fact that $\mathbb{H}^1_{(g,A)}$ is finite-dimensional follows from the ellipticity of the deformation complex. $\mathrm{d}_{(g,A)}\bar{\mathcal{E}} \colon T_{(g,A)}\mathrm{Conf}(P) \to T_{(g,A)}\mathrm{Conf}(P)$ is self-adjoint, whence:

$$T_{(g,A)}\mathrm{Conf}(P) = \mathrm{Ker}(\Delta^{(1)}_{(g,A)}) \oplus \mathrm{Im}(\mathrm{d}_{(g,A)}\bar{\mathcal{E}}) \oplus \mathrm{Im}(\mathrm{d}_e\Phi_{(g,A)})$$

$$T_{(g,A)}\mathrm{Conf}(P) = \mathrm{Ker}(\Delta^{(2)}_{(g,A)}) \oplus \mathrm{Im}(\mathrm{d}_{(g,A)}\bar{\mathcal{E}}) \oplus \mathrm{Im}(\mathrm{d}_e\Phi_{(g,A)})$$

We obtain an isomorphism between $\mathrm{Ker}(\Delta^{(1)}_{(g,A)})$ and $\mathrm{Ker}(\Delta^{(2)}_{(g,A)})$ which immediately implies that $\mathbb{H}^1_{(g,A)}$ and $\mathbb{H}^2_{(g,A)}$ are isomorphic as finite dimensional vector spaces. □

Essential deformations $\mathbb{E}_{(g,A)} = \mathrm{Ker}(\mathrm{d}_{(g,A)}\mathcal{E}) \cap \mathrm{Ker}\,\mathrm{d}_e\Phi^*_{(g,A)} \simeq \mathbb{H}^1_{(g,A)}$.

Theorem

Let (g, A) be an Einstein-Yang-Mills pair and $\mathbb{S} \subset T_{(g,A)}\mathrm{Conf}(P)$ a slice around (g, A). Then, there exists an analytic closed submanifold $\mathcal{Z}_{(g,A)} \subset \mathbb{S}$ of $\mathbb{S}$ such that:

$$T_{(g,A)}\mathcal{Z} = \mathrm{Ker}(\mathrm{d}_{(g,A)}\mathcal{E}) \cap \mathrm{Ker}\,\mathrm{d}_e\Phi^*_{(g,A)}$$

and $\mathcal{E}^{-1}(0) \cap \mathbb{S}$ is an analytic subset of $\mathcal{Z}_{(g,A)}$.

Proof.

By the Hodge decomposition we have $\mathrm{Im}(\mathrm{d}_{(g,A)}\mathcal{E}_{\mathbb{S}}) = \mathrm{Im}(\mathrm{d}_{(g,A)}\mathcal{E}) \subset T_{(g,A)}\mathrm{Conf}(P)$. Denote by $\mathrm{P}\colon T_{(g,A)}\mathrm{Conf}(P) \to \mathrm{Im}(\mathrm{d}_{(g,A)}\mathcal{E})$ the natural projection. Then:

$$\mathrm{P} \circ \mathcal{E}_{\mathbb{S}}\colon \mathbb{S} \to \mathrm{Im}(\mathrm{d}_{(g,A)}\mathcal{E})$$

is a smooth map that has a surjective derivative at $(g, A) \in \mathrm{Conf}(P)$. We realize this map as a projective limit of Sobolev spaces and maps, applying the inverse function theorem in the Hilbert category and proving that the local model for the inverse of zero does not depend on the Sobolev norm and consists only of smooth elements. □

Essential deformations I

Proposition

A pair $(h,a) \in T_{(g,A)}\mathrm{Conf}(P)$ is an essential deformation of (g,A) if and only if:

$$\frac{1}{2}(\nabla^{g*}\nabla^{g})h - \mathrm{R}^{g}_{o}(h) - \delta^{g}\nabla^{g*}h - \frac{1}{2}\nabla^{g}\mathrm{d}\mathrm{Tr}_{g}(h) - \kappa\Big(F_A \circ_{h,c} F_A + h \circ_g (F_A \circ_{g,c} F_A) - 2F_A \circ_{g,c} \mathrm{d}_A a\Big) - \frac{\kappa}{n-2}\Big(2\langle F_A, \mathrm{d}_A a\rangle_{g,c} - g(F_A \circ_{g,c} F_A, h)\Big)g = 0$$

$$\mathrm{d}^{g*}_{A}\mathrm{d}_A a - a \lrcorner^{\mathfrak{g}}_{g} F_A - \frac{1}{2}\mathrm{d}\mathrm{Tr}_{g}(h)\lrcorner_{g} F_A + \mathrm{d}^{g*}_{A}(F_A)^{g}_{h} = 0\,, \quad 2\nabla^{g*}h = a\lrcorner^{c}_{g}F_A\,, \quad \mathrm{d}^{g*}_{A}a = 0$$

Deformations of the EYM system do not decouple even for *pure* metric or YM deformations: $a \in \Omega^1(M,\mathfrak{g}_P)$ defines $(0,a) \in \mathbb{H}^1_{(g,A)}$ if and only if:

$$\mathrm{d}^{g*}_{A}\mathrm{d}_A a = a\lrcorner^{\mathfrak{g}}_{g}F_A\,, \quad \mathrm{d}^{g*}_{A}a = 0\,, \quad F_A \circ_{g,c} \mathrm{d}_A a = \frac{1}{n-2}\langle F_A, \mathrm{d}_A a\rangle_{g,c}\, g\,, \quad a\lrcorner^{c}_{g}F_A = 0$$

By Koiso, solutions to the first and second equations above correspond to essential deformations of A as a YM connection. Hence, essential deformations of the form $(0,a)$ correspond to the subset of the essential deformations of A as a YM connection that satisfies the third and fourth equations above.

Essential deformations II

The previous slide implies:

Lemma

Let $(h, a) \in \mathrm{Ker}(\mathrm{d}_{(g,A)}\mathcal{E}) \subset T_{(g,A)}\mathrm{Conf}(P)$ *be an infinitesimal deformation of* (g, A)*. Then, the following equations hold according to the dimension* n*:*

$$\Delta_g \mathrm{tr}(h) + (\nabla^{g*}\nabla^{g*})h + \kappa\, g((F_A \circ_{g,c} F_A)^o, h^o) = 0\,, \qquad n = 4$$

$$\Delta_g \mathrm{tr}(h) + (\nabla^{g*}\nabla^{g*})h + \frac{s^g}{n}\mathrm{tr}_g(h) = 2\kappa\frac{n-4}{2-n}\langle \mathrm{d}_A a, F_A\rangle_{g,c} + \frac{2\kappa}{2-n} g((F_A \circ_{g,c} F_A)^o, h^o)\,,$$

where $(F_A \circ_{g,c} F_A)^o$ *and* h^o *denote the trace-less projections of* $(F_A \circ_{g,c}, F_A)$ *and* h*.*

Corollary

Let $(h, a) \in \mathrm{Ker}(\mathrm{d}_{(g,A)}\mathcal{E}) \subset T_{(g,A)}\mathrm{Conf}(P)$ *be an infinitesimal deformation of an Einstein-Yang-Mills pair* (g, A)*. Then, the following equations hold:*

$$\int_M \frac{s^g}{n}\mathrm{tr}_g(h)\nu_g = \frac{2\kappa}{2-n}\int_M g((F_A \circ_{g,c} F_A)^o, h^o)\nu_g\,, \quad 0 = [2\nabla^{g*}h^o - a \lrcorner^c_g F_A] \in H^1(M, \mathbb{R})$$

In particular, if $n = 4$ *we have* $\int_M g((F_A \circ_{g,c} F_A)^o, h^o)\nu_g = 0$*.*

The trace of a deformation $(h, a) \in T_{(g,A)}\mathrm{Conf}(P)$ does not in general decouple. Define $(h^o, a) \in T^o_{(g,A)}\mathrm{Conf}(P)$ to be *completable* if there exists $f \in C^\infty(M)$ such that $(fg/4 + h^o, a)$ is an essential deformation of (g, A).

Theorem

Let (g, A) be an anti-self-dual Einstein-Yang-Mills pair on a principal bundle P over a four-dimensional manifold M. An unobstructed pair $(h^o, a) \in T^o_{(g,A)}\mathrm{Conf}(P)$ is a completable essential deformation of (g, A) if and only if:

$$\frac{1}{2}\nabla^{g*}\nabla^g h^o - \mathrm{R}^g_o(h^o) - 2\delta^g\nabla^{g*}h^o - \frac{1}{6}\nabla^{g*}\nabla^{g*}h^o g + \frac{1}{2}\delta^g(a\lrcorner^c_g F_A)$$
$$-\kappa\Big(F_A \circ_{h^o,c} F_A + \frac{1}{2}|F_A|^2_{g,c}h^o - 2F_A \circ_{g,c} \mathrm{d}_A a + \langle F_A, \mathrm{d}_A a\rangle_{g,c} g\Big) = 0$$
$$\mathrm{d}^{g*}_A \mathrm{d}_A a - a\lrcorner^{\mathfrak{g}}_g F_A + \mathrm{d}^{g*}_A (F_A)^g_{h^o} = 0\,, \qquad \mathrm{d}^{g*}_A a = 0$$

If that is the case, the completed essential deformation (h, a) satisfies $4h = fg + 4h^o$, where $\mathrm{d}f = 4\nabla^{g*}h^o - 2\,a\lrcorner^c_g F_A$.

Essential deformations in four dimensions II

Natural examples of Ricci flat metrics coupled to instantons in $4d$: zero-slope holomorphic vector bundles over K3 surfaces or complex tori. These have their own deformation problem based on first-order differential operators: $\mathbb{E}_{(g,A)} \subset \mathbb{H}^1_{(g,A)}$.

Natural question: Is $\mathbb{E}_{(g,A)} = \mathbb{H}^1_{(g,A)}$?

Corollary

Let $(h,a) \in \mathbb{H}^1_{(g,A)}$. *If* $\mathrm{d}_A(F_A)^g_{h^o} = 0$ *and* $\mathrm{Ker}(\mathrm{d}_A) \cap \mathrm{Ker}(\mathrm{d}_A^{g*}) \subset \Omega^2(M, \mathfrak{g}_P)$ *vanishes then* $(h,a) \in \mathbb{E}_{(g,A)}$.

- Construct examples of Einstein-Yang-Mills pairs whose Yang-Mills connection is not an instanton.
- Classify homogenous Riemannian manifolds carrying Einstein-Yang-Mills pairs.
- Study the vector space of essential deformations of an Einstein-Yang-Mills pair and the rigidity of Einstein-Yang-Mills pairs.
- Study the second variation of the Einstein-Yang-Mills functional and the stability of Einstein-Yang-Mills pairs.
- Extend the current results to supergravity.