Dynamical RG flows and universality in classical multifield cosmological models

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Outline

FLRW cosmology with scalar matter

Definition

An n-dimensional scalar triple is an ordered system $(\mathcal{M}, \mathcal{G}, \Phi)$, where:

- $(\mathcal{M}, \mathcal{G})$ is a connected and borderless Riemannian n-manifold (called scalar manifold)
- $\Phi \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ is a smooth function (called scalar potential).

Assumptions

1. $(\mathcal{M}, \mathcal{G})$ is complete (this ensures conservation of energy)
2. $\Phi > 0$ on $\mathcal{M}$ (this avoids certain technical problems but can be relaxed)

Each scalar triple defines a cosmological model:

$$\mathcal{S}_{\mathcal{M},\mathcal{G},\Phi}[g,\varphi] = \int_{\mathbb{R}^4} \mathrm{d}^4x \sqrt{|g|}\left[\frac{M^2}{2}R(g) - \frac{1}{2}\mathrm{Tr}_g\varphi^*(\mathcal{G}) - \Phi\circ\varphi\right] .$$

Define the *rescaled Planck mass* through $M_0 \stackrel{\text{def.}}{=} \sqrt{\frac{2}{3}}M$, where M is the reduced Planck mass. Take g to describe a simply-connected and spatially flat FLRW universe:

$$\mathrm{d}s_g^2 := -\mathrm{d}t^2 + a^2(t)\mathrm{d}\vec{x}^2 \quad (x^0 = t \ , \ \vec{x} = (x^1, x^2, x^3) \ , \ a(t) > 0 \ \forall t)$$

and φ to depend only on the cosmological time $\varphi = \varphi(t)$.

Define the *Hubble parameter* through $H(t) \stackrel{\text{def.}}{=} \frac{\dot{a}(t)}{a(t)}$. When $H > 0$, the equations of motion are equivalent with the cosmological equation:

$$\nabla_t \dot{\varphi}(t) + \frac{1}{M_0}\left[||\dot{\varphi}(t)||^2_{\mathcal{G}} + 2\Phi(\varphi(t))\right]^{1/2} \dot{\varphi}(t) + (\mathrm{grad}_{\mathcal{G}}\Phi)(\varphi(t)) = 0 \quad ,$$

together with the condition

$$H(t) = H_\varphi(t) \stackrel{\text{def.}}{=} \frac{1}{3M_0}\left[||\dot{\varphi}(t)||^2_{\mathcal{G}} + 2\Phi(\varphi(t))\right]^{1/2} = \frac{1}{3M_0}\mathcal{H}(\dot{\varphi}(t)) \quad ,$$

where the *rescaled Hubble function* $\mathcal{H} : T\mathcal{M} \to \mathbb{R}_{>0}$ is defined through:

$$\mathcal{H}(u) \stackrel{\text{def.}}{=} \sqrt{||u||^2 + 2\Phi(\pi(u))} \quad \forall u \in T\mathcal{M}$$

and $\pi : T\mathcal{M} \to \mathcal{M}$ is the bundle projection.

The solutions $\varphi : I \to \mathcal{M}$ of the cosmological equation (where I is a non-degenerate interval) are called cosmological curves.

The cosmological equation defines an autonomous and dissipative geometric dynamical system on $T\mathcal{M}$. The flow $\Pi : \mathcal{D} \to T\mathcal{M}$ (with $\mathcal{D} \subset \mathbb{R} \times T\mathcal{M}$) of this dynamical system is called cosmological flow.

Main ideas and results

Main idea

We study the cosmological dynamical system using ideas inspired by the Wilsonian theory of RG flows in critical phenomena.

Main results

- We introduce a dynamical RG flow for this system, which relates the UV and IR limits. The UV and IR limit correspond to fast and slow variation of cosmological curves (high and low frequency modes of $\varphi(t)$)
- We consider UV and IR scale expansions, which expand the cosmological flow around its UV and IR limits. These expansions were not studied previously in the multifield model literature.
- The leading approximants of the UV and IR expansions recover the geodesic flow of $(\mathcal{M}, \mathcal{G})$ and certain reparameterization of the gradient flow of $(\mathcal{M}, \mathcal{G}, V)$, where $V \stackrel{\mathrm{def.}}{=} M_0\sqrt{2\Phi}$ is called the *classical effective potential of the model*.
- We define UV and IR cosmological universality classes, which allow for natural classifications of multifield models.
- IR cosmological universality classes depend only on the conformal equivalence class of $(\mathcal{M}, \mathcal{G}, V)$.

Some applications

- We show that two-field models whose scalar field metric has constant Gaussian curvature are infrared universal. In particular, generalized α-attractor models are IR universal among hyperbolizable two-field models.
- We study the "phases" arising in IR universality classes of tame hyperbolizable two field models, characterizing all of them explicitly.
  - The IR phases arising from non-plane ends are qualitatively different from those of models based on the Poincare disk.
  - The phases associated to Freudenthal ends of the scalar manifold have exotic behavior different from that of hyperbolic dynamical systems.
  - We compare the first order IR approximation for such models with numerical studies.
  - Cusp ends of hyperbolic surfaces are particularly interesting, leading to highly complex fast turn cosmological curves that are not well-approximated by the first order of the IR expansion.

The similarity group

Multifield cosmological models admit a universal two-parameter group of similarities, which relate the cosmological curves of models with the same target manifold $\mathcal{M}$ but different parameters $(M_0, \mathcal{G}, \Phi)$.

Definition

Let $\epsilon > 0$. The *ϵ-scale transform* of a curve $\varphi : I \to \mathcal{M}$ is the curve $\varphi_\epsilon : I_\epsilon \to \mathcal{M}$ defined through:

$$I_\epsilon \stackrel{\text{def.}}{=} \epsilon I = \{\epsilon t | t \in I\} \ \ , \ \ \varphi_\epsilon(t) \stackrel{\text{def.}}{=} \varphi(t/\epsilon) \ \ \forall t \in I_\epsilon \ \ .$$

The cosmological equation is invariant under:

- **Parameter homotheties**:

$$\mathcal{G} \to \lambda \mathcal{G} \ \ , \ \ \Phi \to \lambda \Phi \ \ , \ \ M_0 \to \lambda^{1/2} M_0 \ \ (\lambda > 0)$$

- **Scale similarities**:

$$\varphi \to \varphi_\epsilon \ \ , \ \ \Phi \to \Phi_\epsilon \stackrel{\text{def.}}{=} \Phi/\epsilon^2 \ \ (\epsilon > 0) \ \ .$$

Definition

The *RG similarity* is the composition of the parameter homothety at parameter $\lambda = \epsilon^2$ with the scale similarity at parameter ϵ:

$$\varphi \to \varphi_\epsilon \ \ , \ \ M_0 \to \epsilon M_0 \ \ , \ \ \mathcal{G} \to \epsilon^2 \mathcal{G} \ \ , \ \ \Phi \to \Phi \ \ (\epsilon > 0) \ \ .$$

The time t cosmological flow of the model with parameters $(\epsilon M_0, \epsilon^2\mathcal{G}, \Phi)$ coincides with the time t/ϵ flow of the model with parameters $(M_0, \mathcal{G}, \Phi)$. Hence the IR and UV limits of the latter can be described equivalently by taking ϵ to zero and infinity in the former. The RG transformations:

$$M_0 \to \epsilon M_0 \quad , \quad \mathcal{G} \to \epsilon^2\mathcal{G} \quad , \quad \Phi \to \Phi$$

play a role similar to that in the theory of critical phenomena.

Main result

The RG flow of the model $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ interpolates between the geodesic flow of $(\mathcal{M}, \mathcal{G})$ (which is recovered in the UV limit $\epsilon \to \infty$) and a reparameterization of the gradient flow of the classical effective potential $V \stackrel{\mathrm{def.}}{=} M_0\sqrt{2\Phi}$ (which is recovered in the IR limit $\epsilon \to 0$).

A *Riemannian homothety line* on $\mathcal{M}$ is a one-dimensional linear subspace $L \subset \mathcal{T}_2(\mathcal{M}) = \Gamma(\mathcal{M}, \mathrm{Sym}^2(\mathcal{M}))$. Its elements positively-homothetic to $\mathcal{G}$ form an open half-line $L_+ \subset L$. The *cosmological homothety plane* defined by L is the set $\Pi(\mathcal{M}, L) \stackrel{\mathrm{def.}}{=} \mathbb{R} \times L \subset \mathbb{R} \times \mathcal{T}_2(\mathcal{M})$, which contains the *cosmological homothety cone* $\mathrm{C}(\mathcal{M}, L) \stackrel{\mathrm{def.}}{=} \mathbb{R}_+ \oplus L_+$. The *cosmological RG action* on $\mathrm{C}(\mathcal{M}, L)$ is the $\mathbb{R}_{>0}$ action defined through:

$$\rho_{\mathrm{RG}}(\epsilon)(M_0, \mathcal{G}) \stackrel{\mathrm{def.}}{=} (\epsilon M_0, \epsilon^2\mathcal{G}) \quad \forall (M_0, \mathcal{G}) \in \mathrm{C}(\mathcal{M}, L) \quad \forall \epsilon > 0 \quad .$$

Setting $\epsilon = e^\lambda$ with $\lambda \in \mathbb{R}$, this action describes the flow on the homothety cone of the Euler vector field E_L defined through:

$$E_L(M_0, \mathcal{G}) \stackrel{\text{def.}}{=} M_0 \oplus 2\mathcal{G} \in \Pi(\mathcal{M}, \mathcal{G}) \ \ \forall (M_0, \mathcal{G}) \in \mathrm{C}(\mathcal{M}, L) \ \ .$$

This flow is called the *cosmological RG flow* of $(\mathcal{M}, L)$. Choosing a reference metric $\mathcal{G}_1 \in L_+$ induces coordinates w_1, w_2 on $\mathrm{C}(\mathcal{M}, L)$ given by:

$$M_0 = w_1 \ \ , \ \ \mathcal{G} = w_2 \mathcal{G}_1 \ \ ,$$

which identify $\Pi(\mathcal{M}, L)$ and with $\mathbb{R}^2$ and $\mathrm{C}(\mathcal{M}, L)$ with the first quadrant. Then ρ_{RG} identifies with the action:

$$\rho_{\mathrm{RG}}(\epsilon)(w_1, w_2) = (\epsilon w_1, \epsilon^2 w_2) \ \ \forall \epsilon > 0$$

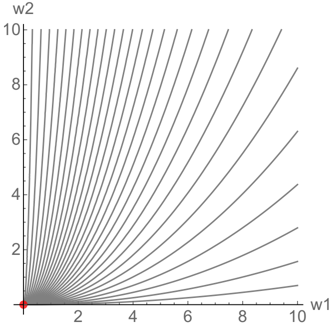
and E_L identifies with the vector field E on $\mathbb{R}^2_{>0}$ given by:

$$E(w_1, w_2) = (w_1, 2w_2) \ \ \forall w_1, w_2 > 0.$$

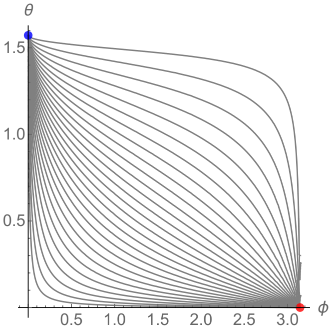
The integral curves of the RG flow identify with:

$$w_1(\lambda) = e^\lambda w_1(0) \ \ , \ \ w_2(\lambda) = e^{2\lambda} w_2(0) \ \ .$$

The limit $\lambda \to +\infty$ recovers the UV limit $\epsilon \to +\infty$ while $\lambda \to -\infty$ corresponds to the IR limit $\epsilon \to 0$. These correspond to the fixed points of the RG flow on the one-point compactification of the homothety cone, which are the apex of the cone and the point at infinity.



() Integral curves of the RG flow on the homothety cone.



() Integral curves of the RG flow on the one-point compactification of the homothety cone.

Figure: Integral curves of the RG flow on the homothety cone and on its one-point compactification. In the second figure we identified the compactification of the homothety plane with the unit sphere by stereographic projection. The RG flow has fixed points at $(\phi,\theta)=(\pi,0)$ (the red dot) and $(\phi,\theta)=(0,\frac{\pi}{2})$ (the blue dot), which correspond respectively to the apex of the homothety cone and its point at infinity, i.e. to the south and north poles of the sphere. These fixed points give the IR limit (red dot) and the UV limit (blue dot).

The rescaled cosmological equation

A curve $\varphi : I \to \mathcal{M}$ satisfies the cosmological equation of the model $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ iff φ_ϵ satisfies the ϵ-*rescaled cosmological equation*:

$$\epsilon^2 \nabla_t \frac{\mathrm{d}\varphi_\epsilon(t)}{\mathrm{d}t} + \frac{\epsilon}{M_0} \left[\epsilon^2 \left|\left|\frac{\mathrm{d}\varphi_\epsilon(t)}{\mathrm{d}t}\right|\right|_{\mathcal{G}}^2 + 2\Phi(\varphi_\epsilon(t))\right]^{1/2} \frac{\mathrm{d}\varphi_\epsilon(t)}{\mathrm{d}t} + (\mathrm{grad}_{\mathcal{G}}\Phi)(\varphi_\epsilon(t)) = 0 \ .$$

Dividing by ϵ^2, the latter is equivalent with:

$$\nabla_t \frac{\mathrm{d}\varphi_\epsilon(t)}{\mathrm{d}t} + \frac{1}{M_0} \left[\left|\left|\frac{\mathrm{d}\varphi_\epsilon(t)}{\mathrm{d}t}\right|\right|_{\mathcal{G}}^2 + 2\Phi_\epsilon(\varphi_\epsilon(t))\right]^{1/2} \frac{\mathrm{d}\varphi_\epsilon(t)}{\mathrm{d}t} + (\mathrm{grad}_{\mathcal{G}}\Phi_\epsilon)(\varphi_\epsilon(t)) = 0 \ ,$$

where $\Phi_\epsilon \stackrel{\mathrm{def.}}{=} \Phi/\epsilon^2$. Hence φ satisfies the cosmological equation of the scalar triple $(\mathcal{M}, \mathcal{G}, \Phi)$ iff φ_ϵ satisfies the cosmological equation of the scalar triple $(\mathcal{M}, \mathcal{G}, \Phi_\epsilon)$.

Remark

The UV limit $\epsilon \to \infty$ amounts to taking the overall scale of Φ to zero, while the IR limit $\epsilon \to 0$ amounts to taking the overall scale of Φ to infinity.

The UV expansion. When $\epsilon \gg 1$, expand φ_ϵ in positive powers of $\frac{1}{\epsilon^2}$. Then $\varphi(t) \stackrel{\mathrm{def.}}{=} \varphi_\epsilon(\epsilon t)$ is a solution of the cosmological equation which is expanded in non-negative powers of Φ. This amounts to treating Φ as small, Taylor expanding the reduced Hubble function $\mathcal{H} = \sqrt{||u||^2 + 2\Phi(\pi(u))}$ as:

$$\mathcal{H}(u) = ||u||_{\mathcal{G}} \left[1 + \frac{2\Phi(\pi(u))}{||u||^2_{\mathcal{G}}}\right]^{1/2} = ||u||_{\mathcal{G}} \left[1 + \frac{\Phi(\pi(u))}{||u||^2_{\mathcal{G}}} - \frac{1}{2}\left(\frac{\Phi(\pi(u))}{||u||^2_{\mathcal{G}}}\right)^2 + \ldots\right]$$

and seeking solutions φ of the cosmological equation expanded in powers of Φ.

The IR expansion. When $\epsilon \ll 1$, expand φ_ϵ in positive powers of ϵ. Equivalently, expand $\varphi(t)$ in powers of $\frac{1}{\sqrt{2\Phi}}$. This amounts to treating Φ as large and expanding the cosmological equation as:

$$\frac{1}{\sqrt{2\Phi(\varphi(t))}} \nabla_t \dot{\varphi}(t) + \frac{1}{M_0}\left[1 + \left(\frac{||\dot{\varphi}(t)||^2_{\mathcal{G}}}{\sqrt{2\Phi(\varphi(t))}}\right)^2 - \frac{1}{8}\left(\frac{||\dot{\varphi}(t)||_{\mathcal{G}}}{\sqrt{2\Phi(\varphi(t))}}\right)^4 + \ldots\right]^{1/2} \dot{\varphi}(t) + (\mathrm{grad}_{\mathcal{G}}\sqrt{2\Phi})(\varphi(t)) = 0$$

Then one seeks solutions expanded in powers of $\frac{1}{\sqrt{2\Phi}}$.

Remark

The small expansion parameter multiplies the highest order term in the last equation. The first order approximant φ_{IR} is obtained by solving a *first order* ODE, hence the first IR approximant of the cosmological flow is confined to a closed submanifold of $T\mathcal{M}$.

The leading IR approximant

To leading order in the IR expansion, φ is approximated by the solution φ_{IR} of the equation:

$$\frac{\mathrm{d}\varphi_{\mathrm{IR}}(t)}{\mathrm{d}t} + (\mathrm{grad}_{\mathcal{G}} V)(\varphi_{\mathrm{IR}}(t)) = 0$$

which satisfies:

$$\varphi_{\mathrm{IR}}(0) = \varphi(0) \quad .$$

Consider the *gradient flow shell*

$$\mathrm{Grad}_{\mathcal{G}} V \stackrel{\mathrm{def.}}{=} \mathrm{graph}(-\mathrm{grad}_{\mathcal{G}} V) = \{u \in T\mathcal{M} | u = -(\mathrm{grad}_{\mathcal{G}} V)(\pi(u))\} \subset T\mathcal{M} \quad .$$

of the effective scalar triple $(\mathcal{M}, \mathcal{G}, V)$.

Proposition

The cosmological flow $\Pi : \mathcal{D} \to \mathcal{M}$ *of the model is approximated to first order of the IR expansion by the map* $\Pi_{\mathrm{IR}} : \mathcal{D}_V \to \mathrm{Grad}_{\mathcal{G}} V$ *defined through:*

$$\Pi_{\mathrm{IR}}(t, u) \stackrel{\mathrm{def.}}{=} -(\mathrm{grad}_{\mathcal{G}} V)(\Pi_V(t, \pi(u)))$$

where $\Pi_V : \mathcal{D}_V \to \mathcal{M}$ *is the gradient flow of the effective scalar triple* $(\mathcal{M}, \mathcal{G}, V)$*, whose maximal domain of definition we denote by* $\mathcal{D}_V \subset \mathbb{R} \times \mathcal{M}$*.*

The first order IR approximation of a cosmological curve φ is most precise when $|t| \ll 1$ for those cosmological curves which satisfy $\dot{\varphi}(0) \in \mathrm{Grad}_{\mathcal{G}} V$.

Definition

A cosmological curve φ is called *infrared optimal* if its orbit meets the gradient flow shell $\mathrm{Grad}_{\mathcal{G}} V$ of the effective scalar triple $(\mathcal{M}, \mathcal{G}, V)$.

Suppose that $\varphi : I \to \mathcal{M}$ is an infrared optimal cosmological curve and let $t_0 \in I$ be such that $\dot{\varphi}(t_0) = -(\mathrm{grad}_{\mathcal{G}} V)(\varphi(t_0))$. Shifting t by a constant we can assume that $t_0 = 0$ without generality. Then the first order IR approximant of φ satisfies $\varphi_{\mathrm{IR}}(0) = \varphi(0)$ and $\dot{\varphi}_{\mathrm{IR}}(0) = \dot{\varphi}(0)$.

Thus φ_{IR} osculates in first order to φ at $t = 0$ and hence approximates φ to first order in t for $|t| \ll 1$. Notice that the covariant accelerations of φ and φ_{IR} need not agree at $t = 0$ and hence the approximation need not hold to second order in t.

Remark

One can work out the consistency conditions of the leading IR approximation. This approximation implies the first slow roll condition, but is far from equivalent to it.

Conformal invariance in the infrared

Proposition

The gradient flow of a scalar potential V defined on a Riemannian manifold $(\mathcal{M}, \mathcal{G})$ is invariant under Weyl rescalings of the metric $\mathcal{G}$ up to reparameterization of the gradient flow curves.

Definition

Two scalar triples $(\mathcal{M}_1, \mathcal{G}_1, V_1)$ and $(\mathcal{M}_2, \mathcal{G}_2, V_2)$ are called *conformally equivalent* if there exists a smooth conformal diffeomorphism $f : (\mathcal{M}_1, \mathcal{G}_1) \to (\mathcal{M}_2, \mathcal{G}_2)$ such that $V_1 = V_2 \circ f$. In this case, f is called a (smooth) *conformal equivalence* between the two triples. A *conformal automorphism* of a scalar triple $(\mathcal{M}, \mathcal{G}, V)$ is a conformal equivalence from $(\mathcal{M}, \mathcal{G}, V)$ to itself.

Proposition

The gradient flows of conformally-equivalent scalar triples are smoothly topologically equivalent.

Definition

Consider two scalar triples $(\mathcal{M}_1, \mathcal{G}_1, V_1)$ and $(\mathcal{M}_2, \mathcal{G}_2, V_2)$. A diffeomorphism $f : (\mathcal{M}_1, \mathcal{G}_1, V_1) \to (\mathcal{M}_2, \mathcal{G}_2, V_2)$ is called a *gradient equivalence* if:

$$f_\sharp(\Omega^{-1}\mathrm{grad}_{\mathcal{G}_1} V_1) = \mathrm{grad}_{\mathcal{G}_2} V_2$$

for some positive smooth function $\Omega : \mathcal{M}_1 \to \mathbb{R}_{>0}$. The two scalar triples are called *gradient equivalent* if there exists a gradient equivalence between them.

Example

A conformal equivalence of scalar triples $f : (\mathcal{M}_1, \mathcal{G}_1, V_1) \to (\mathcal{M}_2, \mathcal{G}_2, V_2)$ is a gradient equivalence.

Definition

Consider two classical cosmological models parameterized by $\mathfrak{M}_i = (M_{0i}, \mathcal{M}_i, \mathcal{G}_i, \Phi_i)$ and let $V_i \stackrel{\mathrm{def.}}{=} M_{0i}\sqrt{2\Phi_i}$ $(i = 1, 2)$. The models are called *IR equivalent* and we write $\mathfrak{M}_1 \sim_{\mathrm{IR}} \mathfrak{M}_2$ if there exists a smooth gradient equivalence

$$f : (\mathcal{M}_1, \mathcal{G}_1, V_1) \to (\mathcal{M}_2, \mathcal{G}_2, V_2) \quad .$$

The equivalence classes of $\sim_{\mathrm{IR}}$ are called *IR cosmological universality classes*.

Theorem (Poincaré)

The Weyl equivalence class of any Riemannian metric $\mathcal{G}$ on a borderless connected surface Σ contains a unique complete metric G (called the *uniformizing metric* of $\mathcal{G}$) of constant Gaussian curvature $K = -1$, 0 or $+1$.

The case $K = -1$ is generic – then $\mathcal{G}$ (and its conformal class) is called *hyperbolizable* and G is called the *hyperbolization* of $\mathcal{G}$. The cases $K = +1$ and $K = 0$ occur only for seven topologies, as follows:

- When $K = +1$, the surface Σ must be diffeomorphic with the 2-sphere S^2 or with the real projective plane $\mathbb{RP}^2 \simeq \mathrm{S}^2/\mathbb{Z}_2$. Both of these surfaces admit a unique metric of unit Gaussian curvature.
- When $K = 0$, the surface Σ must be diffeomorphic with the 2-torus T^2, the Klein bottle $\mathrm{K}^2 = \mathbb{RP}^2 \times \mathbb{RP}^2 \simeq \mathrm{T}^2/\mathbb{Z}_2$, the open annulus A^2 (which is diffeomorphic with the open cylinder and with the twice punctured sphere), the open Möbius strip $\mathrm{M}^2 \simeq \mathrm{A}^2/\mathbb{Z}_2$ (which is diffeomorphic with the once-punctured real projective plane) or with the plane $\mathbb{R}^2$.

For the remaining topologies, any metric $\mathcal{G}$ defined on Σ is hyperbolizable. The plane $\mathbb{R}^2$, the open annulus A^2 and the open Möbius strip M^2 are the only three surfaces which admit *two* types of uniformizing metrics, namely a Riemannian metric defined on one of these surfaces is uniformized either by a complete flat metric or by a hyperbolic metric depending on its conformal class.

Proposition

Up to (curve-dependent) increasing reparameterization, the IR behavior of the cosmological flow of a two-field model with scalar triple $(\Sigma, \mathcal{G}, \Phi)$ and rescaled Planck mass M_0 is described by the gradient flow of the scalar triple (Σ, G, V), where G is the uniformizing metric of $\mathcal{G}$ and $V \stackrel{\mathrm{def.}}{=} M_0\sqrt{2\Phi}$ is the classical effective scalar potential of the model.

Two-field models whose complete scalar manifold metric $\mathcal{G}$ has constant negative curvature are called *generalized two-field α-attractor models*.

Corollary

Generalized two-field α-attractor models are IR universal among two-field cosmological models with hyperbolizable target manifold.

This gives a conceptual reason to single out generalized two-field α-attractor models for special study.

Tame hyperbolizable two-field models

Consider a hyperbolizable model parameterized by $(M_0, \Sigma, \mathcal{G}, \Phi)$, where Σ is a connected borderless surface. Let G be the hyperbolization of $\mathcal{G}$ and $V \stackrel{\mathrm{def.}}{=} M_0\sqrt{2\Phi}$. Let $\widehat{\Sigma}$ be the end compactification of Σ, where each point of the set:

$$\mathrm{Ends}(\Sigma) \stackrel{\mathrm{def.}}{=} \widehat{\Sigma} \setminus \Sigma$$

corresponds to a Freudenthal end.

Definition

A hyperbolizable two-dimensional scalar triple $(\Sigma, \mathcal{G}, \Phi)$ is called *tame* if it satisfies the following conditions:

1. Σ is oriented and *topologically finite* in the sense that its fundamental group $\pi_1(\Sigma)$ is finitely-generated. This implies that Σ has finite genus and a finite number of ends and that its end compactification $\widehat{\Sigma}$ is a compact smooth surface. Notice that (Σ, G) need not have finite area.
2. The scalar potential Φ is *globally well-behaved*, i.e. Φ admits a smooth extension $\hat{\Phi}$ to $\widehat{\Sigma}$. We require that $\hat{\Phi}$ is strictly positive on $\widehat{\Sigma}$, which means that the limit of Φ at each end of Σ is a *strictly* positive number.
3. The extended potential $\hat{\Phi}$ is a Morse function on $\widehat{\Sigma}$ (in particular, Φ is a Morse function on Σ).

A two-field cosmological model with tame scalar triple is called tame.

Since Σ is topologically finite implies, the set of ends $\mathrm{Ends}(\Sigma)$ is finite. The condition that $\hat{\Phi}$ is Morse implies that the set:

$$\mathrm{Crit}\hat{\Phi} \stackrel{\mathrm{def.}}{=} \{c \in \widehat{\Sigma} | (\mathrm{d}\hat{\Phi})(c) = 0\}$$

is finite. Since $\hat{\Phi}$ is strictly positive on $\widehat{\Sigma}$, the classical effective potential $V = M_0\sqrt{2\Phi}$ has smooth extension to $\widehat{\Sigma}$ given by $\widehat{V} = M_0\sqrt{2\hat{\Phi}}$ and we have:

$$\mathrm{Crit}\,\widehat{V} = \mathrm{Crit}\hat{\Phi} \quad .$$

The critical points of V coincide with the *interior critical points* of $\widehat{V}$ (and $\hat{\Phi}$):

$$\mathrm{Crit}\,V = \mathrm{Crit}\Phi = \Sigma \cap \mathrm{Crit}\,\widehat{V} = \Sigma \cap \mathrm{Crit}\hat{\Phi} \quad .$$

Let:

$$\mathrm{Crit}_\infty V = \mathrm{Crit}_\infty \Phi \stackrel{\mathrm{def.}}{=} \mathrm{Ends}(\Sigma) \cap \mathrm{Crit}\,\widehat{V} = \mathrm{Ends}(\Sigma) \cap \mathrm{Crit}\hat{\Phi}$$

be the set of *critical ends*. We have the disjoint union decomposition:

$$\mathrm{Crit}\,\widehat{V} = \mathrm{Crit}\,V \sqcup \mathrm{Crit}_\infty V \quad .$$

An end of Σ which is not a critical point of $\hat{\Phi}$ (and hence of $\widehat{V}$) is called a *noncritical end*.

We denote interior critical points by $\mathbf{c}$ and arbitrary critical points of $\widehat{V}$ by c. We denote by $\mathbf{e}$ the ends of Σ.

Any end $\mathbf{e}$ of Σ admits an open neighborhood $U_\mathbf{e} \subset \widehat{\Sigma}$ diffeomorphic with a disk such that G has a canonical form when restricted to $\dot{U}_\mathbf{e} \stackrel{\text{def.}}{=} U_\mathbf{e} \setminus \{\mathbf{e}\} \subset \Sigma$. Namely, there exist *semigeodesic polar coordinates* $(r,\theta) \in \mathbb{R}_{>0} \times \mathrm{S}^1$ defined on $\dot{U}_\mathbf{e}$ in which:

$$\mathrm{d}s_G^2|_{\dot{U}_\mathbf{e}} = \mathrm{d}r^2 + f_\mathbf{e}(r)\mathrm{d}\theta^2 \quad ,$$

where:

$$f_\mathbf{e}(r) = \begin{cases} \sinh^2(r) & \text{if } \mathbf{e} = \text{plane end} \\ \frac{1}{(2\pi)^2} e^{2r} & \text{if } \mathbf{e} = \text{horn end} \\ \frac{\ell^2}{(2\pi)^2} \cosh^2(r) & \text{if } \mathbf{e} = \text{funnel end of circumference } \ell > 0 \\ \frac{1}{(2\pi)^2} e^{-2r} & \text{if } \mathbf{e} = \text{cusp end} \end{cases} .$$

The end $\mathbf{e}$ corresponds to $r \to \infty$. Setting $\omega \stackrel{\text{def.}}{=} \frac{1}{r}$, we have:

$$\mathrm{d}s_G^2|_{\dot{U}_\mathbf{e}} = \frac{\mathrm{d}\omega^2}{\omega^4} + f_\mathbf{e}(1/\omega)\mathrm{d}\theta^2 \quad ,$$

with:

$$f_\mathbf{e}(1/\omega) = \tilde{c}_\mathbf{e} e^{\frac{2\epsilon_\mathbf{e}}{\omega}} \left[1 + \mathrm{O}\left(e^{-\frac{2}{\omega}}\right)\right] \quad \text{for} \quad \omega \ll 1 \quad ,$$

where:

$$\tilde{c}_\mathbf{e} = \begin{cases} \frac{1}{4} & \text{if } \mathbf{e} = \text{plane end} \\ \frac{1}{(2\pi)^2} & \text{if } \mathbf{e} = \text{horn end} \\ \frac{\ell^2}{(4\pi)^2} & \text{if } \mathbf{e} = \text{funnel end of circumference } \ell > 0 \\ \frac{1}{(2\pi)^2} & \text{if } \mathbf{e} = \text{cusp end} \end{cases}$$

$$\epsilon_\mathbf{e} = \begin{cases} +1 & \text{if } \mathbf{e} = \text{flaring (i.e. plane, horn or funnel) end} \\ -1 & \text{if } \mathbf{e} = \text{cusp end} \end{cases}$$

Principal canonical coordinates centered at an end

Consider *canonical Cartesian coordinates* centered at an end **e**, which are defined on $U_{\mathbf{e}}$ by:

$$x + \mathbf{i}y \stackrel{\text{def.}}{=} \frac{1}{r} e^{\mathbf{i}\theta} \quad .$$

The *canonical polar coordinates* (ω, θ) centered at **e** are defined through:

$$\omega \stackrel{\text{def.}}{=} |x + \mathbf{i}y| = \frac{1}{r}$$

In such coordinates, the end **e** corresponds to $\omega = 0$, i.e. $(x, y) = (0, 0)$.

Definition

A canonical Cartesian coordinate system (x, y) for (Σ, G) centered at the critical end **e** is called *principal* for V if the tangent vectors $\epsilon_x = \frac{\partial}{\partial x}\big|_{\mathbf{e}}$ and $\epsilon_y = \frac{\partial}{\partial y}\big|_{\mathbf{e}}$ form a principal basis for V at **e**.

In a principal coordinate system (x, y) centered at **e**, the Taylor expansion of $\widehat{V}$ at **e** has the form:

$$\widehat{V}(x, y) \;=\; \widehat{V}(\mathbf{e}) + \frac{1}{2}\omega^2 \Big[\lambda_1(\mathbf{e}) \cos^2\theta + \lambda_2(\mathbf{e}) \sin^2\theta\Big] + \mathrm{O}(\omega^3) \;,$$

where $\lambda_1(\mathbf{e})$ and $\lambda_2(\mathbf{e})$ are the principal values of V at **e** and $\omega = \sqrt{x^2 + y^2}$, $\theta = \arg(x + \mathbf{i}y)$.

Definition

The *critical modulus* $\beta_{\mathbf{e}}$ of (Σ, G, V) at the critical end $\mathbf{e}$ is the ratio:

$$\beta_{\mathbf{e}} \stackrel{\text{def.}}{=} \frac{\lambda_1(\mathbf{e})}{\lambda_2(\mathbf{e})} \in [-1, 1] \setminus \{0\} \quad ,$$

where $\lambda_1(\mathbf{e})$ and $\lambda_2(\mathbf{e})$ are the principal values of (Σ, G, V) at $\mathbf{e}$. The sign factors:

$$\epsilon_i(\mathbf{e}) \stackrel{\text{def.}}{=} \mathrm{sign}(\lambda_i(\mathbf{e})) \in \{-1, 1\} \quad (i = 1, 2)$$

are called the *characteristic signs* of (Σ, G, V) at $\mathbf{e}$. V is called *circular* at $\mathbf{e}$ if $\lambda_1(\mathbf{e}) = \lambda_2(\mathbf{e})$.

Proposition

There exists a principal Cartesian canonical coordinate system (x, y) for (Σ, G, V) at every critical end $\mathbf{e}$*. When V is circular at* $\mathbf{e}$*, these coordinates are determined by V and G up to an $\mathrm{O}(2)$ transformation. When V is not circular at* $\mathbf{e}$*, these coordinates are determined by V and G up to the action of the subgroup Δ of $\mathrm{O}(2)$.*

When V is not circular at $\mathbf{e}$ (i.e. when $\lambda_1(\mathbf{e}) \neq \lambda_2(\mathbf{e})$), the geodesic orbits of $(\dot{U}_{\mathbf{e}}, G)$ given by $(\theta - \theta_{\mathbf{e}}) \mod 2\pi \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ will be called the *principal*

Definition

A system of *special Cartesian canonical coordinates* for (Σ, G, V) centered at the noncritical end $\mathbf{e}$ is a system of canonical Cartesian coordinates (x, y) centered at $\mathbf{e}$ which satisfies the conditions:

$$(\mathrm{d}\widehat{V})(\mathbf{e})(v_y) = 0 \ \text{ i.e. } \ (\partial_y \widehat{V})(\mathbf{e}) = 0$$

and:

$$(\mathrm{d}\widehat{V})(\mathbf{e})(v_x) > 0 \ \text{ i.e. } \ (\partial_x \widehat{V})(\mathbf{e}) > 0 \ .$$

Given such coordinates, we set $\mu_{\mathbf{e}} \stackrel{\text{def.}}{=} (\mathrm{d}\widehat{V})(\mathbf{e})(v_x) = (\partial_x \widehat{V})(\mathbf{e})$.

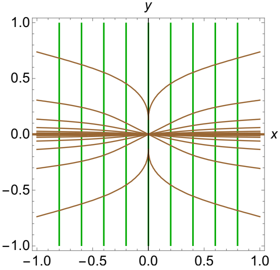
For $\theta \not\in \{0, \pi\}$, the unoriented gradient flow orbits are given by:

$$\frac{1}{4}\gamma_2\left(\frac{2\epsilon_{\mathbf{e}}}{\omega}\right) = A + \tilde{c}_{\mathbf{e}} \log|\sin\theta| \ ,$$

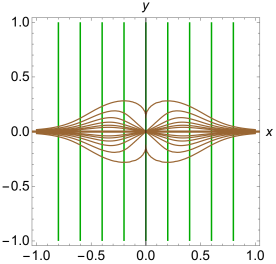
where A is an integration constant and:

$$\gamma_2(v) \stackrel{\text{def.}}{=} 1 - e^{-v} - ve^{-v} = \int_0^v we^{-w}\mathrm{d}w \ \ (\text{with } v \in \mathbb{R})$$

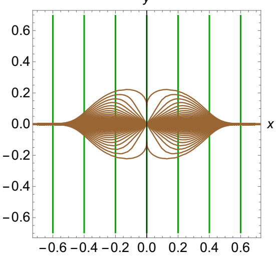
is the lower incomplete Gamma function of order 2.



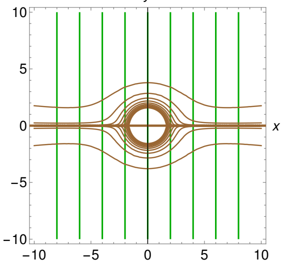
() Plane end.



() Horn end.

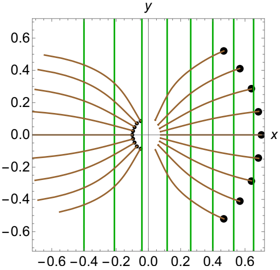


() Funnel end with $\ell = 1$.

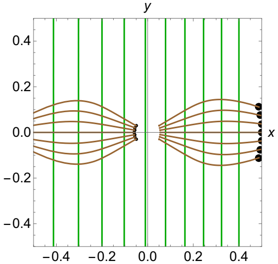


() Cusp end.

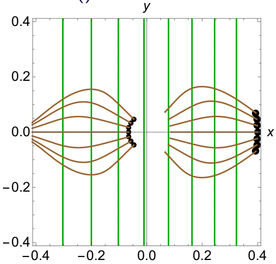
Comparison with infrared optimal cosmological curves near noncritical ends



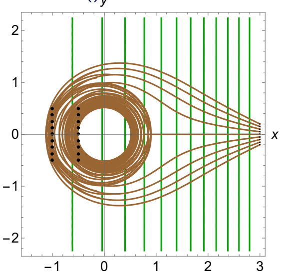
() Plane end.



() Horn end.



() Funnel end with $\ell = 1$.



() Cusp end.

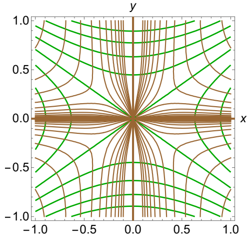
In special canonical coordinates a $\theta \notin \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$, the non-special unoriented gradient flow orbits are given by:

$$\frac{1}{4}[\lambda_1(\mathbf{e}) - \lambda_2(\mathbf{e})]\gamma_2\left(\frac{2\epsilon_{\mathbf{e}}}{\omega}\right) = A + \tilde{c}_{\mathbf{e}}\left[\,\lambda_1(\mathbf{e})\log|\sin\theta| - \lambda_2(\mathbf{e})\log|\cos\theta|\,\right] \quad ,$$

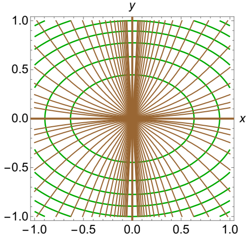
where A is an integration constant.

Proposition

The unoriented orbits of the asymptotic gradient flow of (Σ, G, V) near a critical end $\mathbf{e}$ are determined by the hyperbolic type of the end (i.e. by $\epsilon_{\mathbf{e}}$ and $\tilde{c}_{\mathbf{e}}$) and by the critical modulus $\beta_{\mathbf{e}}$, while the orientation of the orbits is determined by the critical signs $\epsilon_i(\mathbf{e})$, which satisfy $\epsilon_1(\mathbf{e})\epsilon_2(\mathbf{e}) = 1$.

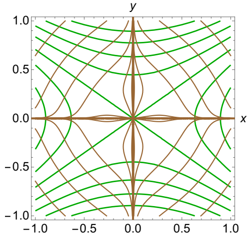


() For $\beta_{\mathbf{e}} = -1/2$

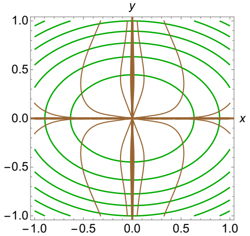


() For $\beta_{\mathbf{e}} = 1/2$

Figure: Critical plane end.

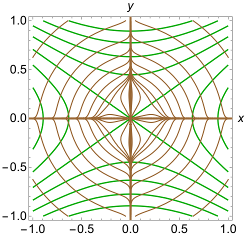


() For $\beta_{\mathbf{e}} = -1/2$.

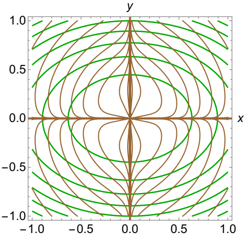


() For $\beta_{\mathbf{e}} = 1/2$.

Figure: Critical horn end

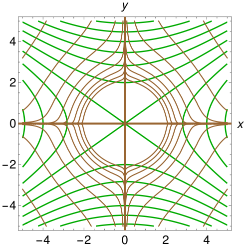


() For $\beta_e = -1/2$.

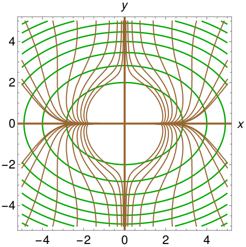


() For $\beta_e = 1/2$.

Figure: Critical funnel end.

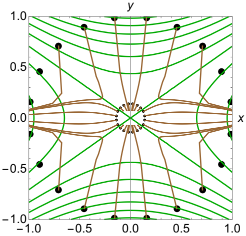


() For $\beta_e = -1/2$.

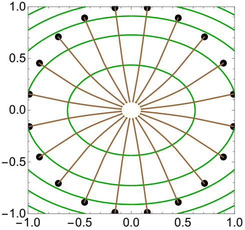


() For $\beta_e = 1/2$.

Figure: Critical cusp end.

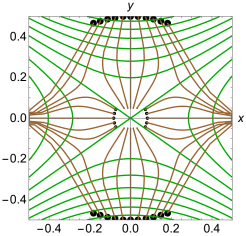


() For $\beta_{\mathbf{e}} = -1/2$

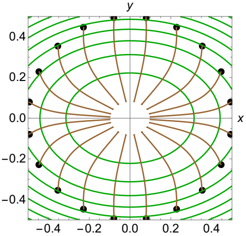


() For $\beta_{\mathbf{e}} = 1/2$

Figure: Critical plane end.

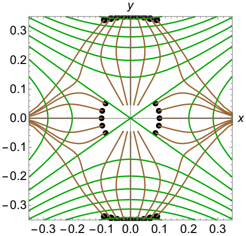


() For $\beta_{\mathbf{e}} = -1/2$.

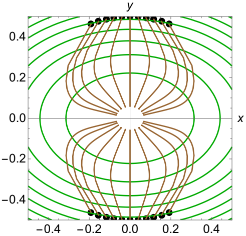


() For $\beta_{\mathbf{e}} = 1/2$.

Figure: Critical horn end.

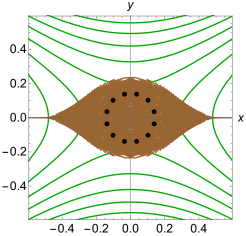


() For $\beta_{\mathbf{e}} = -0.5$.

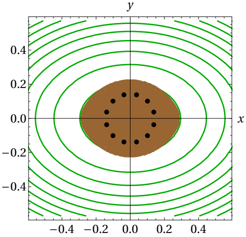


() For $\beta_{\mathbf{e}} = 0.5$.

Figure: Critical funnel end.



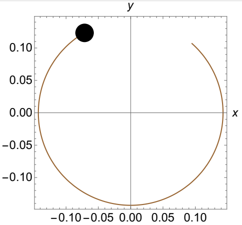
() For $\beta_{\mathbf{e}} = -0.5$.



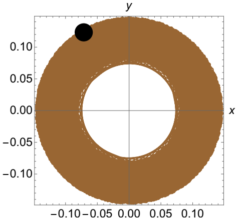
() For $\beta_{\mathbf{e}} = 0.5$.

Figure: Critical cusp end.

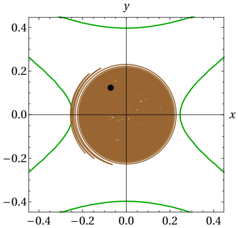
Some numerical cosmological curves near critical cusp ends



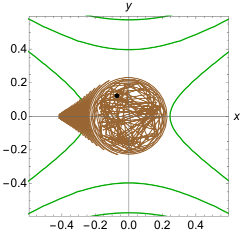
() For $t = 0.000008$.



() For $t = 0.01$.

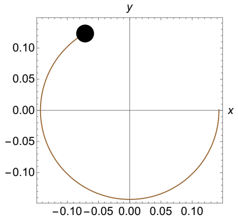


() For $t = 0.5$.

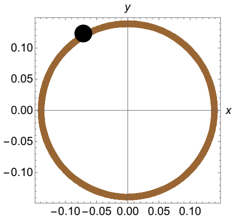


() For $t = 5$.

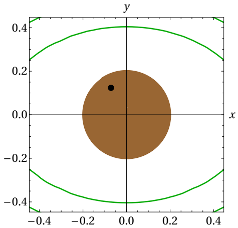
Figure: A numerically computed infrared optimal cosmological orbit of near a critical cusp end **e** for $\beta_{\mathbf{e}} = -1/2$.



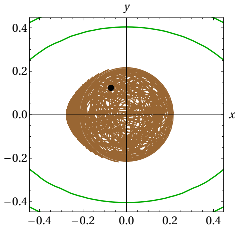
() For $t = 0.00002$.



() For $t = 0.01$.



() For $t = 0.5$.



() For $t = 5$.

Figure: A numerically computed infrared optimal cosmological orbit near a critical cusp end **e** for $\beta_{\mathbf{e}} = +1/2$ when the cusp end is a local minimum of $\widehat{V}$.